One-regular graphs

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In this report I first give a brief survey of one-regular graphs. Then I will talk about a conjecture, that is the existence of one-regular 3-valent graphs of order $4m$ for an odd integer $m$, which was answered by Conder and the author.
Regular permutation groups

- Let $G$ be a permutation group on $\Omega$, that is, $G \leq S_\Omega$.

- $G$ is transitive on $\Omega$: for any two points in $\Omega$ there is a permutation in $G$ mapping one to the other.

- $G$ is regular on $\Omega$: for any two points in $\Omega$ there is one and only one permutation in $G$ mapping one to the other, that is, only the identity element in the transitive subgroup fixes a point.

- A regular permutation group is ‘the smallest possible transitive group’.
Definitions and basic facts

Notation for graphs

- **X**: a simple graph (no loops or multiple edges).
- **V(X), E(X)**: the vertex set and the edge set.
- The automorphism group **Aut(X)** of a graph **X**: the group of all permutations on **V(X)** preserving the adjacency of **X**, that is, mapping an edge to an edge.
- **X** is vertex-transitive or edge-transitive: **Aut(X)** is transitive on **V(X)** or **E(X)**, respectively.
Definitions and basic facts

**Notation for graphs**

- **s-arc**: an \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_{s-1}, v_s)\) of vertices s.t. \(\{v_i, v_{i+1}\} \in E(X), v_{i-1} \neq v_{i+1}\).

- **s-arc-transitive**: \(\text{Aut}(X)\) acts transitively on the set of \(s\)-arcs in \(X\).

- **0-arc-transitive**: vertex-transitive.

- **1-arc-transitive**: arc-transitive or symmetric

- **s-arc-regular graph**: \(\text{Aut}(X)\) acts regularly on the set of \(s\)-arc of \(X\).

- **one-regular graph**: 1-arc-regular graph.
**Cayley graph**

$G$: a finite group, $S \subset G$, $1 \notin S$, $S = S^{-1} = \{ s^{-1} \mid s \in S \}$.

- **Cayley graph** $\text{Cay}(G, S)$: vertex set $V = G$, edge set $E = \{(g, sg) \mid g \in G, s \in S\}$

- $\text{Cay}(G, S)$ is connected $\iff G = \langle S \rangle$.

- **Right regular representation** $R(G)$ of $G$: the permutation group $\{ R(g) \mid g \in G \}$ on $G$, where $R(g) : x \mapsto xg$, $\forall x \in G$ is a permutation on $G$. Clearly, $R(G) \leq \text{Aut}(\text{Cay}(G, S))$, acting regularly on $V(X)$.

- **Characterization**: A graph $X$ is a Cayley graph on $G$ $\iff \text{Aut}(X)$ has a regular subgroup isomorphic to $G$. 

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Basic facts about one-regular graphs

- If an \( s \)-arc-regular graph is not connected, it must be a union of one vertex and a connected \( s \)-arc-regular graph.

- A 2-valent (regular) graph is one-regular if and only if it is a cycle \( C_n \) for some positive integer \( n \geq 3 \). On the other hand, \( C_n \) is \( s \)-regular for any \( s \geq 2 \).

- When one consider one-regular graph, it is supposed that the graph is connected and has valency greater than 2.

- Some examples of cubic \( s \)-regular graphs: the 2-regular complete graph \( K_4 \), the 2-regular three dimensional hypercube \( Q_3 \), the 3-regular Petersen graph \( O_3 \).
One-regular graphs with valency greater than 4

- One may easily obtain a classification of one-regular graphs of prime order by Burnside Theorem. (also see [1, 2]).


- Kwak et al [20] constructed an infinite family of one-regular Cayley graphs on dihedral groups of any even valency.
One-regular graphs based on valencies

One-regular graphs with valency greater than 4

- Kwak et al 2008 [18] constructed an infinite family of one-regular Cayley graphs on dihedral groups of any prescribed valency. In particular, a classification of one-regular Cayley graphs on a dihedral group of valency 5 can be reduced.

- Feng and Li [10] classified one-regular Cayley graphs of prime valency on dihedral groups, and as a result, one-regular graphs of square free order of prime valency were classified.

- Infinitely many one-regular Cayley graphs of valency 6 on dihedral groups were constructed by Hwang, Kwak and Oh [19, 27].
One-regular graphs based on valencies

One-regular graphs with valency 4

- Hwang, Kwak and Oh [19, 27] constructed infinitely many tetravalent one-regular Cayley graphs on dihedral groups.

- Wang, Xu and Zhou [29, 30] classified one-regular Cayley graphs of valency 4 on dihedral groups.

- Note that Du, Malnič and Marušič [8] classified 2-arc-transitive Cayley graphs on dihedral groups.

- Xu [33] give a classification of tetravalent one-regular circulant graphs.

- Xu and Xu [31] give a classification of tetravalent one-regular Cayley graphs on abelian groups.
One-regular graphs with valency 4

- All tetravalent one-regular graphs of order $p$ or $pq$ are circulant, and a classification of such graphs can be easily deduced from [32].

- Zhou and Feng [35, 37] classified tetravalent one-regular graphs of order $2pq$, where $p$ and $q$ are primes.

- An infinite family of tetravalent one-regular Cayley graphs on alternating groups was constructed by Marušič in [22].

- An infinite family of infinite one-regular graphs of valency 4 was constructed by Malnič et al [23].
One-regular graphs with valency 3

- The first one-regular cubic graphs was constructed by Frucht in [28].

- Conder and Dobcsányi [5] classified one-regular (s-regular) cubic graphs of order up to 768.


- Zhou and Feng [36] classified cubic one-regular graphs of square-free order.

- Kutnar and Marušič [17] classified one-regular (s-regular) Cayley graphs of valency 3 on a generalized dihedral group.
One-regular graphs with valency 3

- Feng and Kwak [14] constructed an infinite family of cubic one-regular Cayley graphs on alternating groups.

- Du and Wang [9] proved that there is no cubic one-regular Cayley graphs on $\text{PSL}(2, p)$, where $p \geq 5$ is a prime.

- Feng, Kwak, et al [11, 16, 12, 15, 13] classified cubic one-regular ($s$-regular) graphs of order $2p^2$, $2p^3$, $mp$ and $mp^2$ for $m = 4, 6, 8, 10$, where $p$ is a prime.

- Oh [25, 26] classified cubic one-regular ($s$-regular) graphs of order $14p$ and $16p$. 
A conjecture on one-regular cubic graphs

- By checking all cubic one-regular graphs discovered before, there is no cubic one-regular graphs of order 4 times an odd integer. Then a natural conjecture follows:

- Conjecture [36]: There is no cubic one-regular graphs of order $4m$ for any odd integer $m$.

- However, the conjecture is not true. Recently, Conder and Feng [4] answered the above conjecture negatively by proving the following results.
Main results: Theorems 1 and 2

- **Theorem 1**: Let $X$ be a one-regular cubic graph of order $4m$ where $m$ is odd. Then $X$ is a normal cover of a base graph $Y$, where $Y$ has an arc-regular group of automorphisms that is isomorphic to a subgroup of $\text{Aut}(\text{PSL}(2, q))$ containing $\text{PSL}(2, q)$ for some odd prime-power $q$.

- To state the second result, we need some notation. Let $p$ be an odd prime and let $K = \text{GF}(p^3)$ be the field of order $p^3$. Denote by $\alpha$ the Frobenius automorphism of $K$: $\alpha : x \mapsto x^p$. For any matrix $M \in \text{SL}(2, K)$, denote by $\bar{M}$ the image of $M$ under the natural homomorphism from $\text{SL}(2, K)$ to $\text{PSL}(2, K) = \text{SL}(2, K)/\text{Z}($\text{SL}(2, K))$. 
**Theorem 2**: For any element \( t \in K \) such that \( t^3 \) lies outside the base field \( F = \text{GF}(p) \), let

\[
U = \begin{pmatrix} 1 & -2t \\ t^{-1} & -1 \end{pmatrix}, \quad V = U^\alpha = \begin{pmatrix} 1 & -2t^p \\ t^{-p} & -1 \end{pmatrix},
\]

\[
W = V^\alpha = \begin{pmatrix} 1 & -2t^{p^2} \\ t^{-p^2} & -1 \end{pmatrix}.
\]

Then

1. the images \( \overline{U}, \overline{V} \) and \( \overline{W} \) generate \( \text{PSL}(2, K) \), and
2. the Cayley graph \( \text{Cay}(\text{PSL}(2, K), \{\overline{U}, \overline{V}, \overline{W}\}) \) is a one-regular cubic graph.
Let $A = \text{Aut}(X)$, $P \in \text{Syl}_2(A)$. Then $|A| = 3|V(X)| = 12m$ and $|P| = 4$, so $P \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$P \cong \mathbb{Z}_4 \hookrightarrow N_A(P)/C_A(P) \leq \text{Aut}(P) \cong \mathbb{Z}_2$ (N/C theorem).

$|N_A(P)/C_A(P)| = 2 \iff (P \leq C_A(P)) |N_A(P)|$ is divisible by $2 \times 4 = 8$, contradiction.

$|N_A(P)/C_A(P)| = 1 \iff N_A(P) = C_A(P) \iff$ there is $T \trianglelefteq A$ such that $A = TP$ and $T \cap P = 1$ (Burnside), so $|T| = |A|/|P| = 12m/4 = 3m \iff (|V(X)| = 4m)$ $T$ has four orbits on $V(X) \iff T$ is semiregular on $V(X)$ (Lorimer), contradiction.
Let $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $N$ be the largest normal subgroup of $A$ of odd order. Then $N$ has at least four orbits and $X$ is a normal cover of $X_N$ (Lorimer).

Gorenstein-Walter theorem $\leftrightarrow A/N \cong P$, or $A_7$, or a subgroup of $\text{Aut}(\text{PSL}(2, q))$ containing $\text{PSL}(2, q)$ for some odd $q$.

$A/N \cong P \leftrightarrow |N| = 3m$ and $N$ is semiregular on $V(X)$ (Lorimer), which is impossible.

Clearly, $A/N \not\cong A_7$ because $8 \nmid |A|$.

$A/N \cong$ a subgroup of $\text{Aut}(\text{PSL}(2, q))$ containing $\text{PSL}(2, q)$ for some odd $q$. 
Let $S = \{\overline{U}, \overline{V}, \overline{W}\}$, $G = \text{PSL}(2, K)$, $X = \text{Cay}(G, S)$, and $A = \text{Aut}(X)$.

The fact that $\langle \overline{U}, \overline{V}, \overline{W} \rangle = \text{PSL}(2, K)$ is proved by considering maximal subgroups of $\text{PSL}(2, q)$, which was first given by Dickson [6].

To prove that $X$ is one-regular, it suffices to show that $A = R(G) \rtimes \text{Aut}(G, S)$ and $\text{Aut}(G, S) = \langle \overline{\alpha} \rangle$, where $\overline{\alpha}$ is the automorphism of $G$ induced by $\alpha$.

Xu et al. [34] $\leftrightarrow A = R(G) \rtimes \text{Aut}(G, S)$. Clearly, $|\text{Aut}(G, S)| = 3$ or $6$. The former implies $A = R(G) \rtimes \text{Aut}(G, S)$. We only need to show that the latter cannot happen.
Main Ideas for Theorem 2

**skeleton Proof of Theorem 2**

- Suppose $|\text{Aut}(G, S)| = 6$. Let $B = \langle R(G), \bar{\alpha} \rangle$. Then $|A : B| = 2$ and $B \cong \text{PΣL}(2, K)$. Let $C = C_A(B)$, the centralizer of $B$ in $A$.

- $C \cap B = Z(B) = 1 \iff |C| = 1$ or $2$. Note that $A_1 \cong S_3$.

- $|C| = 2 \iff A = B \times C \iff A_1 \cong A/R(G) \cong \mathbb{Z}_6$, contradiction.

- $|C| = 1 \iff A \lesssim \text{Aut}(B)$ (N/C theorem) $\cong \text{Aut}(\text{PΣL}(2, K)) \cong \text{PGL}(2, K) \iff A \cong \text{PGL}(2, K)$ (order).

- $A_1 \cong A/R(G) \cong \text{PGL}(2, K)/\text{PSL}(2, K) \cong \mathbb{Z}_6$, contradiction.


Y.-Q. Feng, J.H. Kwak, Cubic symmetric graphs of order a small number times a prime or a prime square, *J. Combin. Theory B* **97**(2007), 627-646.


Thanks!