Applications of semiclassical analysis in harmonic analysis

Melissa Tacy

University of Auckland

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What is harmonic analysis?

Key idea is to analyse a function by breaking it up into "simple" functions. The form the simple functions take depends on the situation.

General case Λ a parameter space φ_λ a analysing function. Develop an analysis operator

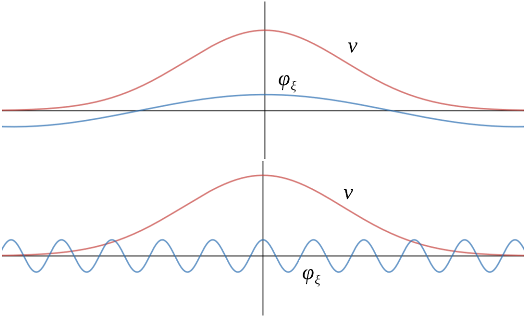
$$T_\lambda[v] = \langle v, \varphi_\lambda \rangle$$

and a synthesis

$$v(x) = \int T_\lambda[v]\varphi_\lambda(x) d\mu(\lambda)$$

for μ a suitable measure on Λ.

Example: Fourier analysis in n dimensions



$$\Lambda = \mathbb{R}^n$$

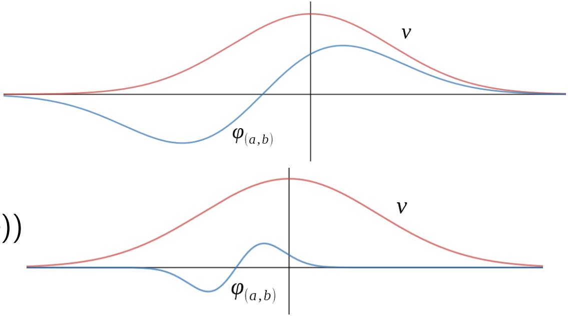
parameter ξ

$$\varphi_\xi = \frac{1}{(2\pi)^{n/2}} e^{i\langle x,\xi\rangle}$$

$$T_\xi[v] = \langle v, \varphi_\xi \rangle = \int v(x)\overline{\varphi_\xi(x)} = \frac{1}{(2\pi)^{n/2}} \int e^{-i\langle x,\xi\rangle} v(x) dx$$

$$v = \int T_\xi[u]\varphi_\xi(x) d\xi = \frac{1}{(2\pi)^{n/2}} \int e^{i\langle x,\xi\rangle} T_\xi[v](\xi) d\xi$$

Example: Wavelet analysis in 1 dimension



$$\Lambda = \mathbb{R}^2$$

parameter (a, b)

$$\varphi_{(a,b)} = |a|^{-1/2} f(a^{-1}(x-b))$$

with $\int f(x)dx = 0$.

$$T_{(a,b)}[v] = \langle v, \varphi_{(a,b)} \rangle = \int v(x)\overline{\varphi_{(a,b)}(x)}dx = \frac{1}{|a|^{1/2}} \int v(x)\overline{f}(a^{-1}(x-b))dx$$

$$v = \iint T_{(a,b)}[v]\varphi_{(a,b)}(x)\frac{dadb}{a^2} = \iint f(a^{-1}(x-b))\, T_{(a,b)}[v]\frac{dadb}{|a|^{3/2}}$$

The importance of L^2 spaces

The "action" of analysis/synthesis takes places in L^2 where we measure the size of a function $v(x)$ by

$$\|v\|_{L^2} = \left(\int |v(x)|^2 dx\right)^{1/2}$$

Equality $v = u$ in this context means

$$\|v - u\|_{L^2} = \left(\int |v(x) - u(x)|^2 dx\right)^{1/2} = 0$$

Convergence $v_j \to u$ in this context means

$$\lim_{j\to\infty} \|v_j - v\|_{L^2} = \lim_{j\to\infty} \left(\int |v_j(x) - v(x)|^2 dx\right)^{1/2} = 0$$

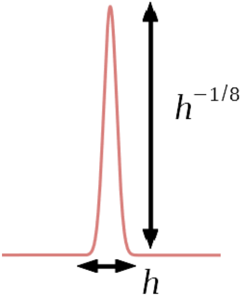
So when we say that v is synthesised from $T_\lambda[v]$ we mean in the L^2 sense.

Convergence in L^2 does not mean pointwise

Consider the function $u : \mathbb{R} \to \mathbb{R}$ with

$$u = h^{-1/8} e^{-\frac{|x|^2}{h^2}}$$

Claim this converges to zero in L^2.



$$\|u - 0\|_{L^2} = \left(\int h^{-1/4} e^{-2\frac{|x|^2}{h^2}} dx \right)^{1/2}$$
$$\approx \left(h^{-1/4} \times h \right)^{1/2}$$
$$\to 0$$

Other ways of measuring size

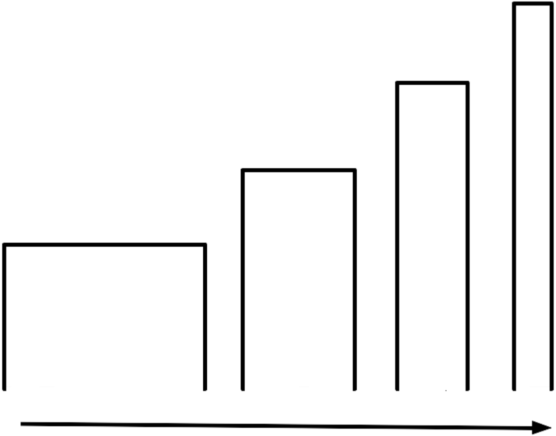
Typically we want to study $v(x)$ in other (than L^2) function spaces. This allows a finer understanding of the behaviour of $v(x)$. Often want to know about concentration of $v(x)$. An important class of functions spaces are the L^p spaces for $1 \leq p \leq \infty$. Where

$$\|v\|_{L^p} = \left(\int |v(x)|^p dx \right)^{1/p} \quad 1 \leq p < \infty$$

$$\|v\|_{L^\infty} = \operatorname{esssup} |v(x)|$$

By knowing about the full family of norms we can understand the types of potential concentrations.

What L^p norms measure

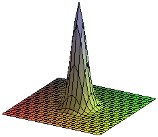


All these functions have L^p norm equal to one. As p increases the L^p norm is less sensitive to area.

Measuring Concentration

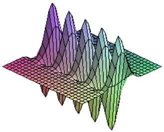
Understanding how the L^p norm of v grows helps us to understand the local features of a function. Some feature that frequently appear in solutions to PDE

Point



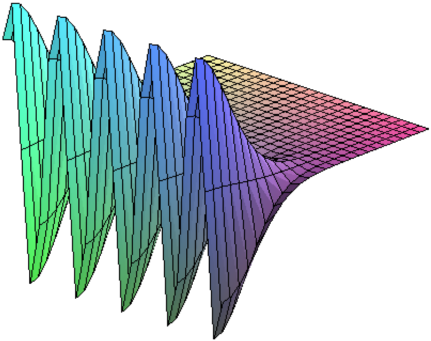
- High L^∞ norm
- Sharp change in L^p norm when $p < \infty$

Tube



- Lower L^∞ norm
- Change in L^p norm more gentle

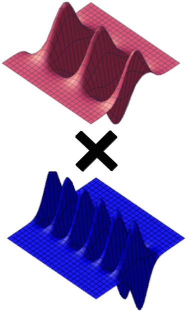
Submanifold Estimates



- Can take cross sections of functions and study the L^p norm of the cross section.
- Because cross sections are lower dimensional we can get very fine information about how $v(x)$ concentrates to lower dimensional objects.

Bilinear estimates

Suppose we want to look at something that has a non-linearity. We then need to consider how production of functions behave so we might ask What happens to $\|uv\|_{L^p}$?



Applications to nonlinear PDE

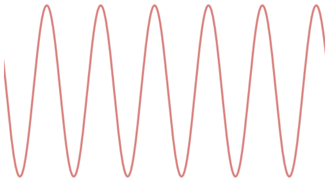
$$i\partial_t v(t,x) + \Delta v(t,x) = \pm|v(t,x)|^{\alpha} v(t,x)$$

Interaction between frequency bands determines regularity properties.

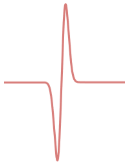
High vs low frequency contributions

Typically an analysis/synthesis will break up into high/low frequency bands.

Fourier analysis: For $e^{i\langle x,\xi\rangle}$ the frequency is $|\xi|$ so high/low frequency corresponds to large/small $|\xi|$.



Wavelet analysis: For $|a|^{-1/2}f(a^{-1}(x-b))$ the frequency is $|a|^{-1}$ so high/low frequency corresponds to small/large $|a|$.



We are usually interested what happens to the high frequency contributions.

What is Semiclassical Analysis?

Term for a collection of techniques to address various PDE that involve a parameter $h \to 0$. For example if we take the Laplacian

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

and consider its eigenfunctions

$$\Delta v = \lambda^2 v$$

when λ is large. Check that $e^{i\lambda\langle x,\xi\rangle}$ for some direction $\xi \in \mathbb{R}^n$, $|\xi| = 1$.

$$\frac{\partial^2}{\partial x_j} e^{i\lambda\langle x,\xi\rangle} = \lambda^2 \xi_j^2 e^{i\lambda\langle x,\xi\rangle}$$

By setting $h = \lambda^{-1}$ and dividing everything through by λ^2 we can re-write this as a semiclassical equation

$$h^2 \Delta e^{\frac{i}{h}\langle x,\xi\rangle} = e^{\frac{i}{h}\langle x,\xi\rangle}$$

Primary objects

Semiclassical differential (and pseudodifferential) operators Built out of combinations of partial derivatives. Use multi-index notation. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ where each $\alpha_j = \mathbb{Z}^+$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

$$D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

Then a semiclassical differential operator can be written as

$$L = \sum_{|\alpha| \leq N} c_\alpha(x) h^{|\alpha|} D^\alpha$$

where $c_\alpha(x)$ is the coefficient associated to D^α (simplest case is constant coefficient where c_α does not depend on x).

The Fourier transform method

Provides the link for constant coefficient equations.
We use a scaled version of the Fourier transform to analyse/synthesise $v(x)$.

$$v(x) = \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x,\xi\rangle} T_\xi[v] d\xi$$

where

$$T_\xi[v] = \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x,\xi\rangle} v(x) dx$$

and see what happens when we apply a differential operator.

$$h\frac{1}{i}\frac{\partial}{\partial x_j} v = \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x,\xi\rangle} \xi_j T_\xi[v] d\xi$$

From differential equations to algebraic equations

So we have a relationship between differentiation of $v(x)$ and multiplication of $T_\xi[v]$.

$$h^{|\alpha|}D^\alpha v(x) \to \xi^\alpha T_\xi[v]$$

where

$$\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}.$$

Suppose that for $\alpha = (\alpha_1, \dots, \alpha_n)$

$$L = \sum_{|\alpha| \leq N} c_\alpha h^{|\alpha|} D^\alpha$$

then

$$L \to \sum_{|\alpha| \leq N} c_\alpha \xi^\alpha$$

So we can solve an algebraic equation on the Fourier side then invert to solve the differential equation $L[u] = 0$.

Semiclassical pseudodifferential operator

Can go step further and generalise to allow us to deal with non-constant coefficient equations. Let $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Define

$$p(x,hD)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} p(x,\xi)u(y)dyd\xi$$

and call $p(x,hD)$ the left (or standard) quantisation of the symbol $p(x,\xi)$. If $p(x,\xi) = p(\xi)$ is independent of x then

$$p(x,hD) = p(hD) = \mathcal{F}_h^{-1} p(\xi)\mathcal{F}_h$$

Need requirements on $p(x,\xi)$ to make sure that it is "polynomial like" but can for example have fractional powers like $p(\xi) = (1+|\xi|^2)^{\frac{2}{3}}$.

Semiclassical calculus

- Tells us how to compose two operators L_1, L_2.
- Tells us when operators are invertible and gives a formula for the inverse.
- Gives a framework in which we can construct explicit local solutions to $Lv = f$ or $(hD_t - L)u = 0$ type equations.
- Allows us to study the high frequency problems from harmonic analysis by studying the analysing functions $\phi_\lambda(x)$ as solutions to partial differential equations.
- Major point of connection are the functions $e^{\frac{i}{h}\langle x,\xi\rangle}$ from a semiclassical perspective we see them as solutions to an eigenfunction equation. From the harmonic analysis perspective they are the "simple" functions that we will use to analyse more general functions $v(x)$.

Evolutions equations and eigenfunctions

When we first learn PDE we learn to separate solutions, look for solutions

$$v(t,x) = f(t)u(x)$$

for the evolution equation.

$$(hD_t - L)v(t,x) = 0$$

In this case have a solution

$$v(t,x) = e^{\frac{i}{h}t}u(x)$$

where

$$Lu = u$$

If you can solve the eigenfunction equation you can construct the solutions to the evolution equation. Want to go the other way. Use evolution equations to solve for eigenfunctions.

From eigenfunctions to evolution equations

Suppose that L is a semiclassical differential operator

$$L = \sum_{|\alpha| \leq N} c_\alpha(x) h^{|\alpha|} D^\alpha$$

and v is an approximate eigenfunction that is

$$(L - 1)v = \text{small}$$

We measure smallness in L^2 so we say that the quasimode error of v is $\|Pv\|_{L^2} = \|(L-1)v\|_{L^2}$. Usually assume that

$$\|Pv\|_{L^2} \leq Ch \|v\|_{L^2}$$

then we say v is an $O_{L^2}(h)$ quasimode of P. Then clearly

$$(hD_t - P)v = hf(x)$$

So v is an approximate solution to the evolution equation.

Duhamel method for describing eigenfunctions

Change our perspective.

$$hf \text{ is an error} \to hf \text{ is an inhomogeneity}$$

We want to think of $v(x)$ as an exact solution to the inhomogeneous equation

$$(hD_t - L)v(x) = hf$$

This perspective changes allows us to use Duhamel principle to write

$$v = U(t)v + \frac{1}{h}\int_0^t U(t-s)[hf(x)]ds$$

Since this is true for any time t we can time average (up to time for which we have a good expression for $U(t)$). Therefore estimating the L^p norms of the function u is the same as analysing the $L^2 \to L^p$ mapping properties of time averages of $U(t)$.

A parametrix for $U(t)$

We can write $U(t)$ as a semiclassical FIO and produce an explicit local representation of it as an oscillatory integral operator.
We use the PDE

$$\begin{cases} (hD_t + p(x, hD))U(t) = 0 \\ U(0) = \mathrm{Id} \end{cases}$$

to produce a parametrix solution

$$U(t)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,y,\xi)} b(x, y, \xi) u(y)$$

$$\partial_t \phi(t, x, y, \xi) = p(x, \nabla_x \phi) \quad \phi(0, x, y, \xi) = \langle x - y, \xi \rangle$$

This parametrix has the advantage of being a nice explicit representation for $U(t)$ in terms of an oscillatory integral. We can always do this up to order one time. This means we are able to use the, well developed, theory of oscillatory integral operators to extract the mapping properties of $U(t)$.

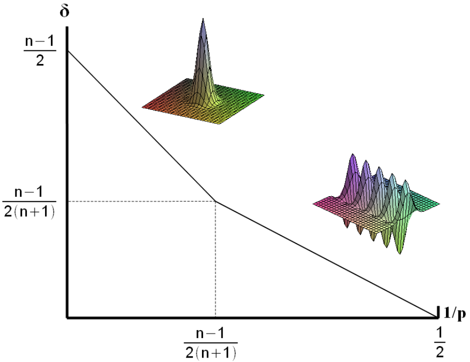
L^p estimates for eigenfunctions

$$(h^2\Delta - 1)v = 0$$

u obeys the estimates due to Sogge

$$\|v\|_{L^p} \lesssim h^{-\delta(n,p)} \|v\|_{L^2}$$

Known to be sharp for spherical harmonics



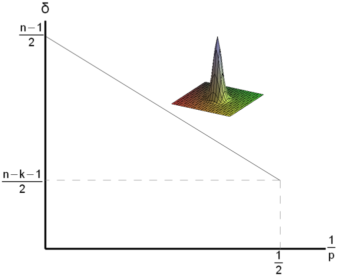
Was extended (by Koch-Tataru-Zworski) to a result about pseudodifferential operators whose symbols obey a curvature assumption.

Submanifold L^p Estimates for Quasimodes

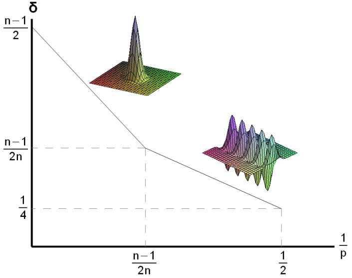
Burq-Gérard-Tzvetkov for Laplace eigenfunction 2007, Tacy 2010 for Laplace like operators and order h quasimodes. If Y has dimension k

$$\|v\|_{L^p(Y)} \lesssim h^{-\delta(n,k,p)} \|v\|_{L^2(M)}$$

$k \leq n-2$



$k = n-1$



Coming up

L^p estimates for joint quasimodes

Where v obeys two semiclassical equations

$$P_1 v = O_{L^2}(h)$$
and
$$P_2 v = O_{L^2}(h)$$

What can we say about

$$\|v\|_{L^p}?$$

Need non-degeneracy conditions on P_1, P_2. This is where the Fourier transform method comes into its own as it allows us to analyse some simple models. More on Wednesday ...