Hurwitz groups with given centre

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Dedicated to Peter Neumann
on the occasion of his 60th birthday

Abstract

A Hurwitz group is any non-trivial finite group which can be $(2,3,7)$-generated, that is, generated by elements x and y satisfying the relations $x^2 = y^3 = (xy)^7 = 1$. In this short paper a complete answer is given to a 1965 question by John Leech, showing that the centre of a Hurwitz group can be any given finite abelian group. The proof is based on a recent theorem of Lucchini, Tamburini and Wilson which states that the special linear group $\mathrm{SL}_n(q)$ is a Hurwitz group for every integer $n \geq 287$ and every prime-power q.

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1 Introduction

The $(2, 3, 7)$ *triangle group* is the abstract group Δ with presentation

$$\Delta = \langle\, x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \,\rangle.$$

A theorem of Hurwitz (1893) states that any compact Riemann surface with genus $g > 1$ admits at most $84(g-1)$ conformal automorphisms, that is, homeomorphisms of the surface onto itself which preserve the local structure. Further, any such surface X with the maximum number of automorphisms must be uniformized by a normal subgroup N of the $(2, 3, 7)$ triangle group Δ, for the latter is the Fuchsian group with fundamental region of smallest hyperbolic area, and the conformal automorphism group of X is then isomorphic to the quotient group Δ/N. Conversely, any non-trivial finite quotient G of Δ gives rise to a compact Riemann surface (of genus $|G|/84 + 1$) with the maximum possible number of automorphisms, and G as its automorphism group. For these reasons, any non-trivial finite quotient of Δ is called a *Hurwitz group*. Further details of their significance may be found in [4].

Every Hurwitz group G is perfect (that is, has trivial abelianisation G/G'), and so it is natural to look among the finite nonabelian simple groups for examples of Hurwitz groups. A search was initiated by Macbeath in the 1960s when he proved in [11] that the simple group $\mathrm{PSL}_2(q)$ is a Hurwitz group precisely when $q = 7$, or q is a prime congruent to ± 1 modulo 7, or $q = p^3$ where p is a prime congruent to ± 2 or ± 3 modulo 7. Graham Higman developed a method of composing coset diagrams to prove that all but finitely many alternating groups are Hurwitz, and the author of this paper took this further to prove in [5] that A_n is Hurwitz far all $n \geq 168$ (and many smaller values of n as well). Many other examples are now known, including $G_2(q)$ for all $q > 4$, certain Ree groups, and some of the sporadic simple groups; a recent summary was given in by Jones in [7].

In a very recent (and amazing) piece of work, Robert Wilson applied clever computational techniques to show that the Monster M is a Hurwitz group, using its 196882-dimensional irreducible representation over GF(2); see [12]. In Wilson's words, this "demonstrates that, against all reasonable expectations, it is now possible to do practical calculations in this enormous group". In particular, it is now known that exactly 12 of the sporadic finite simple groups are Hurwitz groups, namely J_1, J_2, He, Ru, Co_3, Fi_{22}, HN, Ly, Th, J_4, Fi'_{24} and M (see [6] and other references in [7] and [12]).

Also Lucchini, Tamburini and John Wilson have now shown that most finite simple classical groups of sufficiently large dimension are Hurwitz groups. In particular, in [10] they adapted the methods used by the author in [5] to prove that the special linear group $SL_n(q)$ is Hurwitz for every $n \geq 287$ and every prime-power q. We will subsequently call this the Lucchini/Tamburini/Wilson theorem. Further, in [9] Lucchini and Tamburini showed that the classical groups $Sp_{2n}(q), \Omega^+_{2n}(q)$ and $SU_{2n}(q)$ are Hurwitz for every $n \geq 371$ and every prime-power q, and that $\Omega_{2n+7}(q)$ and $SU_{2n+7}(q)$ are Hurwitz for every $n \geq 371$ and every odd prime-power q. It follows that simple (projective) quotients of these groups are all Hurwitz.

Turning now more towards the subject of this paper, we consider the question of what centres are possible in finite quotients of the $(2, 3, 7)$ triangle group. This question was raised in 1965 by Leech [8], who later produced two infinite families of Hurwitz groups with centres of orders 2 and 4, from extensions of 7-generator nilpotent groups by $PSL_2(8)$. In [2] the author of this paper used a similar method to proved the existence of infinitely many Hurwitz groups with a centre of order 3, and in [3] constructed a family of central products of 2-dimensional special linear groups to show that the centre of a Hurwitz group could be an elementary abelian 2-group of arbitrarily large order.

In this paper we use the Lucchini/Tamburini/Wilson theorem (Corollary 1 of the main theorem of [10]) to prove that the centre of a Hurwitz group can be *any* finite abelian group. In fact we prove the following:

Theorem *Given any finite abelian group A, there exist infinitely many Hurwitz groups G such that the centre $Z(G)$ of G is isomorphic to A.*

2 Proof of Theorem

The crux of the proof is quite simple: take a direct product of appropriately chosen special linear Hurwitz groups.

Let A be any finite abelian group. Then A is isomorphic to a direct product of cyclic groups, say $C_{m_1} \times C_{m_2} \times \ldots \times C_{m_s}$, of orders $m_i \geq 2$ for $1 \leq i \leq s$. Now choose any prime p which does not divide $|A| = m_1 m_2 \ldots m_s$. Then p is congruent to a unit in the ring of integers modulo $|A|$ and so there exists a positive integer e for which $p^e \equiv 1 \bmod |A|$. (For example, take $e = \phi(|A|)$.)

Let $q = p^e$. Then for $1 \leq i \leq s$ we know $q \equiv 1$ modulo m_i and therefore $q - 1 = l_i m_i$ for some integer l_i. Further, if k_i is any positive integer coprime to l_i, then letting $n_i = k_i m_i$ we have $\gcd(n_i, q-1) = \gcd(k_i m_i, l_i m_i) = m_i$. In particular, there are infinitely many possibilities for each n_i, and all can be chosen such that the n_i are mutually distinct, and $n_i \geq 287$ for $1 \leq i \leq s$.

Next let $H_i = \mathrm{SL}_{n_i}(q)$ for $1 \leq i \leq s$. By the Lucchini/Tamburini/Wilson theorem, each H_i is a Hurwitz group, so can be generated by two elements X_i and Y_i satisfying the relations $X_i^2 = Y_i^3 = (X_i Y_i)^7 = 1$. Note that the centre $Z(H_i)$ of H_i consists of all the scalar matrices λI_{n_i} of determinant 1, and hence is cyclic of order $\gcd(n_i, q-1) = m_i$. Also $H_i/Z(H_i)$ is isomorphic to the nonabelian simple group $\mathrm{PSL}_{n_i}(q)$, for $1 \leq i \leq s$, and because the n_i were chosen to be distinct, $H_i/Z(H_i) \ncong H_j/Z(H_j)$ whenever $i \neq j$.

Define G to be the direct product $\Pi_{1 \leq i \leq s} H_i$ of these special linear Hurwitz groups, and let $X = X_1 X_2 \ldots X_s$ and $Y = Y_1 Y_2 \ldots Y_s$. As the X_i commute with each other, X has order 2, and similarly Y has order 3. Also the product $XY = X_1 Y_1 X_2 Y_2 \ldots X_s Y_s$ has order 7. Hence the subgroup H generated by X and Y is a Hurwitz group.

We claim that $H = G$. To see this, note first that the natural projection $\pi_i : G \to H_i$ maps (X, Y) to (X_i, Y_i) and therefore maps H onto H_i, for $1 \leq i \leq s$. In particular, H has distinct composition factors isomorphic to $\mathrm{PSL}_{n_i}(q)$ for $1 \leq i \leq s$. Now for the purposes of induction, let us assume that the natural homomorphism $\xi_i : G \to \Pi_{j \neq i} H_j$ (obtained by suppressing the ith term of the product) maps H onto $\Pi_{j \neq i} H_j$. The kernel of ξ_i is isomorphic to a subgroup G_i of $H_i = \mathrm{SL}_{n_i}(q)$ having $\mathrm{PSL}_{n_i}(q)$ as a composition factor, so that $\mathrm{SL}_{n_i}(q) = Z_i G_i$ where Z_i is the centre of $\mathrm{SL}_{n_i}(q)$. But now $Z_i \cap G_i$ is a central subgroup of $\mathrm{SL}_{n_i}(q)$, and $\mathrm{SL}_{n_i}(q)/(Z_i \cap G_i)$ is isomorphic to the direct product $Z_i/(Z_i \cap G_i) \times G_i/(Z_i \cap G_i)$, which forces $Z_i/(Z_i \cap G_i)$ to be trivial (because $\mathrm{SL}_{n_i}(q)$ is perfect). Hence $Z_i \subseteq Z_i \cap G_i \subseteq G_i$, and therefore $G_i = Z_i G_i = \mathrm{SL}_{n_i}(q) = H_i$. As this argument holds for $1 \leq i \leq s$, it follows by induction on the number of factors that $H = \Pi_{1 \leq i \leq s} H_i = G$.

The centre of G is clearly $\Pi_{1 \leq i \leq s} Z(H_i)$, which is a direct product of cyclic groups of orders m_i for $1 \leq i \leq s$; that is, $Z(G) \cong A$ as required.

Note: There are infinitely many possibilities for the prime p, and hence also for the prime-power $q = p^e$, and also infinitely many possibilities for the ranks $n_1, n_2, \ldots n_s$. Moreover, other constructions are possible. For example, by Dirichlet's Theorem on primes in arithmetic progression (see [1]), there are infinitely many possible choices of distinct primes $p_1, p_2, \ldots, p_s$ such that $p_i \equiv 1$ modulo m_i for $1 \leq i \leq s$, and accordingly we may take G to be the direct product of special linear Hurwitz groups $H_i = \mathrm{SL}_{n_i}(p_i)$ with n_i divisible by m_i, and then factor out however much of the centre of this product is surplus to requirements. In this case each $Z(H_i)$ contains a central subgroup K_i of H_i of index m_i in $Z(H_i)$, and hence $Z(G)$ contains a central subgroup $K = \Pi_{1 \leq i \leq s} K_i$ of G of index $m_1 m_2 \ldots m_s$ in $Z(G)$, such that the Hurwitz group G/K has centre $Z(G/K) \cong Z(G)/K \cong \Pi_{1 \leq i \leq s} C_{m_i} \cong A$.

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