Imaging:
An Inverse Scattering Problem

Lecture 2

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Previously...

Find $\hat{m}$ such that

$$\hat{m} = \underset{m}{\operatorname{argmin}}\, \mathcal{J}(m),$$

where

$$\mathcal{J}(m) = \frac{1}{2}\sum_{\ell=1}^{n_s} \left\| P u_\ell - u_\ell^{obs} \right\|_{L^2(\Gamma)}^2$$

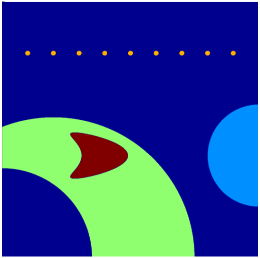
with P projector from Ω to Γ
subject to

$$H(m) u_\ell \;=\; f_\ell\,, \quad \text{in } \Omega, \quad \ell = 1, \ldots, n_s.$$

or

$$\mathcal{J}(m) = \frac{1}{2}\sum_{\ell=1}^{n_s} \left\| P H(m)^{-1} f_\ell - u_\ell^{obs} \right\|_{L^2(\Gamma)}^2.$$

Example:



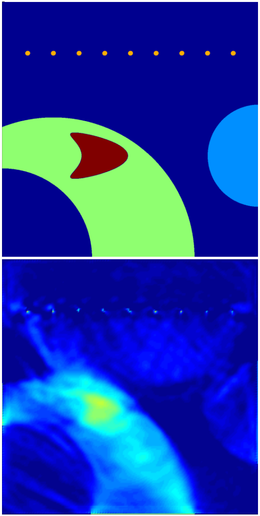
- $n_s = 9$ Gaussian/Dirac sources (orange dots)
- array of receivers $\Gamma := \partial\Omega$
- multiple frequencies $\omega = 8 : 2 : 90$

Strategies:

Frequency stepping
Truncated Gauss-Newton method
No additional regularisation

Algorithm:

1. initialise $m^{(1)}(\vec{x}) \equiv 1$
2. for $\omega_k = \omega_1 < \ldots < \omega_n$
3. while $\|\nabla \mathcal{J}(m^{(k)})\| > tol$
4. solve $\mathcal{B}p = -\nabla \mathcal{J}(m^{(k)})$ (with few iterations of CG)
5. update $m^{(k)} \leftarrow m^{(k)} + \delta p$ (with line search criterion)
6. end while
7. $m^{(k+1)} \leftarrow m^{(k)}$
8. end for



(Nahum, PhD thesis 2016)

Relative L^2-error: 24.55%

Regularisation

Aim: make underdetermined problems determined

Types of regularisation:

- iterative regularisation method, e.g., truncated Newton's methods (Kalterbacher *et al.* 2008, Métivier *et al.* 2013)
- regularisation by penalisation, e.g., Tikhonov regularisation (Tikhonov 1943)
- regularisation by discretisation (Kirsch 1996, de Buhan and Osses 2010)

Modified cost-function

(Tikhonov and Arsenin 1977, Golub *et al.* 1999, Vogel 2002)

$$\mathcal{J}_{Tik}(m) \;=\; \mathcal{J}(m) \;+\; \alpha\mathcal{R}_{Tik}(m),$$

where

- $\alpha > 0$, regularisation parameter
- let D be a differential operator

$$\mathcal{R}_{Tik}(m) = \frac{1}{2}\|Dm\|^2_{L^2(\Omega)}.$$

Typically, $D = I$ or $D = \nabla$.

New minimisation problem: Find $\hat{m}$ such that

$$\hat{m} = \operatorname*{argmin}_{m} \mathcal{J}_{Tik}(m)$$

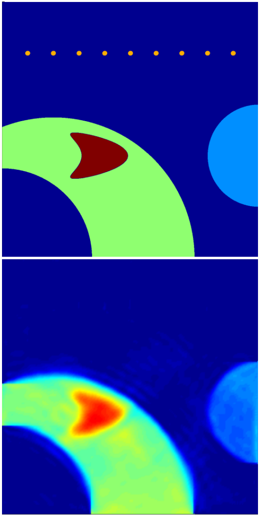
Strategies:

- Frequency stepping
- Truncated Gauss-Newton method
- Tikhonov regularisation with H^1-norm

Algorithm:

1. initialise $m^{(1)}(\vec{x}) \equiv 1$
2. for $\omega_k = \omega_1 < \ldots < \omega_n$
3. while $\|\nabla \mathcal{J}_{Tik}(m^{(k)})\| > tol$
4. solve $\mathcal{B}_{Tik} p = -\nabla \mathcal{J}_{Tik}(m^{(k)})$ (with few iterations of CG)
5. update $m^{(k)} \leftarrow m^{(k)} + \delta p$ (with line search criterion)
6. end while
7. $m^{(k+1)} \leftarrow m^{(k)}$
8. end for

Relative L^2-error: 9.25%



(Nahum, PhD thesis 2016)

Modified cost-function: (Rudin *et al.* 1992, Vogel and Oman 1996)

$$\mathcal{J}_{TV}(m) \;=\; \mathcal{J}(m) \;+\; \alpha \mathcal{R}_{TV}(m),$$

where

- $\alpha > 0$, regularisation parameter
- using L^1-norm

$$\mathcal{R}_{TV}(m) = \frac{1}{2}\|\nabla m\|_{L^1(\Omega)} \quad \text{or} \quad \mathcal{R}_{TV}(m) = \frac{1}{2}\int_{\Omega}\sqrt{|\nabla m|^2 + \epsilon^2}\,dx,$$

with relaxation parameter $\varepsilon > 0$.

New minimisation problem: Find $\hat{m}$ such that

$$\hat{m} = \operatorname*{argmin}_{m} \mathcal{J}_{TV}(m)$$

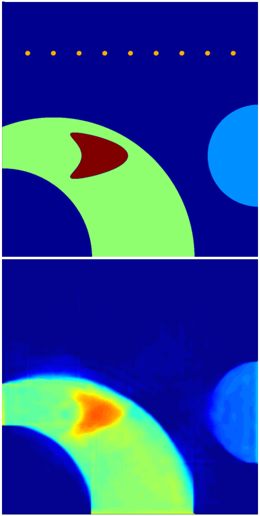
Strategies:

Frequency stepping
Truncated Gauss-Newton method
TV regularisation with $\varepsilon = 10$

Algorithm:

1. initialise $m^{(1)}(\vec{x}) \equiv 1$
2. for $\omega_k = \omega_1 < \ldots < \omega_n$
3. while $\|\nabla \mathcal{J}_{TV}(m^{(k)})\| > tol$
4. solve $\mathcal{B}_{TV} p = -\nabla \mathcal{J}_{TV}(m^{(k)})$ (with few iterations of CG)
5. update $m^{(k)} \leftarrow m^{(k)} + \delta p$ (with line search criterion)
6. end while
7. $m^{(k+1)} \leftarrow m^{(k)}$
8. end for

Relative L^2-error: 10.02%



(Nahum, PhD thesis 2016)

Adaptive Eigenspace Inversion (AEI)

Tikhonov or TV-regularisation: m is described by grid based point values

AEI: expand m as

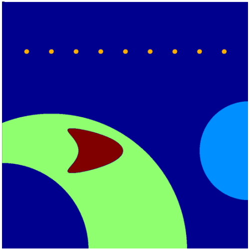
$$m(x) = m_0(x) + \sum_{j=1}^{K} \beta_j \phi_j(x), \quad \beta_j \in \mathbb{R}$$

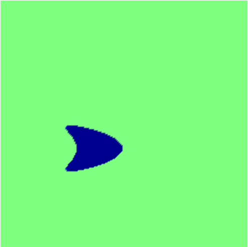
where ϕ_j are the first K eigenfunctions of

$$\begin{cases} -\nabla \cdot \left(\dfrac{1}{\max\{|\nabla m|, \varepsilon\}} \nabla \phi_j(x) \right) = \lambda_j \phi_j(x) & x \in \Omega, \\ \phi_j(x) = 0 & x \in \partial\Omega. \end{cases}$$

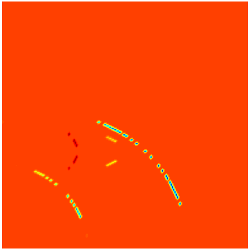
(de Buhan and Osses 2010, de Buhan and MG 2013)

For a **known** parameter m

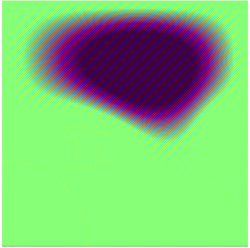




1st eigenfunction



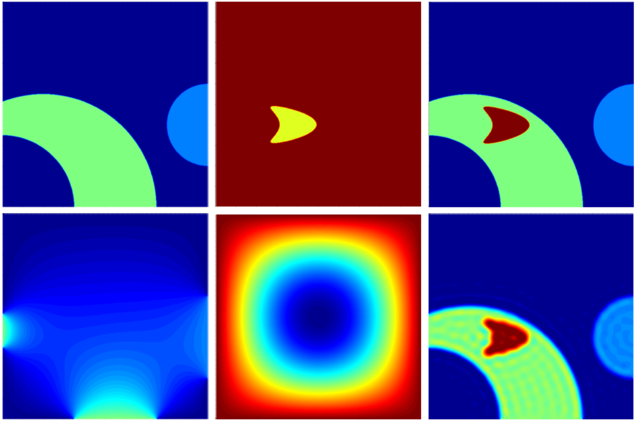
67th eigenfunction



last eigenfunction

Harmonic projection of m in the basis of eigenfunctions when

- Top: m is known in the elliptic operator
- Bottom: $m \equiv 1$, the elliptic operator is the Laplacian



(Grote, MG and Nahum 2017)

Find m s.t.

$$\beta = \underset{\gamma \in \mathbb{R}^K}{\operatorname{argmin}} \; \mathcal{J}(m(\gamma))$$

where

$$\mathcal{J}(m(\beta)) = \frac{1}{2} \sum_{\ell=1}^{n_s} \left\| PH(m(\beta))^{-1} f_\ell - u_\ell^{obs} \right\|^2,$$

with

$$m(x) = m_0(x) + \sum_{j=1}^{K} \beta_j \phi_j(x).$$

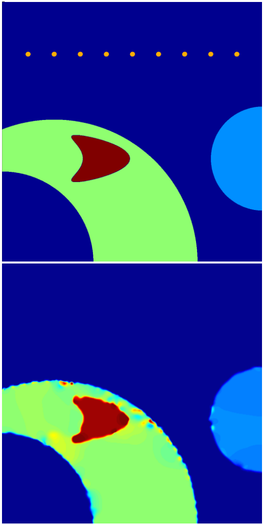
- Optimisation problem with just K control variables
- $K \ll N$, the number of the nodal control variables
- No extra regularisation term

Strategies:

Frequency stepping
Inexact Newton method
Regularisation: AEI

Algorithm:

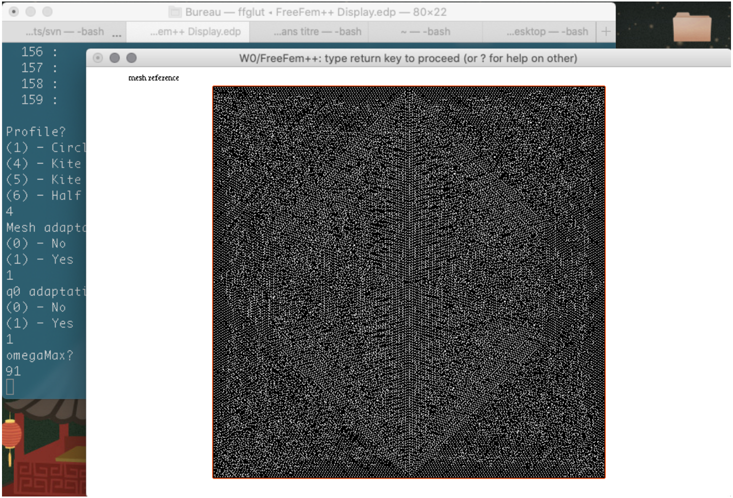
1. initialise $m^{(1)}(\vec{x}) \equiv 1$
2. compute $\phi_j^{(1)}$, $m_0^{(1)}$ and $\beta^{(1)}$ from $m^{(1)}$
3. for $\omega_k = \omega_1 < \ldots < \omega_n$
4. while $\|\nabla \mathcal{J}(m(\beta^{(k)}))\| > tol$
5. solve $\mathcal{B}p = -\nabla \mathcal{J}(m(\beta^{(k)}))$ (with few iterations of CG)
6. update $\beta^{(k)} \leftarrow \beta^{(k)} + \delta p$ (with line search criterion)
7. end while
8. compute $\phi_j^{(k+1)}$ and $m_0^{(k+1)}$ from $m(\beta^{(k)})$
9. update $\beta^{(k+1)} \leftarrow \beta^{(k)}$ in new basis
10. end for

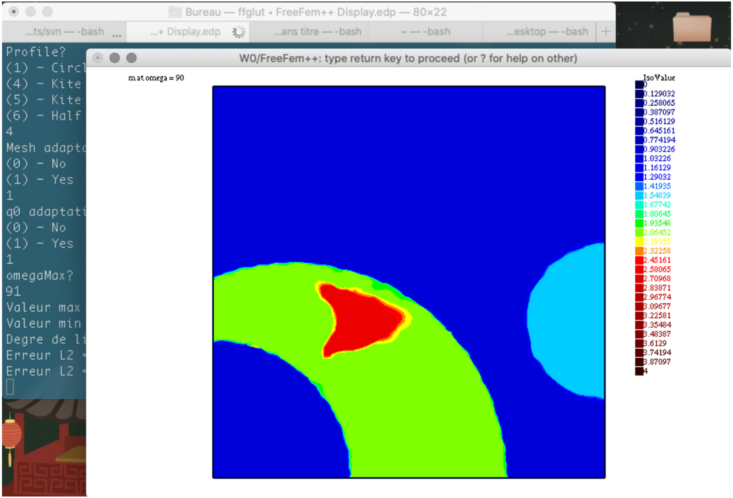


(Nahum, PhD thesis 2016)

Relative L^2-error: 4.65%

Adaptive Eigenspace Inversion
Numerical result – DEMO





Bureau — ffglut ◂ FreeFem++ Display.edp — 80×22
...ts/svn — -bash
...+ Display.edp
...ans titre — -bash
~ — -bash
...esktop — -bash
Profile?
(1) - Circl
(4) - Kite
(5) - Kite
(6) - Half
4
Mesh adapta
(0) - No
(1) - Yes
1
q0 adaptati
(0) - No
(1) - Yes
1
omegaMax?
91
Valeur max
Valeur min
Degre de li
Erreur L2
Erreur L2
W0/FreeFem++: type return key to proceed (or ? for help on other)
mat omega = 90
IsoValue
0
0.129032
0.258065
0.387097
0.516129
0.645161
0.774194
0.903226
1.03226
1.16129
1.29032
1.41935
1.54839
1.67742
1.80645
1.93548
2.06452
2.19355
2.32258
2.45161
2.58065
2.70968
2.83871
2.96774
3.09677
3.22581
3.35484
3.48387
3.6129
3.74194
3.87097
4

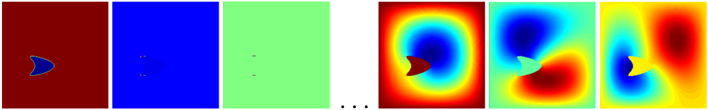
Conclusion

and open questions

Conclusion and open questions

My current work: understand why and how AEI works!

1. link between AEI and TV regularisation
2. choice of K to induce regularisation



Some answers in (Baffet, Grote and Tang, 2020)

3. application to microwave imaging for trees (with real data)

The End.