Constructions for chiral polytopes

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Abstract

An abstract polytope of rank n is said to be *chiral* if its automorphism group has two orbits on flags, with adjacent flags lying in different orbits. In this paper, we describe a method for constructing finite chiral n-polytopes, by seeking particular normal subgroups of the orientation-preserving subgroup of n-generator Coxeter group (having the property that the subgroup is not normalized by any reflection and is therefore not normal in the full Coxeter group). This technique is used to identify the smallest examples of chiral 3- and 4-polytopes, in both the self-dual and non self-dual cases, and then to give the first known examples of finite chiral 5-polytopes, again in both the self-dual and non self-dual cases.

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1 Introduction

In the classical theory of convex geometric polytopes, all the facets and vertex figures are spherical. In the 1970s, Branko Grünbaum [14] proposed the study of of a more general class of polytopes, whose facets or vertex figures might not be spherical. This suggestion was taken further by Ludwig Danzer and Egon Schulte [10, 20, 21] with their investigation of *incidence polytopes*, which are now more commonly known as *abstract polytopes*. Abstract polytopes generalize the classical notion of convex polytopes to more general combinatorial structures. Of particularly interest are those with a high degree of symmetry, together with their geometry and topology.

Chiral polytopes are abstract polytopes that have maximum possible rotational symmetry but no reflection. Note that the assumption of maximum rotational symmetry makes this definition of 'chiral' more restrictive than its use in other contexts such as molecular chemistry, but appears to be the accepted term for such abstract polytopes — see [16, 19, 23, 24] for example. Chiral polytopes occur in pairs, with each member of a pair being the 'mirror image' of the other. This phenomenon does not occur among classical polytopes, but does occur in 3 dimensions. Chiral 3-polytopes are also known as *chiral regular maps*. The first family of such maps was constructed by Heffter [15] in 1898 (and studied again recently by Doro and Wilson [11]). Contributions to the more general study of chiral polytopes were first made by Weber and Seifert [27], and also later by Coxeter [6].

The study of regular maps was initiated by Brahana [2], and progressed by Coxeter [6]. An *orientably-regular* map is now known as a 2-cell embedding of a connected graph (or multigraph) on an orientable surface, with the property that the group of all orientation-preserving automorphisms of the map (or equivalently, the group of all self-homeomorphisms of the surface preserving the embedding) is transitive on the ordered edges of the graph. By connectedness and maximum symmetry, this group can be generated by two particular automorphisms: one that cyclically permutes the edges (and vertices) bounding a given face, and another that cyclically permutes the edges (and faces) incident with a given vertex of that face. Such a map is called *reflexible* if it also possesses an orientation-reversing automorphism that interchanges two adjacent vertices without interchanging the two faces that contains the edge joining them. In that case, the automorphism group is transitive on the incident vertex-edge-face triples (or *flags*) of the map. On the other hand, if no such automorphism exists, then the flags fall into two orbits under the group action, and the map is called *irreflexible*, or chiral.

In 1948 Coxeter [6] enumerated all regular maps (whether reflexible or chiral) on the torus. The smallest example has type $\{4, 4\}$, and is self-dual, with 5 vertices, 10 edges and 5 faces, and automorphism group of order 20. (The two smallest non-self-dual examples have types $\{3, 6\}$ and $\{6, 3\}$, with one being the dual of the other; one has 7 vertices, 21 edges and 14 faces and the other has 14 vertices, 21 edges and 7 faces, and both have automorphism group of order 42.) After Coxeter, several families of chiral regular maps on surfaces of higher genus were found by Sherk [26], Garbe [12], and Bujalance, Conder and Costa [3]. In 1969 Garbe [13] showed that no chiral maps exist on orientable surfaces of genus 2 to 6 (inclusive), and in 2001 Conder and Dobcsányi [4] determined all chiral regular maps on orientable surfaces of genus 7 to 15. Also Schulte [22] has constructed three families of infinite 3-polytopes in ordinary space that are geometrically chiral, and showed that no finite chiral 3-polytope exists in ordinary space.

In 1970 Coxeter [8] used honeycombs to construct a family of chiral 4-polytopes, by forcing the right and left Petrie motions of the polytopes to have different lengths. Later Schulte and Weiss [24] constructed examples from hyperbolic honeycombs, using isometries of hyperbolic 3-space and complex Möbius transformations. Also Nostrand [19] has constructed such chiral 4-polytopes whose faces are cubes or dodecahedra.

In higher dimensions (rank ≥ 5), until now only *infinite* examples of chiral polytopes have been found. In [25], Schulte and Weiss constructed such examples with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ where $p_1 = \infty$ or $p_{n-1} = \infty$.

In this paper, we give the first known examples of finite chiral 5-polytopes. These were found by means of a search for normal subgroups of small index in the orientation-preserving subgroup of a 5-generator Coxeter group, with the property that the subgroup is not normalized by any reflection (and is therefore not normal in the full Coxeter group). The same technique can be used to find the smallest examples of chiral 4-polytopes, in both the self-dual and non self-dual cases. (Here 'smallest' means having the smallest number of flags, and so smallest automorphism group, subject to the given requirements.)

In Section 2 we give background definitions and properties of abstract polytopes and their automorphism groups and their duals, and explain in detail the definitions and properties of directly regular and chiral polytopes. In Section 3 we explain our method for constructing examples of chiral polytopes from particular subgroups of Coxeter groups, and then we apply this method to find the smallest self-dual and non self-dual chiral 4-polytopes in Section 4. Finally, we do the same for chiral 5-polytopes in Section 5.

2 Background definitions and properties

Definition 2.1 *An* abstract polytope *of rank n (otherwise known simply as an n-polytope) is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. The elements of $\mathcal{P}$ are called* faces. *For $-1 \leq j \leq n$, the elements of $\mathcal{P}$ of rank j are called j-*faces, *and a typical j-face is denoted by F_j. The faces of rank $0, 1$ and $n-1$ are usually called the* vertices, edges *and* facets *of the polytope, respectively. We require that $\mathcal{P}$ have a smallest (-1)-face F_{-1}, and a greatest n-face F_n, and that each maximal chain (called a* flag*) of $\mathcal{P}$ contains exactly $n+2$ faces. Two flags are said to be* adjacent *if they differ by just one face. Also, we require that $\mathcal{P}$ is strongly flag-connected, that is, any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive faces Φ_{i-1} and Φ_i are adjacent, and $\Phi \cap \Psi \subseteq \Phi_i$ for all i. Finally, we require the following homogeneity property, which is often called the* diamond condition*: whenever $F \leq G$, with* $\mathrm{rank}(F) = j-1$ *and* $\mathrm{rank}(G) = j+1$*, there are exactly* *two faces H of rank j such that $F \leq H \leq G$.*

These conditions ensure that an abstract polytope $\mathcal{P}$ shares many of the combinatorial properties customarily associated with classical geometric polytopes.

Next, if F and G are faces of $\mathcal{P}$ with $F \leq G$, then we call the set $\{H \mid F \leq H \leq G\}$ a *section* of $\mathcal{P}$, and denote this by G/F. In particular, if F_0 is a vertex, then the section $F_n/F_0 = \{G \in \mathcal{P} \mid F_0 \leq G\}$ is called a *vertex-figure* of $\mathcal{P}$ at F_0. More generally, if F_j is a j-face of $\mathcal{P}$, then the section $F_n/F_j = \{G \in \mathcal{P} \mid F_j \leq G\}$ is called the *co-face* of F_j.

Definition 2.2 *An* automorphism *of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$. The group of all automorphisms of $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$. A polytope $\mathcal{P}$ is said to be* regular *if $\Gamma(\mathcal{P})$ is transitive on the flags of $\mathcal{P}$.*

The automorphism group $\Gamma(\mathcal{P})$ of a regular polytope $\mathcal{P}$ is generated by n involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$, where each ρ_i maps a given *base flag* Φ to the adjacent flag Φ^i (differing from Φ only in its i-face, as determined by the diamond condition applied to the $(i-1)$- and $(i+1)$-faces of Φ). These distinguished generators satisfy (among others) the relations

$$(\rho_i\rho_j)^{p_{ij}} = 1 \quad \text{for} \quad 0 \leq i \leq j \leq n-1, \tag{1}$$

where $p_{ii} = 1$ for all i, and $p_{ji} = p_{ij}$ whenever $|i-j| = 1$, and $p_{ij} = 2$ otherwise.

Letting $p_i = p_{i-1,i} = p_{i,i-1}$ for $1 \leq i < n$, we say that $\mathcal{P}$ is of *type* $\{p_1, \ldots, p_{n-1}\}$, and call $\{p_1, \ldots, p_{n-1}\}$ the *Schläfli symbol* of $\mathcal{P}$.

Furthermore, the generators ρ_i for $\Gamma(\mathcal{P})$ satisfy an additional condition known as the *intersection condition*, namely

$$\langle\, \rho_i : i \in I \,\rangle \cap \langle\, \rho_i : i \in J \,\rangle \;=\; \langle\, \rho_i : i \in I \cap J \,\rangle \quad \text{for every } I, J \subseteq \{0, 1, \ldots, n-1\}. \tag{2}$$

Conversely, if Γ is a permutation group generated by elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ which satisfy the relations (1) and condition (2), then there exists a polytope $\mathcal{P}$ with $\Gamma(\mathcal{P}) \cong \Gamma$; for more details, see [17].

All polytopes of rank 2 are regular, with dihedral automorphism group. Also every polytope of rank 3 is a map (that is, a 2-cell embedding of a connected graph into a closed surface without boundary) but the converse is not true, however, since maps need not satisfy the homogeneity property (diamond condition): for example, it is not always true that every edge has two vertices and lies in exactly two faces of the map.

If the map is orientable and reflexible, then its automorphism group has a subgroup of index 2 consisting of the orientation-preserving automorphisms (and generated by the vertex- and face-rotations). For an arbitrary regular n-polytope $\mathcal{P}$, with automorphism group $\Gamma(\mathcal{P}) = \langle\, \rho_0, \ldots, \rho_{n-1} \,\rangle$, we may consider the subgroup $\Gamma^+(\mathcal{P})$ generated by the 'rotations' $\sigma_j = \rho_{j-1}\rho_j$ for $1 \leq j < n$. This subgroup always has index 1 or 2 in $\Gamma(\mathcal{P})$. When this index is 2, the polytope $\mathcal{P}$ is said to be *directly regular*. All classical regular polytopes are directly regular. A regular 3-polytope (regular map) is directly regular if and only of the corresponding surface is orientable.

We now give the definition of a 'chiral' polytope (having maximum rotational symmetry but no reflections), as follows:

Definition 2.3 *Let $\mathcal{P}$ be an abstract polytope of rank $n \geq 3$. Then $\mathcal{P}$ is* chiral *if its automorphism group $\Gamma(\mathcal{P})$ has precisely two orbits on flags, with adjacent flags being in distinct orbits, and for each flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$, there exist automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ such that each σ_j fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$, and cyclically permutes consecutive j-faces of $\mathcal{P}$ in the rank 2 section $F_{j+1}/F_{j-2} = \{G \mid F_{j-2} \leq G \leq F_{j+1}\}$ of $\mathcal{P}$.*

Given any base flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$, the automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ given by the above definition may be chosen such that σ_j takes Φ to the flag $\Phi^{j,j-1}$ (differing from Φ in its $(j-1)$- and j-faces), for $1 \leq j < n$. These automorphisms then generate $\Gamma(\mathcal{P})$, and satisfy (among others) the relations

$$(\sigma_i \sigma_{i+1} \ldots \sigma_j)^2 = 1 \quad \text{for} \ \ 1 \leq i < j \leq n-1, \tag{3}$$

and if p_i is the order of σ_i for $1 \leq i \leq n-1$, then $\{p_1, \ldots, p_{n-1}\}$ is called the *type* of $\mathcal{P}$. The generators σ_i also satisfy an intersection condition, which is rather intricate and not easy to state for arbitrary rank n; see [23] for precise details. This condition ensures that the group generated by $\sigma_1, \ldots, \sigma_{n-1}$ acts faithfully on the polytope, and that each σ_j acts in the appropriate way on the rank 2 section F_{j+1}/F_{j-2} of $\mathcal{P}$. For rank 5 the intersection condition is equivalent to the following;

$$\begin{aligned}
<\sigma_1> \cap <\sigma_2> = \ &\{1\} = <\sigma_2> \cap <\sigma_3>, \\
<\sigma_1, \sigma_2> \cap <\sigma_2, \sigma_3> &= <\sigma_2>, \\
<\sigma_1, \sigma_2, \sigma_3> \cap <\sigma_2, \sigma_3, \sigma_4> &= <\sigma_2, \sigma_3> \\
<\sigma_1, \sigma_2, \sigma_3> \cap <\sigma_3, \sigma_4> &= <\sigma_3>, \\
<\sigma_1, \sigma_2, \sigma_3> \cap <\sigma_4> = \ &\{1\}.
\end{aligned} \tag{4}$$

Conversely, if Γ is any permutation group generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ which satisfy the relations (3) and the intersection condition, then there exists a polytope $\mathcal{P}$ of rank n which is either directly regular or chiral, of type $\{p_1, \ldots, p_{n-1}\}$ where p_i is the order of σ_i (for $1 \leq i < n$), and with $\Gamma(\mathcal{P}) \cong \Gamma$ if $\mathcal{P}$ is chiral, or $\Gamma^+(\mathcal{P}) \cong \Gamma$ if $\mathcal{P}$ is directly regular. This polytope may be denoted by $\mathcal{P}(\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \rangle)$. The facets and vertex figures of $\mathcal{P}$ are isomorphic to the $(n-1)$-polytopes $\mathcal{P}(\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-2} \rangle)$ and $\mathcal{P}(\langle \sigma_2, \sigma_3, \ldots, \sigma_{n-1} \rangle)$ respectively, and similar isomorphisms exist for other sections of $\mathcal{P}$. Moreover, $\mathcal{P}$ is directly regular if and only if there exists an involutory group automorphism $\rho: \Gamma \to \Gamma$ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2 \sigma_2$, and $\rho(\sigma_i) = \sigma_i$ for $3 \leq i \leq n-1$ (or in other words, acting like conjugation by the generator ρ_1 in the directly regular case; see [23, Theorem 1]).

Chiral polytopes (for which no such additional automorphism exists) occur in pairs of enantiomorphic forms, with one being the 'mirror image' of the other.

All the sections of a chiral polytope $\mathcal{P}$ are either directly regular or chiral, and in fact all the (n–2)-faces (and the co-faces of all edges) of a chiral n-polytope are directly regular; see [23, Proposition 9]). In particular, the 3-faces of a chiral 5-polytope are directly regular.

Furthermore, the *layer graphs* of a chiral n-polytope $\mathcal{P}$ are all edge-transitive, and often vertex- or arc-transitive. Here by a 'layer graph' we mean the (bipartite) incidence graph of $(i-1)$-faces and i-faces, for any i in the range $0 \leq i \leq n$. In particular, when n is even, the layer graph of $(n/2-1)$- and $n/2$-faces is called the *medial layer graph*. The semisymmetric 3-valent graphs occurring as the medial layer graphs of chiral or directly regular 4-polytopes have been investigated in [18].

Finally we define a number of aspects of duality:

Definition 2.4 *The* dual *of the n-polytope $\mathcal{P}$ is the n-polytope $\mathcal{P}^*$ obtained from $\mathcal{P}$ by reversing the partial order, so that j-faces of $\mathcal{P}^*$ are $(n-1-j)$-faces of $\mathcal{P}$ for $-1 \leq j \leq n$, and vice versa. The polytope $\mathcal{P}$ is called* self-dual *if $\mathcal{P}$ is isomorphic to its dual $\mathcal{P}^*$. In that case there exists an incidence-reversing bijection $\delta\colon \mathcal{P} \to \mathcal{P}$, which is called a* duality *of $\mathcal{P}$, and if δ^2 is the identity automorphism of $\mathcal{P}$ then δ is called a* polarity.

If $\mathcal{P}$ is a chiral n-polytope, then $\mathcal{P}$ is said to be properly self-dual *if there exists a duality $\delta : \mathcal{P} \to \mathcal{P}$ mapping a flag Φ of $\mathcal{P}$ to a flag Φ^δ in the same orbit as $\mathcal{P}$ under the automorphism group $\Gamma(\mathcal{P})$, or* improperly self-dual *if $\mathcal{P}$ has a duality mapping the flag Φ to a flag in the other orbit of $\Gamma(\mathcal{P})$.*

Note that in the properly self-dual case, every duality of $\mathcal{P}$ preserves the two orbits of $\Gamma(\mathcal{P})$ on flags, while in the improperly self-dual case, every duality interchanges these two orbits. For chiral 3-polytopes, this notion of duality is more general than that of topological duality for orientably-regular maps: a chiral 3-polytope $\mathcal{P}$ is improperly self-dual whenever the corresponding map $\mathcal{M}$ is isomorphic to its topological dual, and properly self-dual whenever $\mathcal{M}$ is isomorphic to the topological dual of its mirror image. In particular (and slightly at odds with what was suggested in [16]), the chiral map C7.2 of type $\{7, 7\}$ listed in [4, Table 2] is not topologically self-dual as a map, but is properly self-dual as a chiral 3-polytope. This example is discussed in more detail in the next Section.

3 Coxeter groups and small index subgroups

Let $\Lambda = [p_1, \ldots, p_{n-1}]$ be the Coxeter group with Dynkin diagram consisting of a path with n nodes and edges labelled $p_1, p_2, \ldots, p_{n-1}$ (in that order), as in Figure 1.

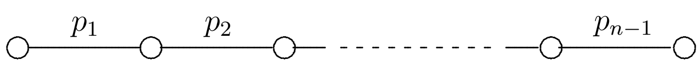

Fig.1: Dynkin diagram for the Coxeter group $[p_1, \ldots, p_{n-1}]$

This is the abstract group generated by n elements $x_0, x_1, \ldots, x_{n-1}$ subject to the defining relations

$$x_i^2 = 1, \quad (x_{i-1}x_i)^{p_i} = 1 \text{ for } 1 \le i < n, \quad (x_j x_j)^2 = 1 \text{ for } 1 \le i+1 < j < n.$$

Also let $\Lambda^+ = [p_1, \ldots, p_{n-1}]^+$ be its 'rotation subgroup', generated by the elements $x_i x_j$ (and therefore consisting of all elements of $[p_1, \ldots, p_{n-1}]$ expressible as words of even length in the generators $x_0, x_1, \ldots, x_{n-1}$), of index 2 in Λ. By Reidemeister-Schreier theory, the generators $y_i = x_{i-1}x_i$ of this subgroup satisfy the defining relations

$$y_i^{p_i} = 1, \quad (y_i y_{i+1} \ldots y_j)^2 = 1 \text{ for } 1 \le i < j \le n-1.$$

If $\Gamma = \Gamma(\mathcal{P})$ is the automorphism group of a directly regular n-polytope $\mathcal{P}$, generated by the involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$ as in Section 2, then there exists an epimorphism $\theta\colon \Lambda \to \Gamma$ taking x_i to ρ_i for all i, and taking the subgroup Λ^+ to the subgroup $\Gamma^+ = \Gamma^+(\mathcal{P})$ generated by the 'rotations' $\sigma_j = \rho_{j-1}\rho_j$ for $1 \le j < n$. In this case, the kernel of $\theta\restriction_{\Lambda^+}$ is a normal subgroup N of Λ that coincides with $\ker\theta$, and $\Lambda/N \cong \Gamma(\mathcal{P})$ while $\Lambda^+/N \cong \Gamma^+(\mathcal{P})$.

On the other hand, if $\Gamma = \Gamma(\mathcal{P})$ is the automorphism group of a chiral n-polytope $\mathcal{P}$, generated by the elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ as in Section 2, then there exists an epimorphism $\psi\colon \Lambda^+ \to \Gamma$ taking $x_{j-1}x_j$ to σ_j for $1 \le j < n$, but this epimorphism ψ does not extend to an epimorphism θ from Λ onto some larger group of automorphisms of $\mathcal{P}$, since $\mathcal{P}$ is chiral. In this case, the kernel of ψ is a normal subgroup K of Λ^+ that is not normal in Λ; in fact its conjugate K^{ρ_0} is another normal subgroup of Λ^+ such that Λ^+/K^{ρ_0} is the automorphism group of the 'mirror image' of $\mathcal{P}$.

It follows from these and earlier observations that chiral n-polytopes can be found by seeking normal subgroups of $\Lambda^+ = [p_1, \ldots, p_{n-1}]^+$ that are not normal in $\Lambda = [p_1, \ldots, p_{n-1}]$.

In particular, if K is such a subgroup of Λ^+, and the natural images of the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ in the quotient Λ^+/K satisfy the appropriate conditions, then a chiral n-polytope $\mathcal{P}$ with automorphism group Λ^+/K can be constructed by taking as 0-faces the (right) cosets of the image of $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle$, as $(n-1)$-faces the (right) cosets of the image of $\langle \sigma_1\sigma_2, \sigma_3, \ldots, \sigma_{n-2} \rangle$, and as j-faces the (right) cosets of the image of $\langle \sigma_1, \sigma_2, \ldots, \sigma_{j-2}, \sigma_{j-1}\sigma_j, \sigma_{j+1}, \ldots, \sigma_{n-2} \rangle$ for $1 \le j \le n-2$, and defining the partial order according to non-empty intersection of cosets; see [24, p. 226] for further details.

To illustrate this, let us consider the case $n = 3$. Here $\Lambda = [p_1, p_2]$ is the full $(2, p_1, p_2)$ triangle group $\langle\, x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = (x_1x_2)^{p_1} = (x_2x_3)^{p_2} = (x_1x_3)^2 = 1 \,\rangle$, and $\Lambda^+ = [p_1, p_2]^+$ is the ordinary $(2, p_1, p_2)$ triangle group $\langle\, y_1, y_2 \mid y_1^{p_1} = y_2^{p_2} = (y_1y_2)^2 = 1 \,\rangle$. When $(p_1, p_2) = (4, 4)$ we have $[4, 4]^+ = \langle\, u, v \mid u^4 = v^4 = (uv)^2 = 1 \,\rangle$, which is an extension of a free abelian group of rank 2 (generated by $[u, v]$ and $[u, v^{-1}]$) by an abelian group of order 8. The non-degenerate quotients of this group are well understood, and give rise to the family of maps denoted by Coxeter as $\{4, 4\}_{(b,c)}$; see [9] for example. The maps in this family are reflexible (and therefore regular) if and only if $bc(b - c) = 0$. When $bc(b - c) \ne 0$, the automorphism group of the chiral map $\{4, 4\}_{(b,c)}$ is isomorphic to the quotient $[4,4]^+/K$ of the ordinary $(2,4,4)$ triangle group $[4,4]^+$ by a subgroup K that is normal in $[4,4]^+$ but not in $[4,4]$.

The smallest chiral example is $\{4,4\}_{(1,2)}$, a map on the torus with 5 vertices, 10 edges and 5 faces. Its automorphism group is a quotient of $[4,4]^+$ of order 20, isomorphic to an extension of C_5 by C_4 (a Frobenius group). This is also the smallest chiral 3-polytope, and (along with every other chiral map $\{4,4\}_{(b,c)}$) is *improperly* self-dual.

Every 0-face (vertex) lies in four 1-faces (edges), and dually, each 2-face (map face) contains four 1-faces. In particular, the number of flags is $5 \times 4 \times 2 = 40$. Also, of course, each 1-face contains two 0-faces and lies in two 2-faces. These numbers can be depicted by the following 'link figure':

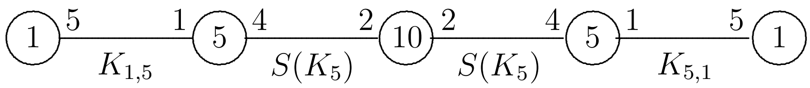

Fig.2: Link figure for the smallest self-dual chiral 3-polytopes, of type $\{4,4\}$

The 0-faces and 1-faces may be taken as the vertices and edges of the complete graph K_5, with natural incidence between them, and then the 2-faces as five particular quadrilaterals (4-cycles) in K_5, as follows. Take a fixed cyclic ordering of the five vertices of K_5, say $(1,2,3,4,5)$. For any vertex a, if (a,b,c,d,e) is this ordering based at the vertex a, then removing the 'parallel' edges $\{b,e\}$ and $\{c,d\}$ from the vertex-deleted subgraph $K_5\backslash\{a\}$ leaves a quadrilateral with edges $\{b,c\}$, $\{c,e\}$, $\{e,d\}$ and $\{d,b\}$. There are five such quadrilaterals, each containing four edges, and each edge lies in two of them, as required.

The between-layer incidence graphs are as indicated in the link figure, where $K_{r,s}$ denotes the complete bipartite graph having parts of size r and s, and $S(K_m)$ denotes the subdivision graph of K_m (the graph on $m + m(m-1)/2$ vertices formed by adding a new vertex to the middle of each edge of K_m).

The two smallest non self-dual chiral 3-polytopes come from a normal subgroup of index 42 in the ordinary $(2,3,6)$ triangle group $[3,6]^+ = \langle\, u, v \mid u^3 = v^6 = (uv)^2 = 1 \,\rangle$ that is not normal in the full $(2,3,6)$ triangle group $[3,6]$. One is a chiral map of type $\{3,6\}$ on the torus, with 7 vertices, 21 edges and 14 faces, and the other is its topological dual (of type $\{6,3\}$, with 14 vertices, 21 edges and 7 faces). Each has automorphism group an extension of C_7 by C_6 (another Frobenius group). In the first one, every 0-face (vertex) lies in six 1-faces (edges), while each 2-face (map face) contains three 1-faces, and the number of flags is $7 \times 6 \times 2 = 14 \times 3 \times 2 = 84$. The link figure is as follows:

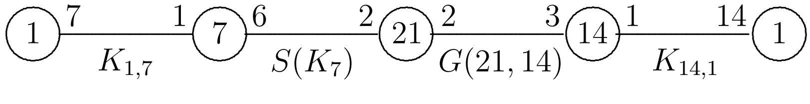

Fig.3: Link figure for the smallest non self-dual chiral 3-polytopes, of type $\{3,6\}$

Here the 0-faces and 1-faces may be taken as the vertices and edges of the complete graph K_7, with natural incidence between them, and then the 2-faces as the 14 lines of two complementary Fano planes (projective planes of order 2), such as those with points $1, 2, 3, 4, 5, 6, 7$, and lines $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$, $\{4, 5, 7\}$, $\{1, 5, 6\}$, $\{2, 6, 7\}$ and $\{1, 3, 7\}$ for one, and $\{1, 2, 6\}$, $\{2, 3, 7\}$, $\{1, 3, 4\}$, $\{2, 4, 5\}$, $\{3, 5, 6\}$, $\{4, 6, 7\}$ and $\{1, 5, 7\}$ for the other. Each edge lies in one line of each plane (and therefore in two of the 2-faces), and correspondingly, each of the 2-faces contains three edges. The between-layer incidence graphs are as indicated in the link figure, with $G(21, 14)$ denoting the subdivision graph of the Heawood graph.

The smallest *properly* self-dual example is the chiral map C7.2 of type $\{7, 7\}$ listed in [4, Table 2]. This lies on a surface of genus 7, and has 8 vertices, 28 edges and 8 faces. Its automorphism group is an extension of an elementary abelian 2-group of order 8 by a cyclic group of order 7. As such, the automorphism group is isomorphic to a quotient of the ordinary $(2, 7, 7)$ triangle group $[7, 7]^+$ by a normal subgroup of index 56 that is not normal in the full $(2, 7, 7)$ triangle group $[7, 7]$. The number of flags is $8 \times 7 \times 2 = 112$, and its link figure is as follows:

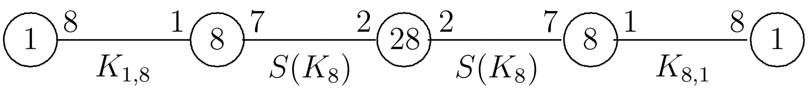

Fig.4: Link figure for the smallest properly self-dual chiral 3-polytopes, of type $\{7, 7\}$

In this case the 0-faces and 1-faces may be taken as the vertices and edges of the complete graph K_8, with natural incidence between them, and then the 2-faces as the faces of a 'cycle double cover' of K_8 using eight cycles of length 7 (such that every edge lies in exactly two of the cycles). One such example can be constructed from the group $\mathrm{AGL}(1, 8)$ of all affine transformations $z \mapsto az + b$ of a field F of order 8, by taking any element of order 7 (regarded as a 7-cycle acting on the elements of F), and letting the faces be given by the eight conjugates of that element under the translation subgroup.

An obvious question is this: how does one find such examples and know they are the smallest? One approach to take is to use a computational procedure or finding normal subgroups of small index in a finitely-presented group, such as that described in [5] and used in [4] to determine chiral maps of genus 2 to 15. This method is based on a systematic enumeration of possible coset tables for the subgroups being sought. A much-improved version (based on enumerating possibilities for a composition series of the quotient) has been developed by Derek Holt and one of his students, and this has now been implemented as the `LowIndexNormalSubgroups` command in MAGMA [1].

This approach allows the determination of all normal subgroups of up to a given index N in the subgroup $\Lambda^+ = [p_1, \ldots, p_{n-1}]^+$ of index 2 in the Coxeter group $\Lambda = [p_1, \ldots, p_{n-1}]$. It is then not difficult to decide for each such subgroup K, whether K is normal in Λ. For example, one can check whether each of the generators of Λ normalizes K. Alternatively, one can use Magma to compute the order of the permutation group induced by Λ on cosets of K; the latter group is isomorphic to $\Lambda/(K \cap K^x)$ for any $x \in \Lambda \backslash \Lambda^+$, and hence its order will be twice that of Λ^+/K if and only if K is normal in Λ. In cases where it is found that K is not normal in Λ, one can then proceed with other computational checks to see whether or not the diamond condition and intersection condition are satisfied. Note that the diamond condition can be translated into an equivalent condition about intersections of cosets of particular subgroups; again see [24] for further details.

In this way, it is theoretically possible to determine whether or not there exists a chiral n-polytope of type $\{p_1, \dots, p_{n-1}\}$ with automorphism group of order at most N, for any given integer $N > 0$.

In practice, as in [4], one can do this without first specifiying the parameters $p_1, \dots, p_{n-1}$, by working within the n-generator Coxeter group $\Sigma = [\infty, \dots, \infty]$, which is generated by elements $x_0, x_1, \dots, x_{n-1}$ subject to the defining relations $x_i^2 = 1$ for $0 \le i < n$ and $(x_j x_j)^2 = 1$ for $1 \le i+1 < j < n$, and then determining the values of the parameters $p_1, \dots, p_{n-1}$ from the orders of the images of the elements $y_i = x_{i-1}x_i$ in the quotient Σ^+/K, for any subgroup K normal in Σ^+ but not in Σ.

Knowing the right value of N (the bound for the index $|\Sigma^+ : K|$) is a challenge, but this can be overcome by a little experimentation using the standard procedure for finding subgroups of small index in Σ^+, and checking the properties of the permutation groups induced by Σ^+ on the cosets. For example, the smallest chiral 5-polytope described in Section 5 can be found from a subgroup of index 6 in Σ^+ in the case $n = 5$, with the permutations on cosets generating the group S_6 of order 720, while the smallest non self-dual example in Section 5 comes from the core of a non-normal subgroup of index 12.

4 Chiral 4-polytopes

Small chiral 4-polytopes can be found by searching for suitable subgroups of small index in the group $\Sigma^+ = [\infty, \infty, \infty, \infty]^+ = \langle\, \sigma_1, \sigma_2, \sigma_3 \mid (\sigma_1\sigma_2)^2 = (\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3)^2 = 1 \,\rangle$. The smallest index of a normal subgroup that satisfies the diamond and intersection conditions, but is not normal in the full Coxeter group $\Sigma = [\infty, \infty, \infty, \infty]$, is 120. There are three pairs of such normal subgroups, and in all cases, the quotient Σ^+/K by the normal subgroup K is isomorphic to the symmetric group S_5.

For one pair, one of the normal subgroups is the normal closure in Σ^+ of the set $\{{\sigma_1}^4$, ${\sigma_2}^4$, ${\sigma_3}^4$, $\sigma_1{\sigma_2}^2{\sigma_1}^{-2}{\sigma_2}^{-1}$, $\sigma_1{\sigma_3}^{-1}{\sigma_1}^2\sigma_2{\sigma_3}^2\sigma_2\}$, or equivalently, the kernel of the epimorphism $\psi\colon \Sigma^+ \to S_5$ given by

$$\sigma_1 \mapsto (1,2,3,4), \quad \sigma_2 \mapsto (1,3,2,5), \quad \sigma_3 \mapsto (1,5,3,4).$$

In this case, the resulting 4-polytope has type $\{4,4,4\}$. It has six 0-faces, fifteen 1-faces, fifteen 2-faces and six 3-faces. The stabilizer of a 0-face is the subgroup generated by $(1,3,2,5)$ and $(1,5,3,4)$, which is a Frobenius group of order 20, as is the stabilizer of a

3-face, generated by $(1,2,3,4)$ and $(1,3,2,5)$. The stabilizer of a 1-face is the subgroup generated by $(1,5)(3,4)$ and $(1,5,3,4)$, which is dihedral of order 8, as is the stabilizer of a 2-face, generated by $(1,2,3,4)$ and $(1,4)(2,3)$.

The i-faces can be labelled as f_i^j, such that $\Phi = \{f_i^1 : -1 \leq i \leq 4\}$ is the base flag, and the generators σ_k induce the following permutations:

$$\begin{aligned}
\sigma_1 \mapsto\ & (f_0^1, f_0^2, f_0^3, f_0^4)(f_1^1, f_1^2, f_1^3, f_1^4)(f_1^5, f_1^6, f_1^7, f_1^8)(f_1^9, f_1^{10})(f_1^{11}, f_1^{12}, f_1^{13}, f_1^{14}) \\
& (f_2^2, f_2^6, f_2^8, f_2^{11})(f_2^3, f_2^{10}, f_2^{15}, f_2^{12})(f_2^4, f_2^7, f_2^{14}, f_2^5)(f_2^9, f_2^{13})(f_3^3, f_3^4, f_3^5, f_3^6)
\end{aligned}$$

$$\begin{aligned}
\sigma_2 \mapsto\ & (f_0^2, f_0^4, f_0^5, f_0^3)(f_1^1, f_1^5, f_1^9, f_1^2)(f_1^3, f_1^{10}, f_1^8, f_1^7)(f_1^4, f_1^6)(f_1^{12}, f_1^{14}, f_1^{15}, f_1^{13}) \\
& (f_2^1, f_2^4, f_2^5, f_2^7)(f_2^2, f_2^{10})(f_2^3, f_2^9, f_2^8, f_2^6)(f_2^{11}, f_2^{13}, f_2^{12}, f_2^{15})(f_3^2, f_3^4, f_3^3, f_3^5)
\end{aligned}$$

$$\begin{aligned}
\sigma_3 \mapsto\ & (f_0^2, f_0^3, f_0^6, f_0^5)(f_1^2, f_1^9, f_1^{11}, f_1^5)(f_1^3, f_1^{13}, f_1^{15}, f_1^6)(f_1^4, f_1^{14}, f_1^8, f_1^{10})(f_1^7, f_1^{12}) \\
& (f_2^1, f_2^2, f_2^3, f_2^4)(f_2^5, f_2^6)(f_2^7, f_2^8, f_2^9, f_2^{10})(f_2^{11}, f_2^{12}, f_2^{13}, f_2^{14})(f_3^1, f_3^2, f_3^3, f_3^4).
\end{aligned}$$

Thus, for example, the orbit of f_1^1 under the stabilizer of the 0-face f_0^1 is $\{f_1^1, f_1^2, f_1^5, f_1^9, f_1^{11}\}$, and this is the set of all 1-faces containing f_0^1.

In fact each 0-face lies in five 1-faces, and dually, each 3-face contains five 2-faces. Similarly each 1-face lies in four 2-faces, and each 2-face contains four 1-faces. In particular, the number of flags is $6 \times 5 \times 4 \times 2 = 240$. These numbers can be depicted by the following link figure:

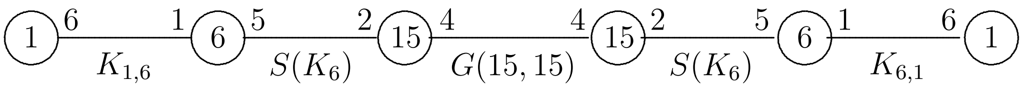

Fig.5: Link figure for the smallest self-dual chiral 4-polytopes, of type $\{4,4,4\}$

The symmetry (of both numbers and subgroups) suggests that this polytope is self-dual, and indeed it is. For example, conjugation by the involution $(1,3)(2,5)$ induces an automorphism of S_5 that interchanges the images of σ_1 and ${\sigma_3}^{-1}$ and inverts the image of σ_2. It follows that this chiral 4-polytope is properly self-dual, and has a polarity. (On the other hand, the polytope is not directly regular because there is no such automorphism inverting the image of σ_1 and interchanging the images of σ_2 and ${\sigma_1}^2\sigma_2$.)

All the facets and vertex figures of this 4-polytope are isomorphic to the chiral 3-polytope associated with the chiral map $\{4,4\}_{(1,2)}$ (with 5 vertices, 10 edges and 5 faces). In fact this is the *universal* 4-polytope having such facets and vertex figures, and may be constructed by glueing together copies of $\{4,4\}_{(1,2)}$ and its enantiomorphic form $\{\{4,4\}_{(2,1)}$, so is isomorphic to the universal $\{\{4,4\}_{(2,1)}, \{4,4\}_{(1,2)}\}$ 4-polytope; see [23].

The medial layer graph (incidence graph of 1- and 2-faces) is interesting, and can be obtained as follows. First, the 0-faces and 1-faces may be taken as the vertices and edges of the complete graph K_6, with natural incidence between them, and similarly the 3- and 2-faces as vertices and edges of a second copy of K_6. Now take any 1-factorisation of K_6

(consisting of a set of five 1-factors, each containing three edges than span the vertices of K_6, such that every edge lies in exactly one of the 1-factors). Given any edge $\{a, b\}$ of K_6, if say $\{\{a, b\}, \{c, d\}, \{e, f\}\}$ is the unique 1-factor of the given 1-factorisation containing $\{a, b\}$, then there are exactly four edges of K_6 that do not contain the vertices a or b and are different from the two edges $\{c, d\}$ and $\{e, f\}$ of that 1-factor, namely the edges $\{c, e\}$, $\{c, f\}$, $\{d, e\}$ and $\{d, f\}$, and these form a quadrilateral. There are 15 such quadrilaterals (one for each edge of each 1-factor). The given 1-factorisation sets ups a one-to-four correspondence between edges of one copy of K_6 and the edges of the other copy of K_6, and this correspondence defines the edges of the medial layer graph, denoted by $G(15, 15)$ in Figure 5, and displayed in Figure 6.

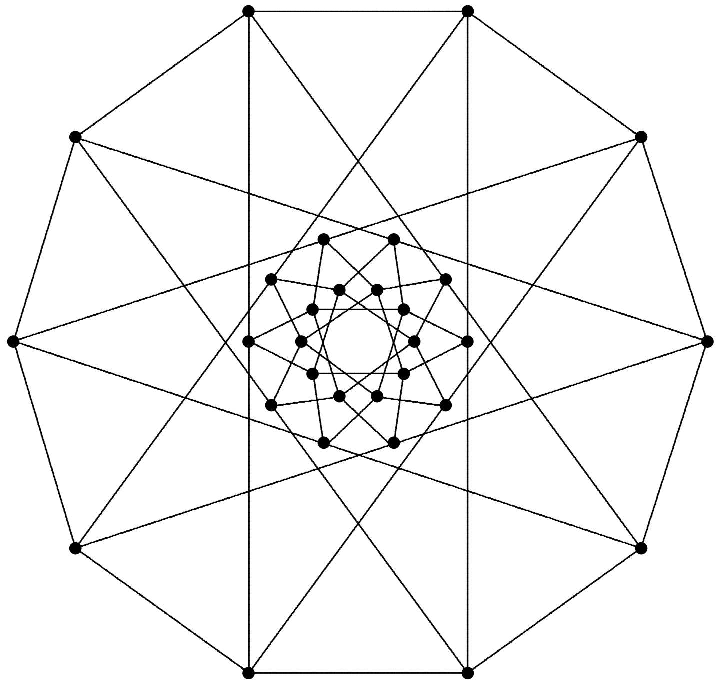

Fig.6: Medial layer graph of a smallest self-dual chiral 4-polytope

For another pair of smallest chiral 4-polytopes, one of the normal subgroups is the normal closure in Σ^+ of the set $\{{\sigma_1}^3, {\sigma_2}^4, {\sigma_3}^4, {\sigma_1}^2\sigma_2{\sigma_3}^{-2}\sigma_1{\sigma_2}^2{\sigma_1}^{-1}{\sigma_3}^{-1}\}$, or equivalently, the kernel of the epimorphism $\psi\colon \Sigma^+ \to S_5$ given by

$$\sigma_1 \mapsto (1, 2, 3), \quad \sigma_2 \mapsto (1, 3, 2, 4), \quad \sigma_3 \mapsto (1, 5, 4, 3).$$

In this case the resulting 4-polytope has type $\{3, 4, 4\}$, and is non self-dual. It has six 0-faces, fifteen 1-faces, twenty 2-faces and five 3-faces. The stabilizers of a 0-face and a 1-face are a Frobenius group of order 20 and a dihedral group of order 8, while the stabilizer of a 2-face is dihedral of order 6, generated by $(1, 2, 3)$ and $(2, 3)(4, 5)$, and the stabilizer of a 3-face is S_4, generated by $(1,2,3)$ and $(1,3,2,4)$. The number of flags is again $6 \times 5 \times 4 \times 2 = 240$, with numbers of inclusions depicted by the following link figure:

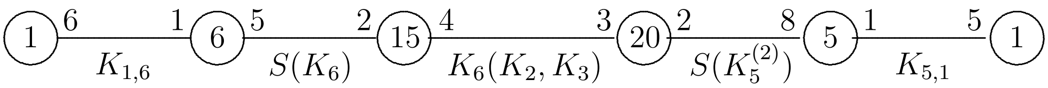

Fig.7: Link figure for the smallest non self-dual chiral 4-polytopes, of type $\{3,4,4\}$

As with the previous example, the vertex figures of this 4-polytope are isomorphic to the chiral 3-polytope associated with the map $\{4,4\}_{(1,2)}$, with 5 vertices, 10 edges and 5 faces. On the other hand, the facets are all isomorphic to the regular 3-polytope of type $\{3,4\}$ associated with the regular embedding of the octahedron in the sphere, with 6 vertices, 12 edges and 8 faces. In fact this 4-polytope is isomorphic to the universal $\{\{3,4\},\{4,4\}_{(1,2)}\}$ 4-polytope.

The 0-, 1- and 2-faces may may be identified with the vertices, edges and triangles (3-cycles) of K_6 respectively, so that the medial layer graph of the polytope is given simply by incidence of edges with triangles in K_6. This is denoted by $K_6(K_2, K_3)$ in Figure 7. The 3-faces may be identified with the five 1-factors of a given 1-factorisation of K_6, and then incidence between 3- and 2-faces may be defined by making a 1-factor $\{\{a,b\},\{c,d\},\{e,f\}\}$ incident to each of the eight triangles consisting of one vertex from each of the three edges in $\{\{a,b\},\{c,d\},\{e,f\}\}$ (or equivalently, making a triangle $\{x,y,z\}$ incident to each of the two 1-factors of the given 1-factorisation in which x, y and z lie in distinct edges). The corresponding incidence graph is also isomorphic to the subdivision graph of a copy of K_5 with doubled edges, denoted by $S(K_5^{(2)})$ in Figure 7.

The final pair of admissible normal subgroups of index 120 in Σ^+ give the dual of this polytope and its mirror image, of type $\{4,4,3\}$.

We believe the smallest *improperly* self-dual chiral 4-polytopes are also of type $\{4,4,4\}$, with automorphism group of order 400. One of a chiral pair of such polytopes is obtainable from a transitive permutation of Σ^+ of degree 10, under which

$$\sigma_1 \mapsto (1,2,3,4)(5,6,7,8), \quad \sigma_2 \mapsto (1,4,3,2)(5,7,9,6), \quad \sigma_3 \mapsto (1,8,2,7)(3,6,10,9)(4,5).$$

Here the stabilizers of a 0-, 1-, 2- and 3-face have orders 40, 8, 8 and 40 respectively, and there are 800 flags, with numbers of inclusions depicted by the following link figure:

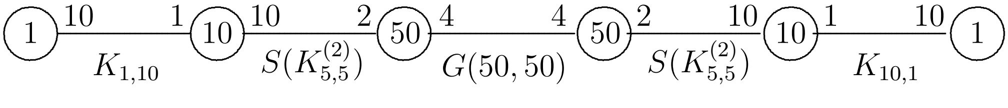

Fig.8: Link figure for an improperly self-dual chiral 4-polytope of type $\{4,4,4\}$

This chiral 4-polytope is a quotient of the universal $\{\{4,4\}_{(3,1)},\{4,4\}_{(3,1)}\}$ 4-polytope. As such, its vertex figures and facets are all isomorphic to the chiral 3-polytope associated with the map $\{4,4\}_{(3,1)}$ (which has 10 vertices, 20 edges and 10 faces).

Its 0-faces and 1-faces may be taken as the vertices and ordered edges of the complete bipartite graph $K_{5,5}$, with natural incidence between them; the graph of incidence between 0- and 1-faces is isomorphic to the subdivision graph of a copy of $K_{5,5}$ with doubled edges, denoted by $S(K_{5,5}^{(2)})$ in Figure 8. The 2- and 3-faces may be identified similarly with the vertices and ordered edges of a second copy of $K_{5,5}$, with natural incidence between them, and then incidence between 1- and 2-faces can be described as follows.

Label the vertices of $K_{5,5}$ as 1 to 10, with 1 to 5 in one part, and 6 to 10 in the other, and let π be the permutation $(1,2,3,4,5)(6,7,8,9,10)$. For any vertex $u \in \{1,\ldots,10\}$, let u^+ and u^{++} be the images of u under π and π^2 respectively, and let u^- and u^{--} be the images of a under π^{-1} and π^{-2} respectively. Now for any ordered edge (a,b) in $K_{5,5}$ with $a < b$, let v be the unique vertex in $\{1,2,3,4,5\}$ such that $v \equiv b-a+1$ mod 5, and let w be the unique vertex in $\{6,7,8,9,10\}$ such that $w \equiv a+b$ mod 5. Now make the 1-face (a,b) from the first copy of $K_{5,5}$ incident with each of the four 2-faces (v^+, w^{++}), (w^{++}, v^-), (v^-, w^{--}), (w^{--}, v^+) from the second copy of $K_{5,5}$, and make its reverse (b,a) from the first copy incident with each of the four 2-faces (w^+, v^{--}), (v^{--}, w^-), (w^-, v^{++}), (v^{++}, w^+) from the second copy. This correspondence defines the edges of the medial layer graph, denoted by $G(50,50)$ in Figure 8.

The graph $G(50,50)$ is too large for drawing a meaningful picture, but it is easily described as a cover of the graph $K_{2,2}^{(2)}$ (a square with doubled edges), with voltage group $\mathbb{Z}_5 \times \mathbb{Z}_5$, as indicated in Figure 9:

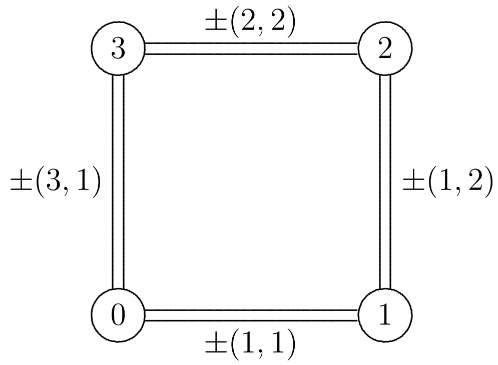

Fig.9: Construction of incidence graph $G(50,50)$ as a $(\mathbb{Z}_5 \times \mathbb{Z}_5)$-cover of $K_{2,2}^{(2)}$

Specifically, the vertices may be taken as the 100 ordered triples (i,j,k) with $i,j \in \mathbb{Z}_5$ and $k \in \mathbb{Z}_4$, with adjacency $\sim$ defined by letting

$$\begin{array}{lll} (i,j,0) \sim (i',j',1) & \text{whenever} & (i',j') = (i,j) \pm (1,1) \bmod 5 \\ (i,j,0) \sim (i',j',3) & \text{whenever} & (i',j') = (i,j) \pm (3,1) \bmod 5 \\ (i,j,2) \sim (i',j',1) & \text{whenever} & (i',j') = (i,j) \pm (1,2) \bmod 5 \\ (i,j,2) \sim (i',j',3) & \text{whenever} & (i',j') = (i,j) \pm (2,2) \bmod 5. \end{array}$$

5 Chiral 5-polytopes

To find examples of chiral 5-polytopes, we can search for suitable subgroups of small index in the group $\Sigma^+ = [\infty, \infty, \infty, \infty, \infty]^+ = \langle\, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \mid (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_3\sigma_4)^2 = (\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3\sigma_4)^2 = (\sigma_1\sigma_2\sigma_3\sigma_4)^2 = 1 \,\rangle$.

We believe the smallest example comes from a normal subgroup K of index 720 in Σ^+, with quotient Σ^+/K isomorphic to the symmetric group S_6, given by

$$\sigma_1 \mapsto (1,2,3), \quad \sigma_2 \mapsto (1,3,2,4), \quad \sigma_3 \mapsto (1,5,4,3), \quad \sigma_4 \mapsto (1,2,3)(4,6,5).$$

This normal subgroup K is the normal closure in Σ^+ of the set $\{{\sigma_1}^3,\ {\sigma_2}^4,\ {\sigma_3}^4,\ {\sigma_4}^3,\ {\sigma_1}^{-1}\sigma_3\sigma_1{\sigma_3}^{-1}{\sigma_2}^{-2},\ {\sigma_3}^{-2}{\sigma_2}^2{\sigma_3}^{-1}\sigma_2,\ {\sigma_4}^{-1}\sigma_3{\sigma_2}^2\sigma_4{\sigma_2}^{-1}\}$.

The resulting 5-polytope has type $\{3,4,4,3\}$. The stabilizer of a 0-face is the subgroup generated by $(1,3,2,4)$, $(1,5,4,3)$ and $(1,2,3)(4,6,5)$, which is isomorphic to S_5 in its representation as PGL$(2,5)$ on 6 points, while the stabilizer of a 4-face is the natural S_5 generated by $(1,2,3)$, $(1,3,2,4)$ and $(1,5,4,3)$. The stabilizer of a 1-face is the subgroup generated by $(1,4)$, $(1,5,4,3)$ and $(1,2,3)(4,6,5)$, which is a wreath product $S_2 \wr S_3$ of order $2^3 \times 3! = 48$, while the stabilizer of a 3-face is the subgroup generated by $(1,2,3)$, $(1,3,2,4)$ and $(1,4)(2,3)(5,6)$, which is isomorphic to $S_4 \times S_2$, also of order 48, and the stabilizer of a 2-face is the subgroup generated by $(1,2,3)$, $(2,3)(4,5)$ and $(1,2,3)(4,6,5)$, which is a subdirect product of two copies of S_3 and has order $(3!)^2/2 = 18$. Hence this 5-polytope has six 0-faces, fifteen 1-faces, forty 2-faces, fifteen 3-faces and six 4-faces.

Each 0-face lies in five 1-faces, and dually, each 4-face contains five 3-faces. Similarly each 1-face lies in eight 2-faces and each 3-face contains eight 2-faces, while each 2-face lies in three 3-faces and also contains three 1-faces. In particular, the number of flags is $6 \times 5 \times 8 \times 3 \times 2 = 1440$. The corresponding link figure is as follows:

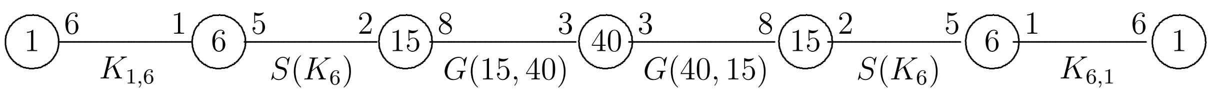

Fig.10: Link figure for a self-dual chiral 5-polytope of type $\{3,4,4,3\}$

Again the symmetry of these numbers suggests that this polytope is self-dual, although the symmetry of subgroups is not quite so obvious. In fact the polytope is improperly self-dual, and has a polarity given by an outer automorphism of S_6 which takes the 3-cycle $(1,2,3)$ to the double 3-cycle $(1,2,3)(4,6,5)$. (On the other hand, the polytope is not directly regular because there is no automorphism of S_6 inverting the image $(1,2,3)$ of σ_1, and interchanging the images $(1,3,2,4)$ and $(1,2,3,4)$ of σ_2 and ${\sigma_1}^2\sigma_2$, while also centralizing the images $(1,5,4,3)$ and $(1,2,3)(4,6,5)$ of σ_3 and σ_4.)

The 0- and 1-faces may may be identified with the vertices and edges of K_6, and the 4- and 3-faces with the vertices and edges of a second copy of K_6. The forty 2-faces can then be identified the triangles (3-cycles) from these two copies of K_6, with incidence defined as follows. Let τ be any outer automorphism of S_6 of order 2. This takes 3-cycles of S_6 to double 3-cycles and vice versa. Now make each edge of the second copy of K_6 (regarded as a 1-face of the polytope) incident to the four triangles containing it in that copy and also to the four triangles containing its analogue in the first copy. In other words, if $\{a,b,c,d,e,f\}$ and $\{a',b',c',d',e',f'\}$ are the two vertex-sets, then make $\{a',b'\}$ incident with $\{a,b,c\}$, $\{a,b,d\}$, $\{a,b,e\}$, $\{a,b,f\}$, $\{a',b',c'\}$, $\{a',b',d'\}$, $\{a',b',e'\}$ and $\{a',b',f'\}$. On the other hand, for any edge $\{a,b\}$ of the first copy of K_6, and any triangle $\{a,b,c\}$ containing it in the first copy, if the outer automorphism τ of S_6 takes the 3-cycle (a,b,c) to the double 3-cycle $(x,y,z)(u,v,w)$, where $\{x,y,z\}$ contains two of the three points a,b,c, and $\{u,v,w\}$ contains the third, then make $\{a,b\}$ incident with $\{x,y,z\}$ and $\{u',v',w'\}$.

The facets of this 5-polytope are all isomorphic to the universal $\{\{3,4\},\{4,4\}_{(2,1)}\}$ 4-polytope, which is non self-dual and chiral of type $\{3,4,4\}$ (like the example described in the previous section), while the vertex-figures are all isomorphic to its dual (the universal $\{\{4,4\}_{(2,1)},\{4,3\}\}$ 4-polytope, of type $\{4,4,3\}$). In fact this example is isomorphic to the universal $\{\{\{3,4\},\{4,4\}_{(2,1)}\},\{\{4,4\}_{(2,1)},\{4,3\}\}\}$ 5-polytope.

One of the smallest *non self-dual* chiral 5-polytopes we have been able to find is one of type $\{3,4,4,6\}$ that comes from a transitive permutation representation of Σ^+ on 12 points, given by

$$\sigma_1 \mapsto (1,2,3)(4,5,6), \quad \sigma_2 \mapsto (1,3,2,7)(4,6,5,8), \quad \sigma_3 \mapsto (1,9,7,3)(4,10,8,6),$$
$$\sigma_4 \mapsto (1,5,3,4,2,6)(7,11,9,8,12,10).$$

These permutations generate a group of order 1440, isomorphic to the direct product $S_6 \times C_2$, with the image of ${\sigma_4}^3$ as the central involution. The stabilizers of a 0-, 1-, 2-, 3- and 4-face have orders 240, 96, 36, 48 and 120 respectively, and the numbers of inclusions are depicted by the link figure below:

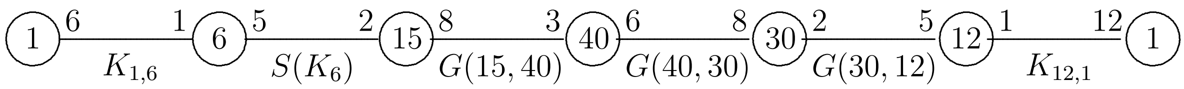

Fig.11: Link figure for a non self-dual chiral 5-polytope of type $\{3,4,4,6\}$

The six 0-faces, fifteen 1-faces and forty 2-faces can be identified with the vertices, edges and oriented triangles of K_6, with natural incidence between them. On the other hand, the thirty 3-faces can be identified with ordered edges of K_6, and the twelve 4-faces with 'marked' vertices a^+ and a^- of K_6, such that every 3-face (u,v) is with the two 4-faces u^+ and v^-. In particular, the incidence graph $G(30,12)$ between 3- and 4-faces is isomorphic to $S(DK_{6,6})$, the subdivision graph of the Kronecker cover of K_6.

Incidence between 2- and 3-faces is the most difficult (but also the most interesting) to define. One way is as follows:

For every unordered triple of vertices of K_6, define a 'good' and 'bad' orientation; for example, if $\min(\{a,b,c\}) = a$ then let (a,b,c) be good if $a < b < c$ and bad if $a < c < b$.

As previously, let τ be any outer automorphism of S_6 of order 2. This takes 3-cycles of S_6 to double 3-cycles and vice versa. Now let (a, b, c) be any oriented triangle in K_6, or equivalently, a 3-cycle in S_6, and let $(x, y, z)(u, v, w)$ be the image of this 3-cycle under the outer automorphism τ, where $\{x, y, z\}$ contains two of the three points a, b, c, and $\{u, v, w\}$ contains the third. Let the 2-face (a, b, c) be incident with each of the six 3-faces naturally associated with 2-element subsets of $\{x, y, z\}$ if (a, b, c) is good, or with those of of $\{u, v, w\}$ if (a, b, c) is bad. This correspondence defines the edges of the incidence graph denoted by $G(40, 30)$ in Figure 11.

The facets of this 5-polytope are again all isomorphic to the universal $\{\{3, 4\}, \{4, 4\}_{(2,1)}\}$ 4-polytope, of type $\{3, 4, 4\}$, while on the other hand, the vertex-figures are all isomorphic to a non self-dual chiral 4-polytope of type $\{4, 4, 6\}$, with five 0-faces, twenty 1-faces, thirty 2-faces and twelve 3-faces, and automorphism group of order 240. The latter is the universal $\{\{4, 4\}_{(2,1)}, \{4, 6 \,|\, 4, 2\}\}$ 4-polytope, where $\{4, 6 \,|\, 4, 2\}$ is a regular quotient of the Petrie-Coxeter polyhedron $\{4, 6 \,|\, 4\}$ having 3-holes of length 2. Accordingly, our 5-polytope is isomorphic to the universal $\{\{\{3, 4\}, \{4, 4\}_{(2,1)}\}, \{\{4, 4\}_{(2,1)}, \{4, 6 \,|\, 4, 2\}\}\}$ 5-polytope.

The smallest *properly* self-dual chiral 5-polytope we have been able to find is rather larger than the above examples. This one has type $\{3, 8, 8, 3\}$, and can be obtained from a transitive permutation representation of Σ^+ on 20 points, given by

$$\begin{aligned}
\sigma_1 &\mapsto (1, 2, 3)(4, 5, 6)(12, 14, 13)(15, 17, 16),\\
\sigma_2 &\mapsto (1, 3, 5, 11, 4, 19, 6, 2)(7, 16, 10, 17, 14, 12, 13, 15)(8, 20)(9, 18),\\
\sigma_3 &\mapsto (1, 13)(2, 17, 11, 16, 18, 20, 19, 15)(3, 12)(4, 10, 5, 14, 6, 8, 9, 7),\\
\sigma_4 &\mapsto (4, 6, 5)(7, 9, 8)(15, 16, 17)(18, 19, 20).
\end{aligned}$$

The automorphism group is the alternating group A_{20}. The polytope has $184, 756$ 0-faces, $1, 810, 194, 946, 560, 000$ $(= |A_{20}|/672)$ 1-faces, $67, 580, 611, 338, 240, 000$ $(= |A_{20}|/18)$ 2-faces, $1, 810, 194, 946, 560, 000$ $(= |A_{20}|/672)$ 3-faces, and $184, 756$ 4-faces. Every 0-face lies in $19, 595, 520, 000$ 1-faces, and dually, every 4-face contains $19, 595, 520, 000$ 3-faces; also every 1-face lies in 112 2-faces, while every 2-face contains three 1-faces, and dually, every 3-face contains 112 2-faces, and every 2-face lies in three 3-faces. Chirality follows from the fact that there is no element of $\mathrm{Aut}(A_{20}) = S_{20}$ that centralizes the images of σ_3 and σ_4 and inverts the image of σ_1. On the other hand, there is an element of S_{20} that takes the images of σ_1 and σ_2 to the inverses of the images of σ_3 and σ_4, and vice versa, namely the permutation (1, 20)(2, 19)(3, 18)(4, 17)(5, 16)(6, 15)(7, 14)(8, 13)(9, 12), and so this chiral 5-polytope is properly self-dual.

Note that there are many other interesting examples that can be found in this way. One is non self-dual chiral 5-polytope of type $\{3, 6, 3, 6\}$ with three 0-faces, 144 1-faces, 576 2-faces, 576 3-faces and 96 4-faces, and automorphism group of order 20736 (which is representable as a subgroup of the symmetric group S_{12}); this has chiral vertex-figures which are obtainable as *proper* quotients of the universal $\{\{6, 3\}_{(3,0)}, \{3, 6\}_{(3,0)}\}$ 4-polytope, and so this 5-polytope is not a universal one. Another example is a non self-dual chiral 5-polytope of type $\{3, 8, 8, 3\}$ with automorphism group A_{12}, in which the stabilizers of a 0-, 1-, 2-, 3- and 4-face are respectively groups of order 95040 (viz. the Mathieu group M_{12}), 192, 18, 672 (viz. $\mathrm{PGL}(2,7) \times \mathrm{C}_2$) and 19958400 (viz. A_{11}). Another is an improperly self-dual chiral 5-polytope of type $\{4,4,4,4\}$ with automorphism group M_{12}.

Finally, a curious reader might ask whether or not there exists a finite chiral 6-polytope with automorphism group S_7, given the chiral 4 and 5-polytopes with automorphism groups S_5 and S_6 presented here, but we have checked for the existence of one and found there is no such chiral 6-polytope (with automorphism group S_7).

References

[1] W. Bosma, J. Cannon and C. Playoust, The MAGMA Algebra System I: The User Language, *J. Symbolic Computation* 24 (1997), 235–265.

[2] H.R. Brahana, Regular maps and their groups, *Amer. J. Math.* 49 (1927), 268–284.

[3] E. Bujalance, M. Conder and A. Costa, Pseudo-real Riemann surfaces and chiral regular maps, preprint.

[4] M. Conder and P. Dobcsányi, Determination of all regular maps of small genus, *J. Combin. Theory Ser. B* 81 (2001), 224–242.

[5] M. Conder and P. Dobcsányi, Applications and adaptations of the low index subgroups procedure, *Math. Comp.* 74 (2005), 485–497.

[6] H.S.M. Coxeter, Configurations and maps, *Rep. Math. Colloquium* 2 (1948), 18–38.

[7] H.S.M. Coxeter, *Regular Polytopes*, 3rd ed., Dover, New York (1973).

[8] H.S.M. Coxeter, *Twisted Honeycombs*, Regional Conf. Ser. In Math., Amer. Math. Soc., Providence, RI (1970).

[9] H.S.M. Coxeter and W.O.J. Moser, *Generators and Relations for Discrete Groups*, 4th ed., Springer Berlin (1980).

[10] L. Danzer and E. Schulte, Reguläre Inzidenzkomplexe, I, *Geom. Dedicata* 13 (1982), 295–308.

[11] S. Doro and S. Wilson, Rotary maps of type $\{6,6\}_4$, *Quart. J. Math. Oxford* 2 (1980), 403–414.

[12] D. Garbe, A generalization of the regular maps of type $\{4,\,4\}_{b,\,c}$ and $\{3,\,6\}_{b,\,c}$, *Canad. Math Bull.* 12 (1969), 293–297.

[13] D. Garbe, Über die Regularen Zerlegungen geschlossener orientierbarer Flächen, *J. Reine. Angw. Math.* 237 (1969), 39–55.

[14] B. Grünbaum, Regularity of graphs, complexes and designs, *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978, pp. 191–197.

[15] L. Heffter, Über Metacyklische Gruppen und Nachbarconfigurationen, *Math. Ann.* 50 (1898), 261–268.

[16] I. Hubard and A. Ivić Weiss, Self-duality of chiral polytopes, *Journal of Combinatorial Theory, Ser. A* 111 (2005), 128–136.

[17] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, Encyclopedia of Math. And its Applic. 92, Cambridge (2002).

[18] B. Monson, T. Pisanski, E. Schulte and A. Ivić Weiss, Semisymmetric graphs from polytopes. *J. Combin. Theory, Ser. A*, to appear.

[19] B. Nostrand, Ring extensions and chiral polytopes, *Proceedings of the 25th Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994)*, Congr. Numer. 102 (1994), 147–153.

[20] E. Schulte, Reguläre Inzidenzkomplexe, II, *Geom. Dedicata* 14 (1983), 33–56.

[21] E. Schulte, Reguläre Inzidenzkomplexe, III *Geom. Dedicata* 14 (1983), 57–79.

[22] E. Schulte, Chiral polyhedra in ordinary space, I, *Discrete and Computational Geometry* 32 (2004), 55–99.

[23] E. Schulte and A. Ivić Weiss, Chiral polytopes, *Applied Geometry and Discrete Mathematics ("The Victor Klee Festschrift")*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 4 (1991), 493–516.

[24] E. Schulte and A. Ivić Weiss, Chirality and projective linear groups, *Discrete Math.* 131 (1994), 221–261.

[25] E. Schulte and A. Ivić Weiss, Free extensions of chiral polytopes, *Can. J. Math.* 47 (1995), 641–654.

[26] F.A. Sherk, A family of regular maps of type $\{6,6\}$, *Canad. Math. Bull.* 5 (1962), 13–20.

[27] C. Weber and H. Seifert, Die beiden Dodekaederräume, *Math. Z.* 37 (1933), 237–253.