Abstract Polytopes: Regular, Semiregular and Chiral

Barry Monson (UNB)

(from projects with Egon Schulte, Daniel Pellicer and Gordon Williams)

SODO – Queenstown, February, 2012

supported in part by the NSERC of Canada
What are abstract polytopes?

An **abstract $n$-polytope** $Q$ is a poset having some of the key structural properties of the face lattice of a convex $n$-polytope, although $Q$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But you can safely think of a finite 3-polytope as a map on a compact surface.
What are abstract polytopes?

An abstract $n$-polytope $Q$ is a poset having some of the key structural properties of the face lattice of a convex $n$-polytope, although $Q$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But you can safely think of a finite 3-polytope as a map on a compact surface.
What are abstract polytopes?

An **abstract** \( n \)-**polytope** \( Q \) is a poset having some of the key structural properties of the face lattice of a convex \( n \)-polytope, although \( Q \)

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But you can safely think of a finite 3-polytope as a *map on a compact surface*.
What are abstract polytopes?

An **abstract** $n$-**polytope** $Q$ is a poset having some of the key structural properties of the face lattice of a convex $n$-polytope, although $Q$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But you can safely think of a finite 3-polytope as a *map on a compact surface*. 

Do we want details?

Barry Monson (UNB), (from projects with Egon Schulte, , Dar Abstract Polytopes: Regular, Semiregular and Chiral
What are abstract polytopes?

An abstract \( n \)-polytope \( Q \) is a poset having some of the key structural properties of the face lattice of a convex \( n \)-polytope, although \( Q \)

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But you can safely think of a finite 3-polytope as a map on a compact surface.
The symmetry of $Q$

is encoded in the group $\Gamma = \Gamma(Q)$ of all order-preserving bijections (= automorphisms) of $Q$.

Each automorphism is det’d by its action on any one flag $\Phi$; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $Q$ is regular if $\Gamma$ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at ⇒
The symmetry of $Q$

is encoded in the group $\Gamma = \Gamma(Q)$ of all order-preserving bijections (\(=\) automorphisms) of $Q$.

Each automorphism is det’ed by its action on any one flag $\Phi$; for a polyhedron, a flag

$$\Phi = \text{incident \{vertex, edge, facet\} triple}$$

Def. $Q$ is regular if $\Gamma$ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at $\Rightarrow$
The symmetry of $Q$ is encoded in the group $\Gamma = \Gamma(Q)$ of all order-preserving bijections (= automorphisms) of $Q$.

Each automorphism is det’ed by its action on any one flag $\Phi$; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $Q$ is regular if $\Gamma$ is transitive on flags.

Examples:
- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at $\Rightarrow$
The symmetry of $Q$

is encoded in the group $\Gamma = \Gamma(Q)$ of all order-preserving bijections (= automorphisms) of $Q$.

Each automorphism is det’d by its action on any one flag $\Phi$; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

**Def.** $Q$ is *regular* if $\Gamma$ is transitive on flags.

**Examples:**
- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at ⇒
Example 1. The regular dodecahedron $D$ (facets removed)

Here $\Gamma(D)$ is the Coxeter group

$$H_3 = \bullet - \bullet - \bullet$$

of order 120. The flags correspond exactly to the triangles in a barycentric subdivision of the surface of $D$. Here is part of that $\Rightarrow$
By transitivity, pick any base flag $= \Phi$ [white]

Then

0-adjacent flag $=: \Phi^0$ [pink]
1-adjacent flag $=: \Phi^1$ [cyan]
2-adjacent flag $=: \Phi^2$ [orange]

For $i = 0, 1, 2$, there is a unique automorphism $ho_i : \Phi \mapsto \Phi^i$.

Then $\Gamma(D) = \langle \rho_0, \rho_1, \rho_2 \rangle$.

Think reflections!
Local data for both polyhedron $\mathcal{P}$ and its group $\Gamma(\mathcal{P})$ reside in the Schl"afli symbol or type $\{p, q\}$.

Platonic solids: $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron), $\{5, 3\}$ (dodecahedron)

Kepler (ca. 1619) $\{\frac{5}{2}, 5\}$ (small stellated dodecahedron), $\{\frac{5}{2}, 3\}$ (great stellated dodecahedron)

Poinsot (ca. 1809) $\{5, \frac{5}{2}\}$ (great dodecahedron), $\{3, \frac{5}{2}\}$ (great isosahedron)

Want to see some?

Barry Monson (UNB), (from projects with Egon Schulte, , Daniel Pellicer and Gordon Williams), , SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada
## The classical convex regular polytopes, their Schläfli symbols and finite Coxeter groups with string diagrams

<table>
<thead>
<tr>
<th>name</th>
<th>symbol</th>
<th># facets</th>
<th>(Coxeter) group</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 4$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>simplex</td>
<td>${3, 3, 3}$</td>
<td>5</td>
<td>$A_4 \simeq S_5$</td>
<td>$5!$</td>
</tr>
<tr>
<td>cross-polytope</td>
<td>${3, 3, 4}$</td>
<td>16</td>
<td>$B_4$</td>
<td>384</td>
</tr>
<tr>
<td>cube</td>
<td>${4, 3, 3}$</td>
<td>8</td>
<td>$B_4$</td>
<td>384</td>
</tr>
<tr>
<td>24-cell</td>
<td>${3, 4, 3}$</td>
<td>24</td>
<td>$F_4$</td>
<td>1152</td>
</tr>
<tr>
<td>600-cell</td>
<td>${3, 3, 5}$</td>
<td>600</td>
<td>$H_4$</td>
<td>14400</td>
</tr>
<tr>
<td>120-cell</td>
<td>${5, 3, 3}$</td>
<td>120</td>
<td>$H_4$</td>
<td>14400</td>
</tr>
<tr>
<td>$n &gt; 4$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>simplex</td>
<td>${3, 3, \ldots, 3}$</td>
<td>$n + 1$</td>
<td>$A_n \simeq S_{n+1}$</td>
<td>$(n + 1)!$</td>
</tr>
<tr>
<td>cross-polytope</td>
<td>${3, \ldots, 3, 4}$</td>
<td>$2^n$</td>
<td>$B_n$</td>
<td>$2^n \cdot n!$</td>
</tr>
<tr>
<td>cube</td>
<td>${4, 3, \ldots, 3}$</td>
<td>$2n$</td>
<td>$B_n$</td>
<td>$2^n \cdot n!$</td>
</tr>
</tbody>
</table>
Schulte (1982) showed that the abstract regular $n$-polytopes $P$ correspond exactly to the string C-groups of rank $n$ (which we often study in their place).

**The Correspondence Theorem.**

**Part 1.** If $P$ is a regular $n$-polytope, then $\Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group.

**Part 2.** Conversely, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group, then we can reconstruct an $n$-polytope $P(\Gamma)$ (in a natural way as a coset geometry on $\Gamma$).

Furthermore, $\Gamma(P(\Gamma)) \cong \Gamma$ and $P(\Gamma(P)) \cong P$. 
Recap: what is a string C-group?

**Means:** having fixed a base flag $\Phi$ in $\mathcal{P}$, for $0 \leq j \leq n - 1$ there is a unique automorphism $\rho_j \in \Gamma(\mathcal{P})$ mapping $\Phi$ to the $j$-adjacent flag $\Phi^j$. These involutions generate $\Gamma(\mathcal{P})$ and satisfy the relations implicit in some string (Coxeter) diagram, like

\[
p_1 \bullet \quad p_2 \bullet \quad \ldots \quad p_{n-1} \bullet,
\]

and perhaps other relations, so long as this *intersection condition* continues to hold:

\[
\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle
\]

(for all $I, J \subseteq \{0, \ldots, n - 1\}$).

Notice that $\mathcal{P}$ then has *Schläfli type* $\{p_1, \ldots, p_{n-1}\}$. 

Barry Monson (UNB), (from projects with Egon Schulte, Daniel Pellicer and Gordon Williams), SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada.
Example 2. A modern look at a classical object

The **small stellated dodecahedron** \( \{\frac{5}{2}, 5\} \) is a Euclidean realization of the map \( \mathcal{M} = \{5, 5 \mid 3\} \). This quotient of the infinite tessellation \( \{5, 5\} \) of \( \mathbb{H}^2 \) is determined by specifying that ‘1st holes’ be triangular, i.e. \( \Gamma(\mathcal{M}) = \langle \rho_0, \rho_1, \rho_2 \rangle \), where
\[
\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0 \rho_1)^5 = (\rho_1 \rho_2)^5 = (\rho_0 \rho_2)^2 = (\rho_0 \rho_1 \rho_2 \rho_1)^3 = 1.
\]
(The ‘extra’ hole relation is red. You can see the 'hole', which is really an anti-hole.)

In contrast, the great icosahedron \( \{3, \frac{5}{2}\} \) is isomorphic to the convex regular icosahedron \( \{3, 5\} \).
There are already too many regular polytopes, so let’s relax the constraints and get even more!

Definition. A polytope $Q$ is said to be semiregular if its facets are regular and $\Gamma(Q)$ is transitive on vertices.

• the classical Archimedean solids, prisms, antiprisms
• any regular polytope

So this is a broad generalization of ‘regular’. One can generalize even more: uniform polytopes inductively have uniform facets, again with $\Gamma(Q)$ vertex-transitive, taking polygons to be uniform to start.

Instead let’s be at least a little more restrictive ...
There are already too many regular polytopes, so let’s relax the constraints and get even more!

**Definition.** A polytope $Q$ is said to be *semiregular* if its facets are regular and $\Gamma(Q)$ is transitive on vertices.
Abstract Semiregular Polytopes

There are already too many regular polytopes, so let’s relax the constraints and get even more!

**Definition.** A polytope $Q$ is said to be *semiregular* if its facets are regular and $\Gamma(Q)$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms
There are already too many regular polytopes, so let’s relax the constraints and get even more!

Definition. A polytope $Q$ is said to be **semiregular** if its facets are regular and $\Gamma(Q)$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms
- any regular polytope
There are already too many regular polytopes, so let’s relax the constraints and get even more!

Definition. A polytope $Q$ is said to be *semiregular* if its facets are regular and $\Gamma(Q)$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms
- any regular polytope

So this is a broad generalization of ‘regular’. One can generalize even more: *uniform* polytopes inductively have uniform facets, again with $\Gamma(Q)$ vertex-transitive, taking polygons to be uniform to start.
Abstract Semiregular Polytopes

There are already too many regular polytopes, so let’s relax the constraints and get even more!

Definition. A polytope $Q$ is said to be \textit{semiregular} if its facets are regular and $\Gamma(Q)$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms
- any regular polytope

So this is a broad generalization of ‘regular’. One can generalize even more: \textit{uniform} polytopes inductively have uniform facets, again with $\Gamma(Q)$ vertex-transitive, taking polygons to be uniform to start.

Instead let’s be at least a little more restrictive ...
Alternating Semiregular Polytopes

**Definition** An abstract semiregular $n$-polytope $S$ is **alternating** if it has two (necessarily compatible) types of regular facets, say $P$ and $Q$, appearing in alternating fashion around each $(n - 3)$-face.

Example 4. The cuboctahedron is an alternating semiregular 3-polytope. Here $k = 2$ each of triangles $P = \{3\}$ and squares $Q = \{4\}$ alternate around each 0-face = vertex. Each rectangular vertex-figure is a ‘geometrically alternating’ polygon, but is abstractly regular of course. A truncated tetrahedron, for example, is semiregular but not alternating.
Definition An abstract semiregular \( n \)-polytope \( S \) is *alternating* if it has two (necessarily compatible) types of regular facets, say \( P \) and \( Q \), appearing in alternating fashion around each \((n - 3)\)-face.

\[ k = 2 \]
\[ \{3\} \text{ and } \{4\} \]

The **cuboctahedron** is an alternating semiregular 3-polytope. Here \( k = 2 \) each of triangles \( P = \{3\} \) and squares \( Q = \{4\} \) alternate around each 0-face = vertex.

Each rectangular vertex-figure is a ‘geometrically alternating’ polygon, but is abstractly regular of course. A truncated tetrahedron, for example, is semiregular but not alternating.
Definition An abstract semiregular $n$-polytope $S$ is alternating if it has two (necessarily compatible) types of regular facets, say $\mathcal{P}$ and $\mathcal{Q}$, appearing in alternating fashion around each $(n - 3)$-face.

Example 4. The cuboctahedron is an alternating semiregular 3-polytope. Here $k = 2$ each of triangles $\mathcal{P} = \{3\}$ and squares $\mathcal{Q} = \{4\}$ alternate around each 0-face = vertex. Each rectangular vertex-figure is a ‘geometrically alternating’ polygon, but is abstractly regular of course.

A truncated tetrahedron, for example, is semiregular but not alternating.
Definition An abstract semiregular \( n \)-polytope \( S \) is alternating if it has two (necessarily compatible) types of regular facets, say \( P \) and \( Q \), appearing in alternating fashion around each \((n - 3)\)-face.

Example 4. The cuboctahedron is an alternating semiregular 3-polytope. Here \( k = 2 \) each of triangles \( P = \{3\} \) and squares \( Q = \{4\} \) alternate around each 0-face = vertex. Each rectangular vertex-figure is a ‘geometrically alternating’ polygon, but is abstractly regular of course. A truncated tetrahedron, for example, is semiregular but not alternating.
** Thing 1 and Thing 2 – build an alternating polytope **

**You are given** – unlimited copies of regular polyhedra $P$ and $Q$ having matching facets

**Your task** – Start with a single $P$. Attach a copy of $Q$ to each $P$-facet, then a copy of $P$ to each remaining ‘exposed’ facet of a $Q$, and so on in alternating fashion with the

**Edge Rule $k$** – close up around each edge after $k$ $P$’s and $k$ $Q$’s.

- Can this be done?
- Is the resulting ‘complex’ $S$ a 4–polytope?
- If so, what is the symmetry group $\text{Aut}(S)$?
You are given – unlimited copies of regular polyhedra \( P \) and \( Q \) having matching facets.

Your task – Start with a single \( P \). Attach a copy of \( Q \) to each \( P \)-facet, then a copy of \( P \) to each remaining ‘exposed’ facet of a \( Q \), and so on in alternating fashion with the

Edge Rule \( k \) – close up around each edge after \( k \) \( P \)'s and \( k \) \( Q \)'s.

- Can this be done?
- Is the resulting ‘complex’ \( S \) a 4–polytope?
- If so, what is the symmetry group \( \text{Aut}(S) \)?
Thing 1 and Thing 2 – build an alternating polytope

You are given – unlimited copies of regular polyhedra $\mathcal{P}$ and $\mathcal{Q}$ having matching facets

Your task – Start with a single $\mathcal{P}$. Attach a copy of $\mathcal{Q}$ to each $\mathcal{P}$-facet, then a copy of $\mathcal{P}$ to each remaining ‘exposed’ facet of a $\mathcal{Q}$, and so on in alternating fashion with the

Edge Rule $k$ – close up around each edge after $k$ $\mathcal{P}$’s and $k$ $\mathcal{Q}$’s.

- Can this be done?
- Is the resulting ‘complex’ $\mathcal{S}$ a 4–polytope?
- If so, what is the symmetry group $\text{Aut}(\mathcal{S})$?

Barry Monson (UNB), (from projects with Egon Schulte, Daniel Pellicer and Gordon Williams), SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada

Abstract Polytopes: Regular, Semiregular and Chiral
You are given – unlimited copies of regular polyhedra $P$ and $Q$ having matching facets

Your task – Start with a single $P$. Attach a copy of $Q$ to each $P$-facet, then a copy of $P$ to each remaining ‘exposed’ facet of a $Q$, and so on in alternating fashion with the

Edge Rule $k$ – close up around each edge after $k$ $P$’s and $k$ $Q$’s.

- Can this be done?
- Is the resulting ‘complex’ $S$ a 4–polytope?
- If so, what is the symmetry group $\text{Aut}(S)$?
Example 5 (rich). Take the regular octahedron $P = \{3, 4\}$ and the regular tetrahedron $Q = \{3, 3\}$.

Assemble $k = 2$ of each around each edge. We get a familiar tessellation $S$ of $\mathbb{E}^3$. This abstract semiregular 4-polytope therefore has a Euclidean realization.
Example 5 (rich). Take the regular octahedron $\mathcal{P} = \{3, 4\}$ and the regular tetrahedron $\mathcal{Q} = \{3, 3\}$.

$\Leftarrow$ Assemble $k = 2$ of each around each edge. We get a familiar tessellation $S$ of $\mathbb{E}^3$. This abstract semiregular 4-polytope therefore has a Euclidean realization.
Example 5 (rich). Take the regular octahedron $P = \{3, 4\}$ and the regular tetrahedron $Q = \{3, 3\}$

$\iff$ Assemble $k = 2$ of each around each edge. We get a familiar tessellation $S$ of $\mathbb{E}^3$.

This abstract semiregular 4-polytope therefore has a Euclidean realization.
Building $S$ from Wythoff’s Construction

The (infinite) Coxeter group $\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ of type $\tilde{B}_3$ has diagram

![Diagram of Coxeter group $\Gamma$]

and acts discretely on Euclidean space $\mathbb{E}^3$. We get $S$ from Wythoff’s construction, as encoded in the modified diagram

![Modified diagram]

Begin with vertex set $= \Gamma$-orbit of the unique point fixed by $\rho_1, \rho_2, \rho_3$; etc.
Keep the regular tetrahedron $Q = \{3, 3\}$ but switch to the regular hemioctahedron $P = \{3, 4\}_3$:

This projective map $P$ has 3 vertices, 6 edges and 4 triangular facets.

We still try to put $k = 2$ of each around each edge. But now our construction is best done using an
Keep the regular tetrahedron $Q = \{3, 3\}$ but switch to the regular hemioctahedron $P = \{3, 4\}_3$:

This projective map $P$ has 3 vertices, 6 edges and 4 triangular facets.

We still try to put $k = 2$ of each around each edge. But now our construction is best done using an
Abstract Version of Wythoff’s Construction (with E. Schulte) in which the vertices, edges, 2-faces (= polygons) and 3-faces (=facets) of the new $S$ are (identified with) right cosets of certain standard subgroups of the new group $\Gamma$ generated by $\rho_0, \rho_1, \rho_2, \rho_3$ and having defining relations

\[ \begin{align*}
\rho_0^2 &= \rho_1^2 = \rho_2^2 = \rho_3^2 = 1 \\
(\rho_0\rho_1)^3 &= (\rho_1\rho_2)^3 = (\rho_1\rho_3)^4 = 1 \\
(\rho_0\rho_2)^2 &= (\rho_0\rho_3)^2 = (\rho_2\rho_3)^2 = 1
\end{align*} \]

and the new projectifying relation

\[ (\rho_0\rho_1\rho_3)^3 = 1. \]

**Remark.** It’s not clear that $S$ ‘survives intact’, since the new relation could destroy polytopality.
Abstract Version of Wythoff’s Construction (with E. Schulte) in which

the vertices, edges, 2-faces (= polygons) and 3-faces (=facets) of the new

$S$ are (identified with) right cosets of certain standard subgroups of the

new group $\Gamma$ generated by $\rho_0, \rho_1, \rho_2, \rho_3$ and having defining relations $\downarrow$

$$
\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = 1
$$

$$(\rho_0 \rho_1)^3 = (\rho_1 \rho_2)^3 = (\rho_1 \rho_3)^4 = 1
$$

$$(\rho_0 \rho_2)^2 = (\rho_0 \rho_3)^2 = (\rho_2 \rho_3)^2 = 1
$$

and the new projectifying relation

$$(\rho_0 \rho_1 \rho_3)^3 = 1.$$

**Remark.** It’s not clear that $S$ ‘survives intact’, since the new relation could destroy polytopality.
But all is well - and we get a finite 4-polytope $S$! (Example 6)

It's true, but not obvious, that $\Gamma$ is now finite. In fact,

- the $\rho_j$'s survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(Q) \simeq S_4$ [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(P)$ (also $\simeq S_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $S$ can't be regular since $P \neq Q$. But it is alternating semiregular – all facets are regular and $\Gamma(S)$ is vertex-transitive.
But all is well - and we get a finite 4-polytope \( S \)!
(Example 6)

It’s true, but not obvious, that \( \Gamma \) is now finite. In fact,

- the \( \rho_j \)'s survive as involutions
- still \( \langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(Q) \simeq S_4 \) [group for tetrahedra]
- still \( \langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(P) \simeq S_4 \) [group for hemioctahedra]
- \( |\Gamma| = 192 \), so there are \( 8 = 192/24 \) facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- \( S \) can’t be regular since \( P \neq Q \). But it is alternating semiregular – all facets are regular and \( \Gamma(S) \) is vertex-transitive.
But all is well - and we get a finite 4-polytope $S$!
(Example 6)

It's true, but not obvious, that $\Gamma$ is now finite. In fact,

- the $\rho_j$'s survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(Q) \simeq S_4$ [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(P)$ (also $\simeq S_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $S$ can't be regular since $P \not\simeq Q$. But it is alternating semiregular – all facets are regular and $\Gamma(S)$ is vertex-transitive.
But all is well - and we get a finite 4-polytope $S$!
(Example 6)

It’s true, but not obvious, that $\Gamma$ is now finite. In fact,

- the $\rho_j$’s survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(Q) (\simeq S_4)$ [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(P)$ (also $\simeq S_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $S$ can’t be regular since $P \not\simeq Q$. But it is alternating semiregular – all facets are regular and $\Gamma(S)$ is vertex-transitive.
But all is well - and we get a finite 4-polytope $S$!
(Example 6)

It’s true, but not obvious, that $\Gamma$ is now finite. In fact,

- the $\rho_j$’s survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \cong \Gamma(Q) \cong S_4$ [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \cong \Gamma(P)$ (also $\cong S_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $S$ can’t be regular since $P \ncong Q$. But it is alternating semiregular – all facets are regular and $\Gamma(S)$ is vertex-transitive.
More on this new $S$

- The polytope $S$ is **universal** for assembling tetrahedra and hemioctahedra face-to-face, with two each alternately surrounding any edge.
- $S$ has a unique, minimal regular cover of Schlӓfli type $\{3, 12, 4\}$ and group order $2^{13} \cdot 3^2 = 73728$.
- But further collapse is possible. Each vertex-figure is a centrally-symmetric cuboctahedron. So let’s collapse these to semiregular hemicuboctahedra by adjoining the relation

$$ (\rho_1 \rho_2 \rho_3)^3 = 1 $$

Polytopality survives this further collapse and we get the
More on this new $S$

- The polytope $S$ is **universal** for assembling tetrahedra and hemioctahedra face-to-face, with two each alternately surrounding any edge.
- $S$ has a unique, minimal regular cover of Schl"afli type $\{3,12,4\}$ and group order $2^{13} \cdot 3^2 = 73728$.
- But further collapse is possible. Each vertex-figure is a centrally-symmetric cuboctahedron. So let's collapse these to semiregular hemicuboctahedra by adjoining the relation

$$ (\rho_1\rho_2\rho_3)^3 = 1 $$

Polytopality survives this further collapse and we get the
More on this new \( S \)

- The polytope \( S \) is **universal** for assembling tetrahedra and hemioctahedra face-to-face, with two each alternately surrounding any edge.
- \( S \) has a unique, minimal regular cover of Schl"afli type \( \{3,12,4\} \) and group order \( 2^{13} \cdot 3^2 = 73728 \).
- But further collapse is possible. Each vertex-figure is a centrally-symmetric cuboctahedron. So let’s collapse these to semiregular **hemicuboctahedra** by adjoining the relation

\[
(\rho_1\rho_2\rho_3)^3 = 1
\]

Polytopality survives this further collapse and we get the
Example 7. The Tomotope $\mathcal{T}$ (w. D. Pellicer, G. Williams)

To visualize $\mathcal{T}$ slice out a $2 \times 2 \times 2$ cube containing eight tetrahedra, a core octahedron and three other octahedra, each split into four identical but non-regular tetrahedra. The latter pieces fit into the twelve ‘dimples’ on the surface of the *stella octangula*:
More on visualizing $\mathcal{T}$

Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and ‘run around’ the belts of those colours.

There are 12 edges, on which $\Gamma(\mathcal{T})$ acts faithfully, and 16 triangular 2-faces.

Aside: $\Gamma(\mathcal{T}) \simeq \mathbb{Z}_2^4 \rtimes S_3$ has order 96 and can be obtained from the crystallographic group $\bar{B}_3$ by reduction mod 2.
Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

You see the 4 vertices, $4 = \frac{8}{2}$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and ‘run around’ the belts of those colours.

There are 12 edges, on which $\Gamma(T)$ acts faithfully, and 16 triangular 2-faces.

Aside: $\Gamma(T) \cong \mathbb{Z}_2^4 \rtimes S_3$ has order 96 and can be obtained from the crystallographic group $\tilde{B}_3$ by reduction mod 2.
Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and ‘run around’ the belts of those colours.

There are 12 edges, on which $\Gamma(\mathcal{T})$ acts faithfully, and 16 triangular 2-faces.

Aside: $\Gamma(\mathcal{T}) \simeq \mathbb{Z}_2^4 \rtimes S_3$ has order 96 an can be obtained from the crystallographic group $\tilde{B}_3$ by reduction mod 2.
Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and ‘run around’ the belts of those colours.

There are 12 edges, on which $\Gamma(T)$ acts faithfully, and 16 triangular 2-faces.

Aside: $\Gamma(T) \cong \mathbb{Z}_2^4 \rtimes S_3$ has order 96 and can be obtained from the crystallographic group $\tilde{B}_3$ by reduction mod 2.
The tomotope has $\mathcal{T}$ has a strange property

does not hold for maps (rank 3 polytopes). It has infinitely many mutually non–isomorphic minimal regular covers.
There is such a finite, minimal regular cover $\mathcal{P}_p$ for each prime $p$. Each of these 4-polytopes has type $\{3, 12, 4\}$.

Intuitively: infinitely many distinct sets of minimal assembly instructions for $\mathcal{T}$ using standardized regular parts.

See The Tomotope, B. Monson, D. Dellicer, G. Williams, to appear in Ars Mathematica Contemporanea.
Egon Schulte and I have proved

**Theorem 1.** A combinatorial Wythoff’s construction works for any group generated by involutions satisfying at least the relations suggested by

\[
\begin{align*}
\rho_0 & \quad \rho_1 & \quad \ldots & \quad p_{n-2} & \quad p_{n-1} & \quad k \\
& & \rho_{n-1} & \quad q_{n-1} & \quad \beta_{n-1}
\end{align*}
\]

and also satisfying an *intersection condition* (akin to that for string C-groups).

Note: \(k\) or various \(p_j\)’s could equal 2: no branch then.

**Definition** Call \(\Gamma\) a *tail-triangle group*. 

**Abstract Polytopes: Regular, Semiregular and Chiral**
Egon Schulte and I have proved

**Theorem 1.** A combinatorial Wythoff’s construction works for any group generated by involutions satisfying at least the relations suggested by

\[
\Gamma : \begin{array}{c}
\rho_0 \quad \rho_1 \quad \cdots \quad \rho_{n-2} \quad \rho_{n-1} \\
\end{array}
\]

and also satisfying an *intersection condition* (akin to that for string C-groups).

Note: $k$ or various $p_j$’s could equal 2: no branch then.

**Definition** Call $\Gamma$ a *tail-triangle group*. 
Egon Schulte and I have proved

**Theorem 1.** A combinatorial Wythoff’s construction works for any group generated by involutions satisfying at least the relations suggested by

\[
\Gamma : \rho_0 \rho_1 \cdots \rho_n \rho_{n-2} \rho_{n-1} \rho_0 \rho_{n-2} p_{n-1} p_n q_{n-1} \beta_{n-1} k
\]

and also satisfying an *intersection condition* (akin to that for string C-groups).

Note: *k* or various *p_j*’s could equal 2: no branch then.

**Definition** Call *Γ* a *tail-triangle group*. 
Theorem 2. For any two regular $n$-polytopes $\mathcal{P}$ and $\mathcal{Q}$ with matching (i.e. isomorphic) facets, there exists a semiregular $(n + 1)$-polytope $\mathcal{S}$ with infinitely many facets of type $\mathcal{P}$ and $\mathcal{Q}$, occurring alternately around each face of corank 2. (In this case, $\Gamma(\mathcal{S})$ contains a certain free product with amalgamation.)

(See *Semiregular polytopes and Amalgamated C-groups*, Adv. in Math., to appear.)
Some almost final words on monodromy ...

The structure of regular covers $\mathcal{R}$ of a general polytope $Q$ has a lot to do with the \textit{monodromy group} $\text{Mon}(Q)$.

$\text{Mon}(Q)$ is a natural permutation group on the flag set of $Q$; this action commutes with the action of automorphisms on flags. $\text{Mon}(Q)$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

The structure of regular covers $\mathcal{R}$ of a general polytope $Q$ has a lot to do with the *monodromy group* $\text{Mon}(Q)$.

$\text{Mon}(Q)$ is a natural permutation group on the flag set of $Q$; this action commutes with the action of automorphisms on flags.

$\text{Mon}(Q)$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

The structure of regular covers $\mathcal{R}$ of a general polytope $Q$ has a lot to do with the *monodromy group* $\text{Mon}(Q)$.

$\text{Mon}(Q)$ is a natural permutation group on the flag set of $Q$; this action commutes with the action of automorphisms on flags. $\text{Mon}(Q)$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

The structure of regular covers $R$ of a general polytope $Q$ has a lot to do with the *monodromy group* $\text{Mon}(Q)$.

$\text{Mon}(Q)$ is a natural permutation group on the flag set of $Q$; this action commutes with the action of automorphisms on flags. $\text{Mon}(Q)$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

**The fall out:** covering and likely other combinatorial questions are complicated.

Some final words on chirality ... Example 8

In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any chiral polytopes $\mathcal{P}$ of higher rank (this meant $> 4$ for $\mathcal{P}$ finite).

We recently noticed that $\text{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$$\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2, 4), \sigma_3 = (1, 5, 4, 3), \sigma_4 = (1, 2, 3)(4, 6, 5).$$

Then $\Gamma$ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3, 4, 4, 3\}$

But $\text{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...
In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any chiral polytopes $\mathcal{P}$ of higher rank (this meant $> 4$ for $\mathcal{P}$ finite).

We recently noticed that $\text{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$$\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2, 4), \sigma_3 = (1, 5, 4, 3), \sigma_4 = (1, 2, 3)(4, 6, 5).$$

Then $\Gamma$ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3, 4, 4, 3\}$

But $\text{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...
In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any \textit{chiral} polytopes $\mathcal{P}$ of higher rank (this meant $\geq 4$ for $\mathcal{P}$ finite).

We recently noticed that $\text{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2, 4), \sigma_3 = (1, 5, 4, 3), \sigma_4 = (1, 2, 3)(4, 6, 5)$. 

Then $\Gamma$ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3, 4, 4, 3\}$.

But $\text{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don’t understand where this leads ...
In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any chiral polytopes $\mathcal{P}$ of higher rank (this meant $> 4$ for $\mathcal{P}$ finite).

We recently noticed that $\text{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2, 4), \sigma_3 = (1, 5, 4, 3), \sigma_4 = (1, 2, 3)(4, 6, 5)$.

Then $\Gamma$ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3, 4, 4, 4, 3\}$

But $\text{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don’t understand where this leads ...
In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any chiral polytopes $\mathcal{P}$ of higher rank (this meant $> 4$ for $\mathcal{P}$ finite).

We recently noticed that $\text{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$$\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2, 4), \sigma_3 = (1, 5, 4, 3), \sigma_4 = (1, 2, 3)(4, 6, 5).$$

Then $\Gamma$ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3, 4, 4, 3\}$.

But $\text{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...
Some Conjectures and a Question

- **Conjecture.** The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)

- **Conjecture.** The monodromy group for any convex polytope with combinatorially regular facets of any type is a string C-group. (This seems likely.)

- **Conjecture.** There is a finite abstract (convex?, chiral?) polytope \( Q \) whose minimal regular covers are all infinite. (Certainly \( \text{Mon}(Q) \) could not then be a string C-group.)

- **Question** Can one always assemble thing 1 and thing 2 for any finite integer \( k \geq 2 \)?
Some Conjectures and a Question

- **Conjecture.** The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)

- **Conjecture.** The monodromy group for any convex polytope with combinatorially regular facets of any type is a string C-group. (This seems likely.)
Conjecture. The monodromy group for any convex polytope is a string C-group.

Is this known? (I have some doubts it is true ...)

Conjecture. The monodromy group for any convex polytope with combinatorially regular facets of any type is a string C-group. (This seems likely.)

Conjecture. There is a finite abstract (convex?, chiral?) polytope $Q$ whose minimal regular covers are all infinite. (Certainly $\text{Mon}(Q)$ could not then be a string C-group.)
Some Conjectures and a Question

• **Conjecture.** The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)

• **Conjecture.** The monodromy group for any convex polytope with combinatorially regular facets of any type is a string C-group. (This seems likely.)

• **Conjecture.** There is a finite abstract (convex?, chiral?) polytope $Q$ whose minimal regular covers are all infinite. (Certainly $\text{Mon}(Q)$ could not then be a string C-group.)

• **Question** Can one always assemble thing 1 and thing 2 for any finite integer $k \geq 2$?
Many thanks to our organizers!


Barry Monson (UNB), (from projects with Egon Schulte, Daniel Pellicer and Gordon Williams), SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada

Abstract Polytopes: Regular, Semiregular and Chiral
Example 3. Two regular star-polyhedra (courtesy Wikipedia)

Small stellated dodec.
\[ \left\{ \frac{5}{2}, 5 \right\} \]

Great icos.
\[ \left\{ 3, \frac{5}{2} \right\} \]
\( \simeq \) convex reg. icos.
The semiregular tessellation $\mathcal{S}$ of $\mathbb{R}^3$
The group $\Gamma(\mathcal{T})$ acts faithfully on edges of $\mathcal{T}$, so we have this permutation representation:

\[
\rho_0 = (5, 10)(6, 9)(7, 12)(8, 11)
\]

\[
\rho_1 = (1, 6)(2, 5)(3, 8)(4, 7)
\]

\[
\rho_2 = (5, 9)(6, 10)(7, 11)(8, 12)
\]

\[
\rho_3 = (5, 8)(6, 7)(9, 12)(10, 11)
\]
The $n$-polytope $Q$ is a poset whose elements (\(=\) faces) satisfy:

- $Q$ has a unique minimal face $F_{-1}$ and maximal face $F_n$.
- Every maximal chain or flag has $n+2$ faces so $Q$ has a strictly monotone rank function onto \{-1, 0, \ldots, n\}.
- $Q$ is strongly flag connected via adjacency in the flag graph; this rules out, for example, the disjoint union of two polyhedra.
- $Q$ satisfies the 'diamond' condition: whenever $F < G$ with rank($F$) = $j - 1$ and rank($G$) = $j + 1$ there exist exactly two $j$-faces $H$ with $F < H < G$. 

\[ \begin{array}{c}
F & H & H' & G \\
\end{array} \]

Barry Monson (UNB), (from projects with Egon Schulte, Daniel Pellicer and Gordon Williams), SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada
The \( n \)-polytope \( Q \)

is a poset whose elements (\( = \) faces) satisfy:

- \( Q \) has a unique minimal face \( F_{-1} \) and maximal face \( F_n \)
The $n$-polytope $Q$

is a poset whose elements (\(=\) faces) satisfy:

- $Q$ has a unique minimal face $F_{-1}$ and maximal face $F_n$
- Every maximal chain or flag has $n + 2$ faces
The $n$-polytope $Q$

is a poset whose elements (= faces) satisfy:

- $Q$ has a unique minimal face $F_{-1}$ and maximal face $F_n$
- Every maximal chain or flag has $n + 2$ faces

so $Q$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
The $n$-polytope $\mathcal{Q}$

is a poset whose elements (\textit{faces}) satisfy:

- $\mathcal{Q}$ has a unique minimal face $F_{-1}$ and maximal face $F_n$
- Every maximal chain or \textit{flag} has $n + 2$ faces
  so $\mathcal{Q}$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
- $\mathcal{Q}$ is strongly flag connected
The $n$-polytope $Q$ is a poset whose elements (= faces) satisfy:

- $Q$ has a unique minimal face $F_{-1}$ and maximal face $F_n$
- Every maximal chain or flag has $n + 2$ faces

so $Q$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
- $Q$ is strongly flag connected

- $Q$ satisfies the ‘diamond’ condition:
The \( n \)-polytope \( Q \)

is a poset whose elements (= faces) satisfy:

- \( Q \) has a unique minimal face \( F_{-1} \) and maximal face \( F_n \)
- Every maximal chain or flag has \( n + 2 \) faces
  so \( Q \) has a strictly monotone rank function onto \( \{-1, 0, \ldots, n\} \)
- \( Q \) is strongly flag connected

- \( Q \) satisfies the ‘diamond’ condition:
  whenever \( F < G \) with \( \text{rank}(F) = j - 1 \) and \( \text{rank}(G) = j + 1 \) there
  exist exactly two \( j \)-faces \( H \) with \( F < H < G \)

\[ \begin{array}{c}
\text{j+1} \\
\text{j} \\
\text{j-1} \\
\end{array} \quad \begin{array}{c}
F \\
H \\
G \\
H' \\
\end{array} \]
The $n$-polytope $Q$

is a poset whose elements (=
\textit{faces}) satisfy:

- $Q$ has a unique minimal face $F_{-1}$ and maximal face $F_n$
- Every maximal chain or \textit{flag} has $n + 2$ faces

so $Q$ has a strictly monotone rank function onto \{-1, 0, \ldots, n\}
- $Q$ is strongly flag connected

via adjacency in the flag graph; this rules out, for example, the
disjoint union of two polyhedra
- $Q$ satisfies the ‘diamond’ condition:

whenever $F < G$ with rank($F$) = $j - 1$ and rank($G$) = $j + 1$ there
exist exactly two $j$-faces $H$ with $F < H < G$

\begin{center}
\begin{tikzpicture}
\node[style=face] (F) at (0,0) {$F$};
\node[style=face] (H) at (1,1) {$H$};
\node[style=face] (H') at (1,-1) {$H'$};
\node[style=face] (G) at (2,0) {$G$};
\node[style=face] (j-1) at (-1,0) {$F$};
\node[style=face] (j) at (0,1) {$H$};
\node[style=face] (j+1) at (1,2) {$G$};
\node[style=face] (j) at (0,-1) {$H'$};
\end{tikzpicture}
\end{center}
So when is $\text{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\text{Mon}(Q)$ is a string C-group if

- $Q$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $Q$ is regular of any rank (in which case $\Gamma(Q) \cong \text{Mon}(Q)$); or
- all facets of $Q$ are regular quotients of one particular regular facet (or dually); or
- $Q$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).

Thus: if $Q$ is any simplicial (or simple) convex polytope, then $\text{Mon}(Q)$ is a string C-group.

Example. The cyclic convex 4-polytope $Q$ on 6 vertices thereby has a regular cover of Schl"afli type $\{3, 3, 12\}$; $\text{Mon}(Q)$ is a string C-group of order $2^6 \cdot 3^7$. 

Barry Monson (UNB), (from projects with Egon Schulte, Daniel Pellicer and Gordon Williams), SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada
So when is $\text{Mon}(Q)$ a string $C$-group?

Some results below may be well-known, others new:

$\text{Mon}(Q)$ is a string $C$-group if

- $Q$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $Q$ is regular of any rank (in which case $\Gamma(Q) \cong \text{Mon}(Q)$); or
- all facets of $Q$ are regular quotients of one particular regular facet (or dually); or
- $Q$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).

Thus: if $Q$ is any simplicial (or simple) convex polytope, then $\text{Mon}(Q)$ is a string $C$-group.

Example. The cyclic convex 4-polytope $Q$ on 6 vertices thereby has a regular cover of Schlaffli type $\{3,3,12\}$; $\text{Mon}(Q)$ is a string $C$-group of order $2^6 \cdot 3^7$.  

Barry Monson (UNB), (from projects with Egon Schulte, , Daniel Pellicer and Gordon Williams), , SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada
So when is $\text{Mon}(Q)$ a string $C$-group?

Some results below may be well-known, others new:

$\text{Mon}(Q)$ is a string $C$-group if

- $Q$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $Q$ is regular of any rank (in which case $\Gamma(Q) \simeq \text{Mon}(Q)$); or
- all facets of $Q$ are regular quotients of one particular regular facet (or dually); or
- $Q$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).

Thus: if $Q$ is any simplicial (or simple) convex polytope, then $\text{Mon}(Q)$ is a string $C$-group.

Example. The cyclic convex 4-polytope $Q$ on 6 vertices thereby has a regular cover of Schlafli type $\{3, 3, 12\}$; $\text{Mon}(Q)$ is a string $C$-group of order $2^6 \cdot 3^7$. 

Barry Monson (UNB), (from projects with Egon Schulte, , Dar Abstract Polytopes: Regular, Semiregular and Chiral
So when is $\text{Mon}(Q)$ a string $C$-group?

Some results below may be well-known, others new:

$\text{Mon}(Q)$ is a string $C$-group if

- $Q$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $Q$ is regular of any rank (in which case $\Gamma(Q) \simeq \text{Mon}(Q)$); or
- all facets of $Q$ are regular quotients of one particular regular facet (or dually); or
- $Q$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).

Thus: if $Q$ is any simplicial (or simple) convex polytope, then $\text{Mon}(Q)$ is a string $C$-group.

Example. The cyclic convex 4-polytope $Q$ on 6 vertices thereby has a regular cover of Schl"afli type $\{3, 3, 12\}$; $\text{Mon}(Q)$ is a string $C$-group of order $2^6 \cdot 3^7$. 

Barry Monson (UNB), (from projects with Egon Schulte, Daniel Pellicer and Gordon Williams), SODO – Queenstown, February, 2012, supported in part by the NSERC of Canada.
So when is $\text{Mon}(Q)$ a string $C$-group?

Some results below may be well-known, others new:

$\text{Mon}(Q)$ is a string $C$-group if

- $Q$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $Q$ is regular of any rank (in which case $\Gamma(Q) \simeq \text{Mon}(Q)$); or
- all facets of $Q$ are regular quotients of one particular regular facet (or dually); or
- $Q$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).

**Thus**: if $Q$ is any simplicial (or simple) convex polytope, then $\text{Mon}(Q)$ is a string $C$-group.

**Example.** The cyclic convex 4-polytope $Q$ on 6 vertices thereby has a regular cover of Schl"afli type $\{3, 3, 12\}$; $\text{Mon}(Q)$ is a string $C$-group of order $2^6 \cdot 3^7$. 
So when is $\text{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\text{Mon}(Q)$ is a string C-group if

- $Q$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $Q$ is regular of any rank (in which case $\Gamma(Q) \simeq \text{Mon}(Q)$); or
- all facets of $Q$ are regular quotients of one particular regular facet (or dually); or
- $Q$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).

**Thus**: if $Q$ is any simplicial (or simple) convex polytope, then $\text{Mon}(Q)$ is a string C-group.

**Example.** The cyclic convex 4-polytope $Q$ on 6 vertices thereby has a regular cover of Schlafli type $\{3, 3, 12\}$; $\text{Mon}(Q)$ is a string C-group of order $2^6 \cdot 3^7$. 

Barry Monson (UNB), (from projects with Egon Schulte, , Dar) Abstract Polytopes: Regular, Semiregular and Chiral