A TETRAVALENT HALF-ARC-TRANSITIVE GRAPH WITH NONABELIAN VERTEX STABILIZER

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Abstract

A construction is given of a 4-valent $\frac{1}{2}$-arc-transitive graph with vertex stabilizer isomorphic to the dihedral group D_8. The graph has 10752 vertices and is the first known example of a 4-valent $\frac{1}{2}$-arc-transitive graph with nonabelian vertex stabilizer.

1 Introduction

Throughout this paper graphs are assumed to be finite, simple and, unless specified otherwise, connected and undirected (but with an implicit orientation of the edges when appropriate). For the group-theoretic concepts and notation not defined here we refer the reader to [3, 10].

Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* if its automorphism group $\mathrm{Aut}\,X$ acts transitively on $V(X)$, $E(X)$ and $A(X)$ respectively. We say that X is $\frac{1}{2}$-*arc-transitive* provided it is vertex- and edge- but not arc-transitive. More generally, by a $\frac{1}{2}$-*arc-transitive* action of a subgroup $G \leq \mathrm{Aut}\,X$ on X we mean a vertex- and edge- but not arc-transitive action of G on X. In this case we say that the graph X is $(G, \frac{1}{2})$-*arc-transitive*, and we say that the graph X is $(G, \frac{1}{2}, H)$-*arc-transitive* when it needs to be stressed that the vertex stabilizers G_v (for $v \in V(X)$) are isomorphic to a particular subgroup $H \leq G$.

[1]Partially supported by the N.Z. Marsden Fund, grant no. UOA 810

[2]Supported in part by the "Ministrstvo za znanost in tehnologijo Republike Slovenije", project no. J1-101-04965-99.

For further references on $\frac{1}{2}$-arc-transitive graphs (which are also referred to as $\frac{1}{2}$-transitive graphs) see the survey paper [6].

Clearly the smallest admissible valency of a $\frac{1}{2}$-arc-transitive graph is 4, which is also the valency of the smallest $\frac{1}{2}$-arc-transitive graph, a graph on 27 vertices constructed by Holt [4]. Now in all of the known examples of 4-valent $\frac{1}{2}$-arc-transitive graphs, vertex stabilizers are abelian groups, or more precisely, elementary abelian 2-groups. On the other hand, $\frac{1}{2}$-arc-transitive graphs with nonabelian vertex stabilizers are known to exist: for example the graphs constructed by Bouwer [2] (in settling a question raised by Tutte in [9] about the existence of a $\frac{1}{2}$-arc-transitive graph of valency $2k$ for every $k \geq 2$) happen to have vertex stabilizers which induce the symmetric group S_k on the neighbourhood of the vertex. (This fact is not explicitly mentioned by Bouwer but it may be easily deduced from his construction.)

The object of this paper is to construct a $\frac{1}{2}$-arc-transitive graph of valency 4 with vertex stabilizer isomorphic to D_8.

How does one come about such a construction? The answer may be seen from the following general comments which provide a connection between graphs admitting $\frac{1}{2}$-arc-transitive group actions and transitive permutation groups having non-self-paired suborbits.

Let G be a transitive permutation group acting on a set V and let $v \in V$. There is a 1-1 correspondence between the set of *suborbits* of G, that is, the set of orbits of the stabilizer G_v on V, and the set of *orbitals* of G, that is, the set of orbits in the natural action of G on $V \times V$, with the trivial suborbit $\{v\}$ corresponding to the diagonal $\{(v,v) : v \in V\}$. For a suborbit W of G, let $\Delta = \Delta_W$ be the corresponding orbital of G. Then the *orbital graph* $X(G,V;W)$ *of* (G,V) *relative to* W is the graph with vertex set V and arc set Δ.

The paired orbital of an orbital Δ is $\Delta^t = \{(v,w) : (w,v) \in \Delta\}$. The orbital Δ is said to be *self-paired* if $\Delta^t = \Delta$, and *non-self-paired* otherwise; in the latter case $\Delta \cap \Delta^t = \emptyset$. This notion of (non)-self-pairedness also carries over to suborbits in a natural way, and it is important to note that for a non-self-paired suborbit W of G, the orbital graph $X(G,V;W)$ is an oriented graph, whereas the underlying undirected graph $X^*(G,V;W)$ admits a $\frac{1}{2}$-arc-transitive action of G.

In the specific instance (of the situation described above) where $V = \mathcal{H}$ is the set of right cosets of a subgroup H of G and W is a non-self-paired suborbit of length 2 in the action of G on $\mathcal{H}$ (by right multiplication), it follows that $X^*(G, \mathcal{H}, W)$ is a 4-valent $(G, H, \frac{1}{2})$-arc-transitive graph. In view of these remarks, our construction of a 4-valent $\frac{1}{2}$-arc-transitive graph with the desired properties will be based on a group G whose action on the set of right cosets of a subgroup $H \leq G$ isomorphic to D_8 gives rise to a non-self-paired suborbit of length 2. We shall prove the following result:

Theorem 1.1 *There exists a transitive permutation group G of degree 32 and order 86016, generated by two elements a and b of orders 8 and 24 respectively, such that if $H = \langle p, q, r \rangle$ where $p = a^{-1}b$ and $q = a^{-1}pa$ and $r = a^{-1}qa$, then:*

(i) *$H \cong D_8$;*

(ii) *$\{Ha, Hb\}$ is a non-self-paired suborbit of the right action of G on the set $\mathcal{H}$ of right cosets of H in G;*

(iii) *the underlying graph X^* of the orbital graph $X(G, \mathcal{H}; \{Ha, Hb\})$ is a connected 4-valent $(G, H, \frac{1}{2})$-arc-transitive graph on 10752 vertices.*

In fact G may be taken as the subgroup of S_{32} generated by

$a =$ (1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12, 13, 14, 15, 16) (17, 18, 19, 20, 21, 22, 23, 24)(25, 26, 27, 28, 29, 30, 31, 32), and

$b =$ (1, 2, 11, 18, 21, 28, 27, 22, 5, 14, 15, 16, 9, 10, 3, 26, 29, 20, 19, 30, 13, 6, 7, 8)(4, 23, 32, 17, 12, 31, 24, 25);

in which case we have

$p =$ (3, 11)(4, 26)(5, 23)(6, 14)(12, 18)(13, 31)(17, 25) (19, 21)(20, 30)(22, 28)(24, 32)(27, 29),

$q =$ (4, 12)(5, 27)(6, 24)(7, 15)(13, 19)(14, 32)(17, 25) (18, 26)(20, 22)(21, 31)(23, 29)(28, 30), and

$r =$ (5, 13)(6, 28)(7, 17)(8, 16)(14, 20)(15, 25)(18, 26) (19, 27)(21, 23)(22, 32)(24, 30)(29, 31).

Following theory developed in [8], these permutations a and b were chosen in such a way that the relations $(a^{-1}b)^2 = (a^{-2}b^2)^2 = a^{-3}b^3a^{-3}aba = 1$ are satisfied, forcing $\langle p, q, r \rangle \cong D_8$, and moreover, so that a and b generate cyclic groups of different orders and with trivial intersection of the corresponding normalizers, so as to avoid additional automorphisms of X^*

arising from group automorphisms. This was achieved with the help of the LowIndexSubgroups process in the MAGMA system [1].

We remark that there is an alternative description, more geometric in nature, for the graph given in Theorem 1.1. Let Y be the *Cayley digraph* $Cay(G;a,b)$ of the group G relative to the given generating set $S=\{a,b\}$. This has vertex set G and arcs of the form (g,gs), for $g \in G, s \in S$. It may be seen that the alternating cycles in Y — that is, cycles whose vertices are alternately heads and tails (in $Cay(G,\{a,b\})$) of the incident edges — have length 4 and decompose $E(Y)$. Let $Al(Y)$ denote the intersection graph of these alternating cycles, together with the orientation inherited from that of $Cay(G,\{a,b\})$ in the natural way. (See Section 2 for more precise definitions of this and related concepts.) Then it is easily seen that this operation may be repeated on both $Al(Y)$ and $Al^2(Y)$ to produce the graph $Al^3(Y)$, which turns out to be isomorphic to the graph $X^*(G,\mathcal{H},\{Ha,Hb\})$ defined above: the theory developed in [7] implies that $Al^3(Y)$ admits a $\frac{1}{2}$-arc-transitive action of G with vertex stabilizers isomorphic to H, and adjacency in $Al^3(Y)$ corresponds to the "action" of the elements a and b.

In Section 2 we provide further graph- and group-theoretic background to our construction and some elementary observations about the orbital graph $X=X(G,\mathcal{H};\{Ha,Hb\})$ and the Cayley graph $Y=Cay(G;a,b)$. A detailed analysis of the cycles of length 8 in the underlying graph of X is carried out in Section 3, and the proof of Theorem 1.1 is given in Section 4.

The strategy behind the proof of Theorem 1.1 is briefly as follows. First we show (in Section 3) that every 3-arc (directed 3-path) in X lies in a unique directed 8-cycle, and that every cycle of length 8 in X^* underlies one of these. Then we show (in Section 4) that every automorphism of X^* which fixes the vertices of a 3-arc in X fixes every vertex of X^*, and use the fact that the group G acts regularly on the set of these 3-arcs to deduce that $G=\operatorname{Aut} X^*$ (and hence that X^* is $\frac{1}{2}$-arc-transitive).

An alternative but similar proof may be obtained by showing $G=\operatorname{Aut} Y$ and then using the isomorphism $X^*(G,H;\{Ha,Hb\}) \cong Al^3(Y)$ and analysis of cycles in Y to show every automorphism of $Al^3(Y)$ must preserve the orientation of edges, and so on, however we do not provide the details here.

2 Further background and observations

Let $G = \langle a, b \rangle$ and $H = \langle p, q, r \rangle$ be as defined in the introductory section. The first observation we make is that p, q and r are involutions with the property that $(pr)^2 = q$, and hence the subgroup H is dihedral of order 8.

Next, the group G itself has order 86016. This is not quite so easy to see, but can be verified with the help of MAGMA [1], which we used in finding and analysing the group G and the orbital graph X.

The group G is transitive but imprimitive of degree 32, with two blocks of imprimitivity consisting of all odd and all even integers in $\{1, 2, \ldots, 32\}$ respectively. These two blocks are interchanged by each of the generators a and b and preserved by each of p, q and r; indeed the set-wise stabilizer of each block is the subgroup K generated by a^2, b^2 and $p = a^{-1}b$.

The latter subgroup K acts transitively but imprimitively on each of the two blocks of size 16, with 8 blocks of size 2, and in each case with kernel of order 2. (In fact the blocks are $\{1, 9\}$, $\{3, 11\}$, $\{5, 13\}$, $\{7, 15\}$, $\{17, 25\}$, $\{19, 27\}$, $\{21, 29\}$, $\{23, 31\}$, and $\{2, 10\}$, $\{4, 12\}$, $\{6, 14\}$, $\{8, 16\}$, $\{18, 26\}$, $\{20, 28\}$, $\{22, 30\}$, $\{24, 32\}$, and the kernels are generated by the involutions $(ab)^7$ and $(ab^9)^7$ respectively.) Moreover, in each case the action of the subgroup K on the 8 blocks of size 2 is equivalent to the action of the affine general linear group $\mathrm{AGL}(3, 2)$ on a 3-dimensional vector space over $\mathrm{GF}(2)$, of order 1344, and with kernel of order 16.

For example, in the case of the block containing odd integers the 8 blocks of size 2 can be labelled $B_1, \ldots, B_8$ in a natural way such that a^2, b^2 and p permute them respectively as $(B_1, B_2, B_3, B_4)(B_5, B_6, B_7, B_8)$, $(B_1, B_2, B_7, B_6, B_3, B_4)(B_5, B_8)$ and $(B_3, B_8)(B_6, B_7)$, and the kernel of this blocks action is generated by b^{12}, $(ba^2b^3)^2$, $(a^3b^6a)^2$ and $(ba^4b^5)^2$.

Thus K has order $|K| = |\mathrm{AGL}(3, 2)| \times 16 \times 2 = 1344 \times 32 = 43013$, and G has order 86016. In particular, since the subgroup H has order 8 it follows that the orbital graph $X = X(G, \mathcal{H}; \{Ha, Hb\})$ has 10752 vertices.

Next, the vertices of the orbital graph $X = X(G, \mathcal{H}; \{Ha, Hb\})$ are the right cosets Hg (for $g \in G$), and an arc joins coset Hx to coset Hy if and only if $xy{-}1 \in HaH$ (or equivalently, $yx^{-1} \in Ha^{-1}H$). Note that the double coset HaH is the union of the two right cosets Ha and Hb, since $b = ap \notin Ha$ while $aq = pa \in Ha$ and $ar = qa \in Ha$. Similarly $Ha^{-1}H$ is the union of cosets Ha^{-1} and $Ha^{-1}r$. It follows that every vertex Hg of

X is joined by an arc to each of the vertices Hag and $Hapg$, and similarly, each of the vertices $Ha^{-1}g$ and $Ha^{-1}rg$ is joined by an arc to Hg in X. Thus X is regular of valency 4. Also since every element of G is expressible as a word in the set $\{a, ap, a^{-1}, a^{-1}r\}$, the graph X is connected.

The group G acts vertex-transitively on X by right multiplication of cosets (with an element $g \in G$ taking Hx to Hxg), and in this action the stabilizer of the vertex H is the subgroup H itself. In particular, this vertex-stabilizer is dihedral of order 8 and has two orbits of length 2 on the vertices adjacent to H: the out-neighbourhood $\{Ha, Hap\}$ and the in-neighbourhood $\{Ha^{-1}, Ha^{-1}r\}$. Hence the underlying graph X^* of X admits a $\frac{1}{2}$-arc-transitive action of G.

Further, we have the following:

Proposition 2.1 *The group G acts transitively, indeed regularly, on 3-arcs (directed paths of length 3) in the orbital graph $X = X(G, \mathcal{H}; \{Ha, Hb\})$.*

PROOF. The stabilizer in G of the arc (H, Ha) is the subgroup of order 4 generated by q and r, and the stabilizer in G of the 2-arc (H, Ha, Ha^2) is the cyclic subgroup of order 2 generated by r. As the latter element interchanges the two out-neighbours of the vertex Ha^2 (namely Ha^3 and $Hapa^2$), it follows that the action of G is regular on 3-arcs in X. ■

Also since the element a has order 8, and $a^k \notin H$ for $1 \le k \le 4$, there is an obvious (directed) 8-cycle in the graph X, namely

$$(H, Ha, Ha^2, Ha^3, Ha^4, Ha^5, Ha^6, Ha^7).$$

In the next section we will show that every cycle of length 8 in the undirected graph X^* lies in the same orbit under the action of G as the undirected form of this one — or in other words, that all 8-cycles in X are directed ones.

In fact 8 is the girth (the length of the shortest cycles) of X^*. Moreover, as each vertex Hx is adjacent only to $Hax, Hatx, Ha^{-1}x$ and $Ha^{-1}vx$, any alternating cycle in X has to correspond to a word of the form $(ata^{-1}v)^k$ for some k, lying in H; and as the element $ata^{-1}v$ has order 6 and its cube does not lie in H, the smallest such k is 6, and hence all the alternating cycles in X have length 12, corresponding to the relator $(ata^{-1}v)^6$. Thus X has radius 6 as defined in [5]. Also X^* has diameter 13. These observations may easily be verified using vertex-transitivity and with help of MAGMA [1].

Before proceeding, we explain the background to the connection between the underlying graph X^* of the orbital graph $X = X(G, \mathcal{H}, \{Ha, Hb\})$ and the Cayley graph $Y = Cay(G; a, b)$, by describing two operators on balanced oriented 4-valent graphs.

For a balanced oriented graph T of valency 4, let the *partial line graph* $U = Pl(T)$ of T be the balanced oriented 4-valent graph with vertex set $A = A(T)$ such that there is an arc in U from $x \in A$ to $y \in A$ if and only if xy is a directed 2-path (a 2-arc) in T. Note that the arc set of U decomposes into alternating 4-cycles, no two of which intersect in more than one vertex.

To define the the inverse operator Al, let the vertex set of $Al(U)$ be the set of alternating cycles (of length 4) in U, with two such cycles adjacent in $Al(U)$ if and only if they have a common vertex in U. The orientation of the edges of $Al(U)$ is inherited from that of the edges of U in a natural way. Letting C_v and C_w be the two alternating 4-cycles in U corresponding to two adjacent vertices v and w in $Al(U)$, we orient the edge $[v, w]$ in $Al(U)$ from v to w if and only if the two arcs in U with the tail in $u \in C_v \cap C_w$ have their heads in C_w. Observe that $Al(Pl(T) = T$ for every balanced oriented graph T of valency 4. Moreover, $Pl(Al(U)) = U$ provided the graph U has the properties assumed above.

These two operators may also be applied to (undirected) graphs whenever an accompanying oriented graph is (perhaps implicitly) associated with the undirected graph in question. A typical situation is presented by a 4-valent graph admitting a $\frac{1}{2}$-arc-transitive group action and its two accompanying balanced oriented graphs, or by a Cayley graph arising from a set of non-involutory generators, for each of which one of the two possible orientations is prescribed. Again, the operators Al and Pl are mutual inverses in this case too. In particular, with X^* and Y defined as above, the graph X^* is isomorphic to $Al^3(Y)$, and conversely $Pl^3(X^*)$ is isomorphic to Y.

3 Analysis of cycles

To begin this section we prove the following:

Proposition 3.1 *Every 3-arc in the orbital graph $X = X(G, \mathcal{H}; \{Ha, Hb\})$ lies in a unique directed 8-cycle in X.*

PROOF. By Proposition 2.1, the group G has a single orbit on 3-

arcs in X, hence all we need do is prove that a particular 3-arc, say $T = (H, Ha, Ha^2, Ha^3)$, can be extended in just one way to a directed 8-cycle in X. Now by definition of adjacency in X, any vertex at the end of a directed 8-path emanating from the vertex H has to be of the form Hw for some element $w \in G$ expressible as a word of length 8 in $\{a, b\}$ with positive exponents, and if this 8-path extends the given 3-arc T then the word has to end in a^3. Further, for this to be a directed 8-cycle extending T we require $Hw = H$ and therefore $w \in H$; in other words we require a word of length 8 in $\{a, b\}$ with positive exponents and ending in a^3 to give an element of H. Since $b = ap$ this word must be of the form $ap^{k_1}ap^{k_2}ap^{k_3}ap^{k_4}ap^{k_5}a^3$ where $k_i \in \{0, 1\}$ for $1 \leq i \leq 5$. There are 32 such words, and an easy (but tedious calculation) reveals that the only one which produces an element of H is the word a^8 (and this is the identity element). Hence the given 3-arc has a unique extension to a directed 8-cycle, namely $(H, Ha, Ha^2, Ha^3, Ha^4, Ha^5, Ha^6, Ha^7)$. ■

The calculation referred to in the above proof can be carried out with the help of MAGMA [1]. Also some of the cases are equivalent to others (for example a^2pa^6 is equivalent to a^3pa^5), and so the number of cases to check can be reduced, but we omit the details. Similar arguments apply to other 8-cycles in the underlying graph X^* of X, and in order to obtain Proposition 3.2 below (which is crucial to the proof of our main theorem), it is helpful to take the following approach.

Let W be a simple walk of length s in X, not necessarily directed. To each internal vertex v of W, assign one of the symbols A^+, A^- or D depending on whether v is respectively the tail of both, the head of both, or the tail of one and the head of the other of the two arcs incident with v in W. The resulting sequence of elements of the set $\{A^+, A^-, D\}$, of length s if W is a closed walk and length $s - 1$ otherwise, may be called the *code* of W. For example, the walk $(H, Ha, Ha^2, Ha^{-1}ra^2)$ has code DA^-, while the directed 8-cycle $(H, Ha, \ldots, Ha^7)$ has code D^8. Note that if each occurrence of the symbol D is deleted from the code, a sequence is obtained in which the symbols A^+ and A^- alternate. (Also the walk is directed if its code contains only D's, or alternating if its code contains no symbol D at all.) Two codes of the same length will be said to be equivalent if they are associated with walks W_1 and W_2 such that W_2 is obtainable from W_1 by a one-step cyclic shift of its vertices (in the case of closed walks) or by reversal (in either case).

Now up to equivalence in any directed graph there are 22 possibilities for the code of a simple closed walk of length 8, and these are listed in Table 1.

Type	Code	Representative word	Number
1	D^8	$ap^{k_1}ap^{k_2}ap^{k_3}ap^{k_4}ap^{k_5}a^3$	32
2	$D^6A^-A^+$	$ap^{k_1}ap^{k_2}ap^{k_3}apa^{-1}ra^3$	8
3	$D^5A^-DA^+$	$ap^{k_1}ap^{k_2}apa^{-1}r^{k_3}a^{-1}ra^3$	8
4	$D^4A^-D^2A^+$	$ap^{k_1}apa^{-1}r^{k_2}a^{-1}r^{k_3}a^{-1}ra^3$	8
5	$D^3A^-D^3A^+$	$apa^{-1}r^{k_1}a^{-1}r^{k_2}a^{-1}r^{k_3}a^{-1}ra^3$	8
6	$D^4A^-A^+A^-A^+$	$ap^{k_1}apa^{-1}rapa^{-1}ra^3$	2
7	$D^3A^-A^+A^-DA^+$	$apa^{-1}r^{k_1}a^{-1}rapa^{-1}ra^3$	2
8	$D^3A^-A^+DA^-A^+$	$apa^{-1}rap^{k_1}apa^{-1}ra^3$	2
9	$D^3A^-DA^+A^-A^+$	$apa^{-1}rapa^{-1}r^{k_1}a^{-1}ra^3$	2
10	$D^2A^-A^+A^-D^2A^+$	$apa^{-1}r^{k_1}a^{-1}r^{k_2}a^{-1}rapa^{-1}ra^2$	4
11	$D^2A^-A^+D^2A^-A^+$	$apa^{-1}rap^{k_1}ap^{k_2}apa^{-1}ra^2$	4
12	$D^2A^-D^2A^+A^-A^+$	$apa^{-1}rapa^{-1}r^{k_1}a^{-1}r^{k_2}a^{-1}ra^2$	4
13	$D^2A^-A^+DA^-DA^+$	$apa^{-1}r^{k_1}a^{-1}rap^{k_2}apa^{-1}ra^2$	4
14	$D^2A^-DA^+A^-DA^+$	$apa^{-1}r^{k_1}a^{-1}rapa^{-1}r^{k_2}a^{-1}ra^2$	4
15	$D^2A^-DA^+DA^-A^+$	$apa^{-1}rap^{k_1}apa^{-1}r^{k_2}a^{-1}ra^2$	4
16	$DA^-DA^+DA^-DA^+$	$apa^{-1}r^{k_1}a^{-1}rap^{k_2}apa^{-1}r^{k_3}a^{-1}ra$	8
17	$D^2A^-A^+A^-A^+A^-A^+$	$apa^{-1}rapa^{-1}rapa^{-1}ra^2$	1
18	$DA^-A^+A^-A^+A^-DA^+$	$apa^{-1}r^{k_1}a^{-1}rapa^{-1}rapa^{-1}ra$	2
19	$DA^-A^+A^-A^+DA^-A^+$	$apa^{-1}rap^{k_1}apa^{-1}rapa^{-1}ra$	2
20	$DA^-A^+A^-DA^+A^-A^+$	$apa^{-1}rapa^{-1}r^{k_1}a^{-1}rapa^{-1}ra$	2
21	$DA^-DA^+A^-A^+A^-A^+$	$apa^{-1}rapa^{-1}rapa^{-1}r^{k_1}a^{-1}ra$	2
22	$A^-A^+A^-A^+A^-A^+A^-A^+$	$a^{-1}rapa^{-1}rapa^{-1}rapa^{-1}ra$	1

TABLE 1: Codes and words representing potential 8-cycles in X^*

In addition, this table includes for each such code a representative element of the group G (expressed as a product of the elements a, ap, a^{-1} and

$a^{-1}r$) which must lie in H if the orbital graph X contains a simple closed walk of the corresponding type. These representative words can be found using the same sort of argument as in the proof of Proposition 3.1.

For example, in the case of the code $D^4A^-D^2A^+$, which is equivalent to code $D^2A^-D^2A^+D^2$, by transitivity of G on 3-arcs we can assume there is a closed walk containing the 3-arc (H, Ha, Ha^2, Ha^3), followed by the reverse of the arc $(Ha^{-1}ra^3, Ha^3)$, followed by the reverse of an arc of the form $(Ha^{-1}r^ka^{-1}ra^3, Ha^{-1}ra^3)$ where $k \in \{0,1\}$, and so on, leading to the conclusion that the subgroup H must contain an element of the form $w = ap^{k_1}apa^{-1}r^{k_2}a^{-1}r^{k_3}a^{-1}ra^3$ with $k_i \in \{0,1\}$ for $1 \le i \le 3$. Accordingly for this type there are $2^3 = 8$ possibilities to check for the word w. This number is given along with the analogous information for the other 21 types in the fourth column of Table 1. Note that in some cases (for example where the code does not contain a D^2 subsequence) the number may be greater than necessary; it is often possible to reduce the number of possibilities using 3-arc transitivity plus a little further local analysis.

Using the information summarised in Table 1 we can now prove the following:

Proposition 3.2 *Every 8-cycle in the underlying graph X^* of the orbital graph $X = X(G, \mathcal{H}; \{Ha, Hb\})$ underlies a directed 8-cycle in X.*

PROOF. Suppose C is any cycle of length 8 in X^*, and let W be the simple closed walk in X which C underlies. Then the code of W is equivalent to one of the 22 codes listed in Table 1, and it follows that for suitable choice of the exponents k_i, the corresponding element given in the third column of Table 1 lies in the subgroup H. Calculation of all such words, however, reveals that the only one which produces an element of H is the word a^8 (corresponding to the code D^8), and hence C must be a directed cycle. ■

4 Proof of Main Theorem

Parts (i) and (ii) and much of part (iii) of Theorem 1.1 were verified in Section 2, and we are now in a strong position to complete the proof. As the action of the group G on the underlying graph X^* of the orbital graph $X = X(G, \mathcal{H}, \{Ha, Hb\})$ has been shown to be $\frac{1}{2}$-arc-transitive, it is sufficient to prove that G is the full automorphism group of X^*. So let us assume the contrary, namely that $G \neq \mathrm{Aut}\, X^*$.

By Proposition 3.1, every 3-arc T of X lies in a unique directed 8-cycle, and by Proposition 3.2, the (undirected) 8-cycle C underlying this is the only 8-cycle in X^* containing the underlying 3-path of T. Moreover, by Proposition 3.2 it also follows that only the 3-paths underlying 3-arcs in X can be extended to 8-cycles in X^*. In particular, this implies that $\mathrm{Aut}\, X^*$ preserves the set of 3-paths underlying 3-arcs in X.

Next by Proposition 2.1, the group G acts regularly on 3-arcs of X, and so the stabilizer in $\mathrm{Aut}\, X^*$ of any 3-path underlying a 3-arc of X must be non-trivial. Hence there exists a non-identity automorphism θ of X^* which fixes a 3-arc of X, say T. Further, if C is the (undirected) 8-cycle underlying the unique directed 8-cycle in X which extends T, then θ also fixes C.

Let $v_0, v_1, \dots v_7$ be the vertices of C, taken in order so that (v_{i-1}, v_i) is an arc of X for each i (modulo 7), and also u_i is the other in-neighbour of v_i in X, and w_i is the other out-neighbour of v_i in X, for $0 \leq i \leq 7$. Consider the effect of θ on the vertices of each of the 3-arcs of the form $(v_{i-2}, v_{i-1}, v_i, w_i)$. As each v_j is fixed by θ, the vertex w_i is either also fixed, or taken to the only other possible neighbour of v_i, namely u_i. But $(v_{i-2}, v_{i-1}, v_i, u_i)$ is not a 3-arc in X, and so the latter case is impossible, and hence θ must fix each w_i. A similar argument shows also that θ fixes the vertices of each 3-arc of the form $(u_i, v_i, v_{i+1}, v_{i+2})$.

Thus θ fixes every vertex at distance 1 in X^* from a vertex of C, and moreover, every such vertex lies on a 3-arc of X which is fixed by θ. By induction (and connectedness), it follows that θ fixes every vertex of X^*, contradiction. Hence proof.

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