L^p estimates for joint eigenfunctions

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13 January 2021

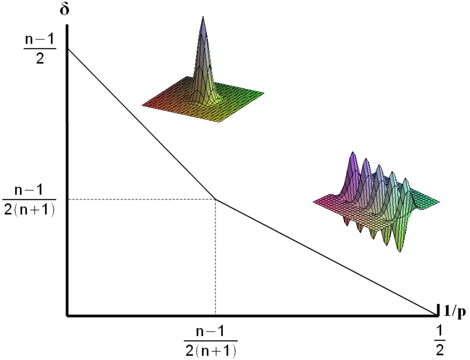
L^p estimates for eigenfunctions

$$(h^2\Delta - 1)v = 0$$

u obeys the estimates due to Sogge

$$\|v\|_{L^p} \lesssim h^{-\delta(n,p)} \|v\|_{L^2}$$

Known to be sharp for spherical harmonics



Was extended (by Koch-Tataru-Zworski) to a result about pseudodifferential operators whose symbols obey a curvature assumption.

Joint eigenfunctions

In a letter to Morawetz, Sarnak asks whether improvements can be made to L^∞ estimates for v

$$P_i v = 0 \quad \Delta - 1 = P_1, \ldots, P_r$$

where the P_i are a set of pseudodifferential operators.

Counterexample on sphere S^2

$$P_1 = \Delta - 1 \quad P_2 = \partial_\varphi$$

must obey Sogge's estimates for $p \geq 6$ as zonal harmonics are invariant under rotations around the north pole.

Positive result, if M is a rank r symmetric space

$$\|v\|_{L^\infty} \lesssim h^{-\frac{n-r}{2}} \|v\|_{L^2}$$

L^p results on symmetric spaces

For eigenfunctions of Δ on symmetric spaces Marshall obtains

$$\|v\|_{L^p} \lesssim h^{-r\delta(n/r,p)} \|v\|_{L^2}$$

(except at $p = \frac{2(n+r)}{n-r}$ where there is logarithmic loss)

Example to keep in mind

$$S^{n/r} \times S^{n/r} \times \cdots \times S^{n/r}$$

$$v = \phi_1(x_1)\phi_2(x_2)\cdots\phi_r(x_r)$$

where each ϕ_i is a spherical harmonic. So eigenfunctions on symmetric spaces have L^p growth no worse than this example.

More general case?

Assume v is a joint quasimode of r semiclassical ΨDOs

$$p_1(x, hD), \dots, p_r(x, hD)$$

obeying a curvature condition.

$$p_i(x, hD)v = \text{Error}$$

- What do we need to assume about the quasimode error?
- What conditions do we need on the $p_i(x, hD)$ to get improvements?
- What curvature condition do we need? What is so important about curvature conditions?

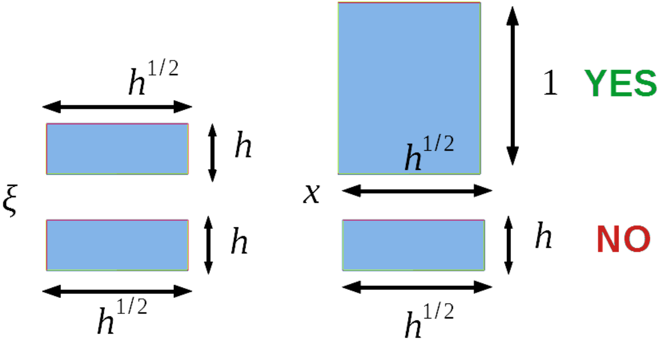
The role of the uncertainty principle

In general terms the uncertainty principle tells us that you cannot concentrate both the Fourier transform of u and u itself. If

$$\mathcal{F}_h[v] = \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x,\xi\rangle} v(x)dx$$

then

$$(\text{concentration scale of } \mathcal{F}_h[v]) \times (\text{ concentration scale of } v) \geq h$$



Uncertainty in solutions to PDE

Suppose we are working with approximate eigenfunctions to Δ in $\mathbb{R}^n$.

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

Using the Fourier transform method we have that if

$$(h^2\Delta - 1)v = hf$$

then

$$(|\xi|^2 - 1)\mathcal{F}_h[v] = h\mathcal{F}_h[f]$$

The scaling on the semiclassical Fourier transform is chose precisely so that $\mathcal{F}_h$ preserves L^2 norms. Therefore

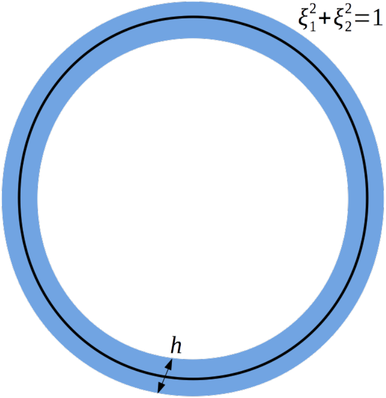
If v satisfies the **differential equation** up to L^2 order h error

The Fourier transform $\mathcal{F}_h[v]$ satisfies the associated **algebraic equation** up to L^2 order h error.
In particular the Fourier transform of v must "live" near the set $|\xi| = 1$.

2D model

We are looking at $P_1 v = p_1(x, hD)v = hf$ where

$$p_1(x, \xi) = \xi_1^2 + \xi_2^2 - 1$$



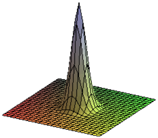
Work on Fourier side. Need

$$(\xi_1^2 + \xi_2^2 - 1)\mathcal{F}_h[v] = O_{L^2}(h)$$

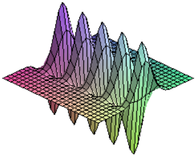
So $\mathcal{F}_h[v]$ can only spread out an order h distance from the $|\xi| = 1$ sphere. So any quasimodes need to have most of their Fourier transform in a very thin annulus.

Let's first ask what this tells us about solutions to just one equation.

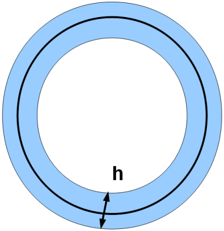
Ways to concentrated $\mathcal{F}_h[v]$

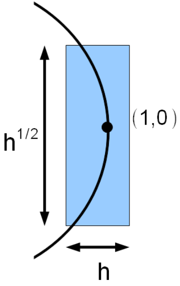


Spread $\mathcal{F}_h v$ evenly throughout annular region



Concentrate $\mathcal{F}_h v$ around one point

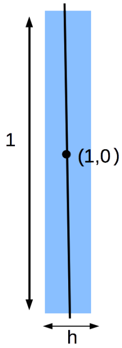




Can produce examples with intermediate spread.

The role of curvature

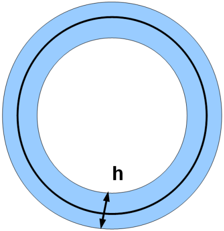
Let's compare our curved example with a flat one. If $Q = hD_{x_1} - 1$ then if v is a order h quasimode $\mathcal{F}_h[v]$ must live near $\xi_1 = 1$.



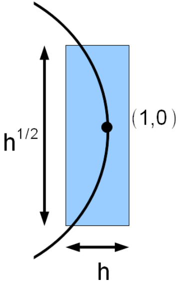
Since $\mathcal{F}_h[v]$ is concentrated in ξ_1 it must be spread out in x_1 (uncertainty principle). But there is no restriction in the second variable. We could make ξ_2 very spread out, this would effect a concentration in x_2.

Compare to the curved case

If we try to spread the Fourier transform throughout the whole annulus (to make the L^∞ norm large) we place restrictions on both x_1 and x_2 concentrations.



If we try to concentrate in a box around a point this is the best we can do. The square root scale is determined by the local curvature near $(1, 0)$.



Degeneracy conditions?

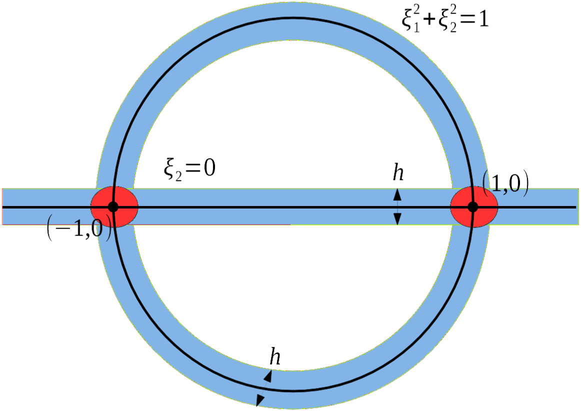
Now we add in a second operator.

$$p_2(x,\xi) = \xi_2 \quad p_2(x,hD) = hD_{x_2}$$

This requires that

$$\xi_2 \mathcal{F}_h[v] = O_{L^2}(h)$$

If v is a joint quasimode $\mathcal{F}_h[v]$ it must live in h^2 balls around $(1,0)$ and $(-1,0)$. This mean the L^∞ norm has to be bounded.



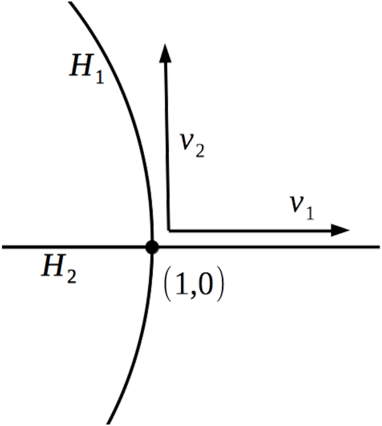
Transversity condition

Consider the hypersurfaces

$$H_1 = \{\xi \mid |\xi|^2 - 1 = 0\}$$

$$H_2 = \{\xi \mid \xi_2 = 0\}$$

Their normal vectors at $(1,0)$, ν_1 and ν_2 are perpendicular. So each one constrains $\mathcal{F}_h$ in one direction. Suggests an assumption that all the $p_i(x,\xi)$ have hypersurfaces as characteristic sets and that the normals are linearly independent.



Sarnak's counter-example

$$\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\varphi}\right) v = -h^{-2} v$$

Converting to a semiclassical ΨDO and dropping non-principal terms

$$(h^2 \sin^2\theta D_\theta^2 + h^2 D_\varphi^2 - \sin^2\theta)v = 0$$

So

$$p_1(\theta, \varphi, \xi, \eta) = \xi^2 \sin^2\theta + \eta^2 - \sin^2\theta$$

$$p_2(\theta, \varphi, \xi, \eta) = \eta$$

Now at $\theta = 0$

$$H_1 = \{(\xi, \eta) \mid \eta^2 = 0\} \quad H_2 = \{(\xi, \eta) \mid \eta = 0\}$$

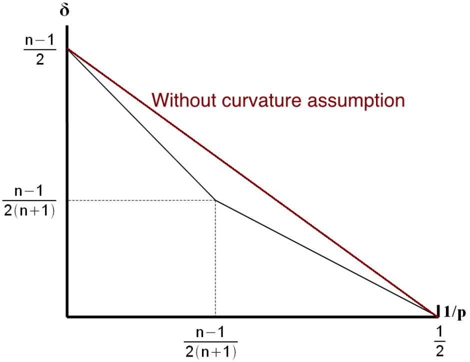
So clearly we need the non-degeneracy to hold for all sets $\{\xi \mid p_i(x_0, \xi) = 0\}$.

Laplace-like condition for $p_1(x, hD)$

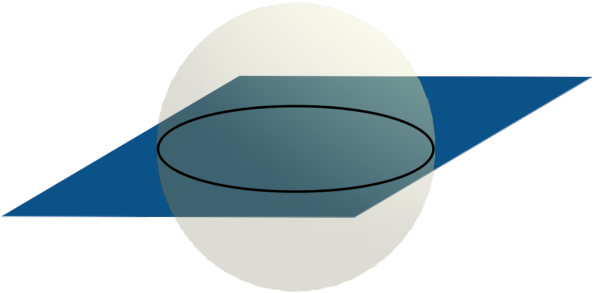
Distinctive piecewise behaviour comes from the curvature of

$$\{\xi \mid |\xi|_{g(x)} = 1\}$$

Need to maintain a curvature assumption on the intersection of characteristic sets.



Cross sections



Joint eigenfunction condition leads to taking cross-sections. So need to maintain curvature for these.

Need

$\gamma S_1 \gamma^\star$ non-degenerate.

- γ be the restriction onto the tangent space of the joint characteristic set
- S_1 the shape operator for $\{\xi \mid p_1(x,\xi) = 0\}$.

Theorem (T 2019)

Suppose v is a strong, joint $O_{L^2}(h)$ quasimode of a set of $p_1(x, hD), \dots, p_r(x, hD)$ where

1. *For any x_0, $\{\xi \mid p_i(x_0, \xi) = 0\}$ is a hypersurface $i = 1, \dots, r$.*
2. *If $\nu_i(x, \xi)$ is the normal to $\{\xi \mid p_i(x, \xi) = 0\}$ then the set of $\nu_i(x, \xi)$ are linearly independent.*
3. *The operator $\gamma S_1 \gamma^\star$ is non-degenerate*

Then

$$\|v\|_{L^p} \lesssim h^{-\delta(n,p,r)} \|v\|_{L^2}$$

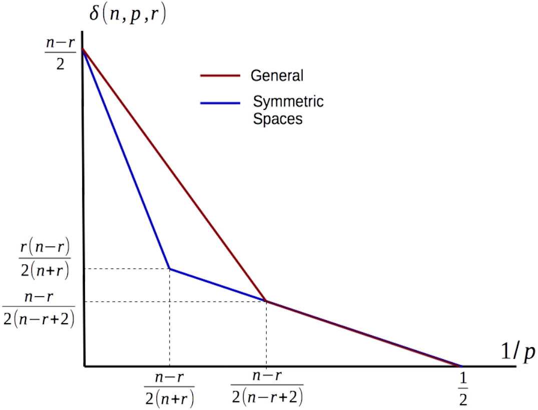
$$\delta(n, p, r) = \begin{cases} \frac{n-r}{2} - \frac{n-r+1}{p} & \frac{2(n-r+2)}{n-r} \leq p \leq \infty \\ \frac{n-r}{4} - \frac{n-r}{2p} & 2 \leq p \leq \frac{2(n-r+2)}{n-r}. \end{cases}$$

Note the curvature condition automatically holds if

$$\{\xi \mid p_1(x_0, \xi) = 0\}$$

has positive definite second fundamental form.

Comparison to symmetric spaces



- Agrees for $p = \infty$ and for $p \leq \frac{2(n-r+2)}{n-r}$.
- Symmetric spaces enjoy slightly better estimates between $\frac{2(n-r+2)}{n-r}$ and ∞.
- Might ask whether better estimates are possible in the general case.

Sharpness

Consider the case

$$p_1(x,\xi) = |\xi|^2 - 1 \quad p_i(x,\xi) = \xi_i, \text{ for } i = 2,\dots,r$$

Then we are essentially looking at solutions

$$v(x) = u(x_1, x_{r+1},\dots,x_n)$$

where u satisfies the $n-r+1$ dimensional eigenfunction equation. So we can apply Sogge's bounds in dimension $n-r+1$. These agree with $\delta(n,p,r)$.

Proof sketch

1. Define a process for successively factoring out $r-1$ of the ξ_i effectively turning the problem into two linked problems
   - One $n-r+1$ dimensional one which will inherit the curvature condition.
   - $r-1$ directions in which we "control the spread of the Fourier transform".
2. We treat the $n-r+1$ dimensional problem with the methods of Koch-Tataru Zworski.
3. We prove uniform L^∞ bounds in all the factored $r-1$ directions.

Factoring out directions

This technique relies on the semiclassical calculus to tell us about inverses. From the semiclassical calculus

$$p(x, hD) \text{ is invertible as an operator}$$
$$\updownarrow$$
$$p(x, \xi) \text{ is invertible as a function (not zero)}$$

Now consider $p(x, hD)v = hf$. If $|p(x, \xi)| > c$ then we can invert to get that u is small. So solutions (or quasimodes) must "live" near the set where $p(x, \xi) = 0$.
For constant coefficient equations this is the same as saying that their Fourier transform has to live near the set where $p(\xi) = 0$.

Inductive process

Write $p_r(x,\xi) = 0$ as a graph

$$\xi_r = q_r(x, \xi_1, \dots, \xi_{r-1}, \xi_{r+1}, \dots, \xi_n)$$

Now update $p_1(x,\xi), \dots, p_{r-1}(x,\xi)$ by substituting for ξ_r in each. Then if

$$p_{r-1}^{(r)}(x,\xi) = p_{r-1}(x, \xi_1, \dots, \xi_{r-1}, q_r, \xi_{r+1}, \dots, \xi_n)$$

write $p_{r-1}^{(r)}(x,\xi)$ as a graph

$$\xi_{r-1} = q_{r-1}(x, \xi_1, \dots, \xi_{r-2}, \xi_{r+1}, \dots, \xi_n)$$

and update $p_1^{(r)}(x,\xi), \dots, p_{r-2}^{(r)}(x,\xi)$ by substituting for ξ_{r-1} in each. Keep going until $\xi_2, \dots, \xi_r$ are all factored out.

The $n - r + 1$ problem

After the inductive process we are left with a function

$$p(x, \xi_1, \ldots, \xi_{r+1}, \xi_n)$$

which is zero on the set

$$\bigcap_{i=1}^{r} \{p_i(x, \xi) = 0\}$$

- If v is a joint quasimode for the $p_1, \ldots, p_r$ then it is a quasimode for $p(x, hD_1, hD_{x_{r+1}}, \ldots hD_{x_n})$.
- Note that no derivatives is the $x_2, \ldots x_r$ directions occur in this equation

So if we freeze the $x_2, \ldots, x_r$ we can estimate $\|v(\cdot, x_2, \ldots, x_r, \cdot)\|_{L^p(\mathbb{R}^{n-r+1})}$ using Koch-Tataru-Zworski.

The $r-1$ remaining directions

Unravel the inductive procedure. Suppose that $v(x)$ satisfies

$$(hD_{x_k} - q(x, hD_{x_1}, \dots, hD_{x_{k-1}}, hD_{x_{k+1}}, \dots hD_{x_n}))v = hf$$

We can think of this as an evolution equation with $x_k = t$ and write

$$v(x_1, \dots, t, \dots x_n) = U(t)v(x_1, \dots, 0, x_n) + \int_0^t U(t,s)fds$$

Since $U(t)$ preserves L^2 mass along time slices we get an L^∞ bound on x_k. By performing this argument successively we can control each of $x_2, \dots, x_r$.

Further work

Can we use weaker assumptions on the intersections of the $\{\xi \mid p_i(x,\xi) = 0\}$ to get improvements for some p?

Consider the case $p_1 = |\xi|^2 - 1$ and $p_2 = \xi_1 - 1$. Since the two curves have order 1 contact we don't have the transversity condition. In the 2D setting I have been able to prove estimates classified by the order at intersection point.
The higher dimensional setting is more complex, requires significant classification work first.