Imaging:
An Inverse Scattering Problem

Lecture 1

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Napier, 10-16 January 2021

Imaging

Douli, 23 weeks

Why are Inverse Problems worth considering?

Principle: from incomplete/partial or corrupted/noisy data, get information about the model,
e.g., ultrasound reflections to understand what is inside the human body

Applications: imaging, parameter estimation, etc...
e.g., half-life of a radioactive nuclei

Inverse (scattering) problems are everywhere:

- geophysics, e.g., deposit prospecting, like gas, oil
- medical imaging, e.g., ultrasound, MRI, photoacoustic
- non-destructive testing, e.g., crack or defect in material
- quality control in primary industry

PHYSICS TODAY

Physics Today > Volume 70, Issue 10 > 10.1063/PT.3.3740
01 October 2017 > page 94

QUICK STUDY

Kasper van Wijk is an associate professor and **Sam Hitchman** is a PhD candidate in the department of physics at the University of Auckland in New Zealand. Both are affiliated with the Dodd–Walls Centre for Photonic and Quantum Technologies.

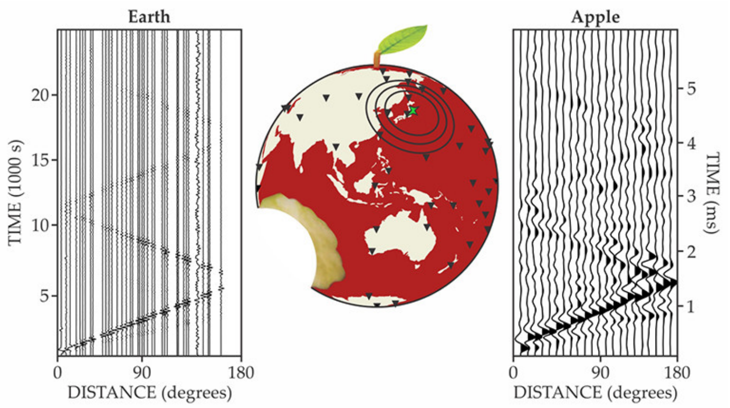
Apple seismology

Kasper van Wijk and Sam Hitchman

Just as an earthquake's seismic waves reveal properties of Earth's interior, elastic surface waves on an apple can tell us about what's going on inside the fruit.

An apple, like Earth, has a core at its center and a thin skin on the outside. In between is the apple's flesh, equivalent to Earth's mantle. Of course, a more careful comparison would uncover important differences between those spheroidal objects. For example, seismic waves reveal that Earth's core is made of a liquid outer core and a solid inner core, whereas the apple core contains derstanding of the depths of our planet that cannot be sampled via drilling.

A similar pattern in the right panel of figure 1 represents elastic waves on the surface of a Braeburn apple. The applequakes we measured were generated via thermoelastic expansion of the apple after a short pulse of laser light heated a small spot on the surface. We used a laser Doppler vibrometer to record



As an apple ages

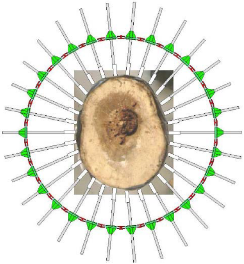
From our "seismic analysis" of waves in an apple, we can estimate average elastic properties. Young's modulus, for example, is closely related to the firmness index, a commonly used parameter in the apple industry to quantify apple firmness.

From van Wijk and Hitchman, **Apple Seismology**, Physics Today 70(10), 2017

Detection of disease in vine tree

Collaboration with:

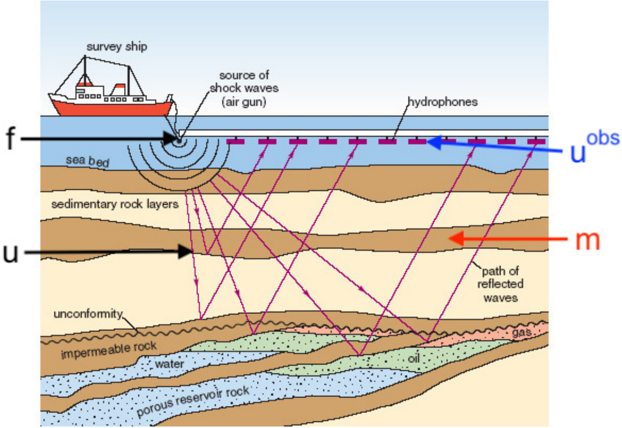
- Dr. Andrew Austin
  Senior Lecturer at Department of Electrical, Computer and Software Engineering, University of Auckland
- Dr. Mark Eltom
  Foundator of Vine Life Limited
- Dr. Ray Simpkin
  Lead Scientist at EMROD Limited from Callaghan Innovation



From Boero *et al.*, **Microwave Tomography for the Inspection of Wood Materials: Imaging System and Experimental Results**, IEEE Transaction on Microwave Theory and Techniques 66(7), 2018

Mathematical formulation

of the inverse problem



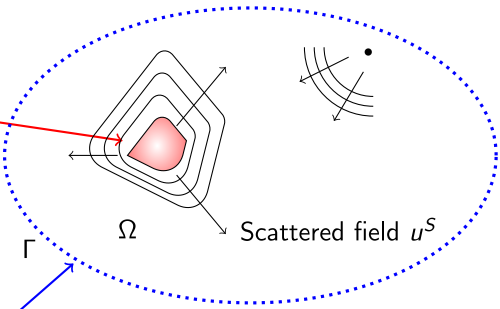
f
u
survey ship
source of
shock waves
(air gun)
hydrophones
u^obs
sea bed
sedimentary rock layers
m
path of
reflected
waves
unconformity
impermeable rock
water
gas
oil
porous reservoir rock

Ambient medium
wave propagation speed
c_0 known

Incident wave u^I
sent in the medium
generated by source f

Unknown inclusion
wave propagation speed
$c(x) \geq c_0$ non constant

model parameter: $m = c^2$



Total field $u^{obs} := u^I + u^S$ recorded on boundary Γ

Aim: Find m such that would "give" us the observations u^{obs}

In practice, make a guess for m and solve the wave equation.
How close to u^{obs} are the predictions u on Γ?

Inverse scattering problem:
Find m such that

$$\text{" } u^{pred} = u^{obs} \quad \text{on } \Gamma \text{ "}$$

or more precisely

$$\|u^{pred} - u^{obs}\|_{L^2(\Gamma)} = 0$$

Inverse scattering problem:
Find m such that

$$\|u^{pred} - u^{obs}\|_{L^2(\Gamma)} = 0$$

Terminology:

	Predictions from simulations	**Observations** from experiment
source f	given	given
medium m	known (guess)	unknown
wavefield u	unknown	known (acquired)
problem	forward	inverse

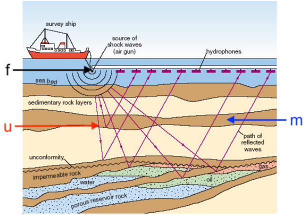
Question 1: What is the forward problem (PDE)?
Question 2: How to get the best guess for m?

Model: wave propagation phenomenon, e.g., Helmholtz equation

Given n_s sources f_ℓ inside a bounded region Ω.
For each source f_ℓ, the scattered field u_ℓ satisfies the Helmholtz equation

$$\begin{cases} -\nabla \cdot (m \nabla u_\ell) - \omega^2 u_\ell & = & f_\ell & \text{in } \Omega\,, \\ \dfrac{\partial u_\ell}{\partial n} & = & \dfrac{i\omega}{\sqrt{m}} u_\ell & \text{on } \partial\Omega. \end{cases}$$

- $m(x)$ is the squared medium velocity
- ω is the time frequency
- $\Omega \subset \mathbb{R}^d$, $d \leq 3$



in short

$$H(m) u_\ell \;=\; f_\ell\,, \quad \text{in } \Omega, \quad \ell = 1, \ldots, n_s$$

Forward problem: Compute the wavefield (prediction) u_ℓ, knowing the medium properties m and the source f_ℓ:

$$H(m)u_\ell \ = \ f_\ell\,, \quad \text{in } \Omega, \qquad \ell = 1, \ldots, n_s$$

Inverse problem: Given the data u_ℓ^{obs} and the source f_ℓ, find the medium properties m, such that:

$$\|u_\ell^{obs} \ - \ PH(m)^{-1} f_\ell\|_{L^2(\Gamma)} = 0\,, \qquad \ell = 1, \ldots, n_s$$

where P projector from Ω to Γ

Difficulty: An inverse problem is usually ill-posed!

Definition

A problem is well-posed if a solution *exists*, is *unique* and *stable* w.r.t. the data.

Why ill-posed?

- localised observations (in space and possibly in time)
- noise, rounding errors
- inaccurate model (PDE)

Therefore, we cannot guarantee

- existence
- uniqueness
- stability

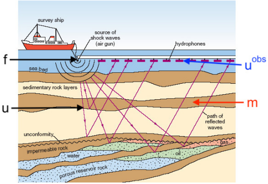
Solving the Inverse Problem

using Optimisation

Using optimisation [1], find $\hat{m}(x)$ s.t.

$$\hat{m} = \underset{m}{\operatorname{argmin}}\, \mathcal{J}(m), \qquad \text{with } \mathcal{J}(m) = \frac{1}{2} \sum_{\ell=1}^{n_s} \Big\| P \underbrace{H(m)^{-1} f_\ell}_{u_\ell} - u_\ell^{obs} \Big\|_{L^2(\Gamma)}^2$$

- $m(x)$ **unknown** squared medium velocity but known outside Ω
- n_s number of sources/shots
- u_ℓ^{obs} measurements at receivers location
- P projector from Ω to Γ



[1] E. Haber, U. Ascher and D. Oldenburg, *Inverse Problems (2000)*

From now, inverse scattering problem $\Leftrightarrow$ minimisation problem

$$\hat{m} = \underset{m}{\operatorname{argmin}}\, \mathcal{J}(m),$$

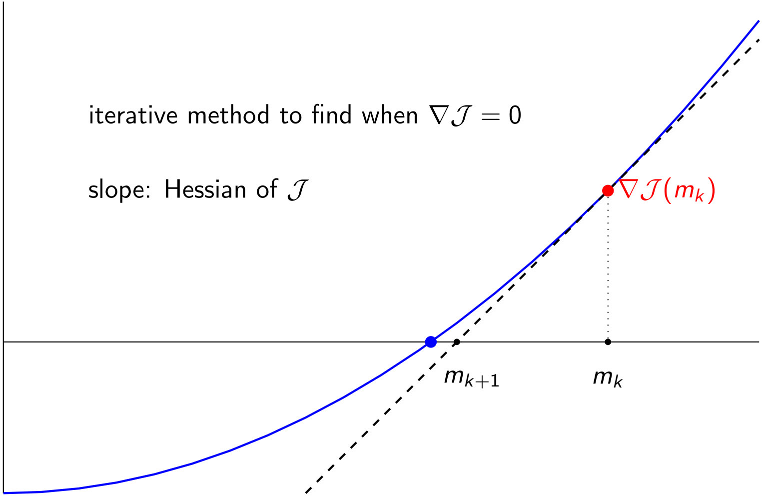
where $\mathcal{J}$ is a convex (quadratic) functional

Therefore, the minimisation is equivalent to

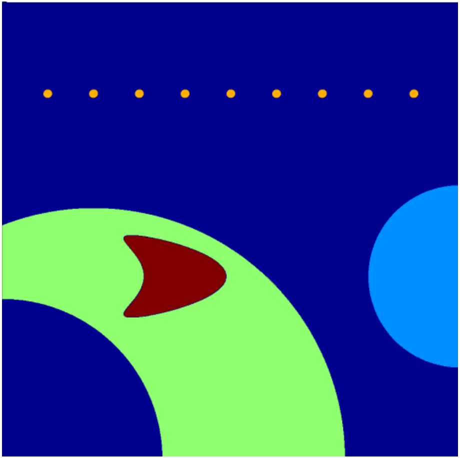
$$\nabla_m \mathcal{J}(\hat{m}) = 0$$

Strategy:

1. derive the gradient
2. use Newton's method


iterative method to find when $\nabla \mathcal{J} = 0$
slope: Hessian of $\mathcal{J}$
$\nabla \mathcal{J}(m_k)$
m_{k+1}
m_k

Example:

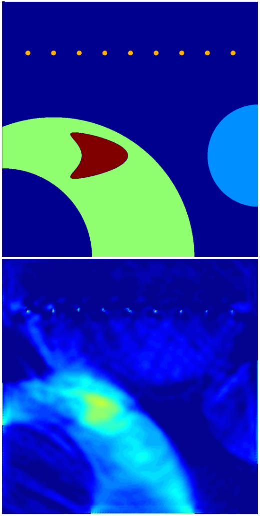


9 sources (orange dots), receivers on boundary

Newton's method
iterative process to update m
Hessian (or approximation) $\mathcal{B}$
tolerance ε

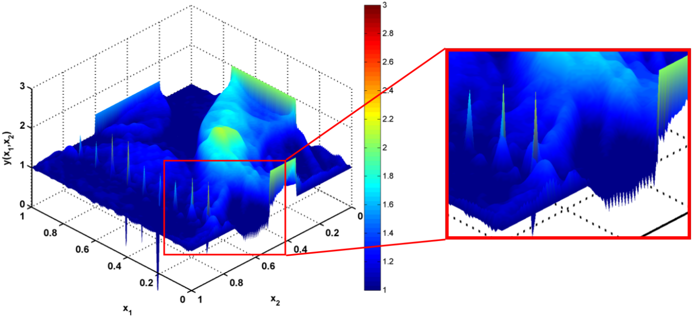
Algorithm:

1. initialize m
2. while $\|\nabla \mathcal{J}(m)\|_2 > \varepsilon$
3. solve $\mathcal{B}p = -\nabla \mathcal{J}(m)$
4. (direction of the variation)
5. update $m := m + \delta\, p$
6. (with line search or step size)
7. end



[1] E. Haber, U. Ascher and D. Oldenburg, *Inverse Problems* (2000)

[2] M. Grote, MG, U. Nahum, *Inverse Problems* (2017)



$\Longrightarrow$ **Regularisation**