Some of my work on buildings

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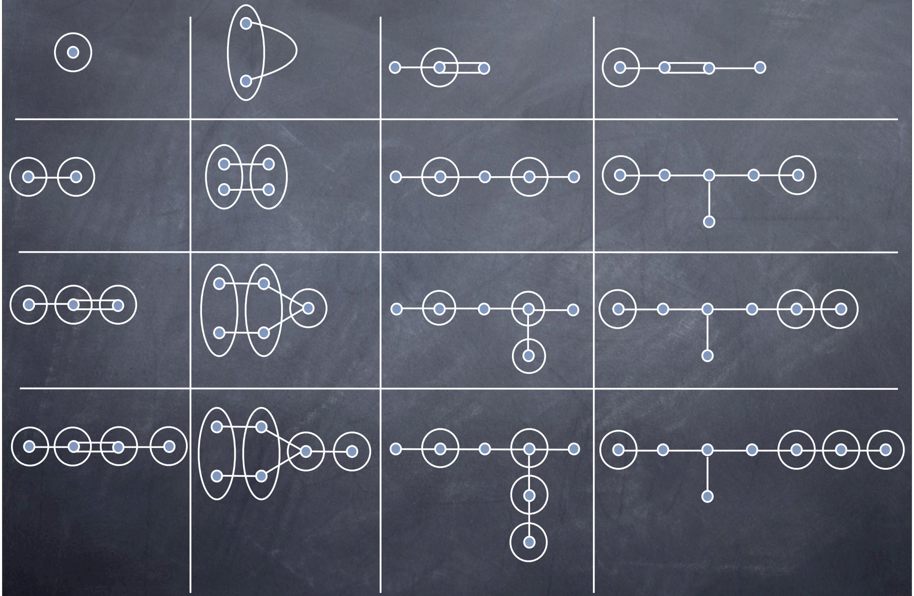
Freudenthal-Tits Magic Square



The second row of the Square

A_2 | $A_2 \times A_2$ | A_5 | E_6

Veronese Variety | Segre Variety | Grassmann Variety | 26-dim $E_{6,1}$ Variety

Projective planes over (quadratic, not necessarily associative) algebras with zero divisors

Algebraic description of the Severi varieties

- Veronese: image of the map
  $\mathbb{P}^2(\mathbb{K}) \to \mathbb{P}^5(\mathbb{K}) : (x, y, z) \mapsto (x^2, xy, xz, y^2, yz, z^2)$
- Segre: image of the map $\mathbb{P}^2(\mathbb{K}) \times \mathbb{P}^2(\mathbb{K}) \to \mathbb{P}^8(\mathbb{K})$:
  $(x, y, z) \times (x', y', z') \mapsto (xx', xy', \cdots, yz', zz')$
- The *line Grassmannian variety* $\mathcal{G}_{5,1}(\mathbb{K})$ of $\mathbb{P}^5(\mathbb{K})$ is the set of points of $\mathbb{P}^{14}(\mathbb{K})$ obtained by taking the images of all lines of $\mathbb{P}^5(\mathbb{K})$ under the Plücker map
  $$\rho(\langle(x_0, x_1, \ldots, x_m), (y_0, y_1, \ldots, y_m)\rangle) = \left(\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \right)_{0 \le i < j \le m}.$$
- The Cartan variety in $\mathbb{P}^{26}(\mathbb{K})$ has a more complicated algebraic description, linked to the 27-dimensional E_6 module.

Axiomatic setup for the second row

- X: point set spanning $\mathbb{P}^N(\mathbb{K})$, $N \in \mathbb{N} \cup \{\infty\}$.
- Ξ: collection of $(d+1)$-dimensional subspaces of $\mathbb{P}^N(\mathbb{K})$, where $|\Xi| \geq 2$ and $1 \leq d < \infty$, such that for each $\xi \in \Xi$, the set $X(\xi) := X \cap \xi$ is a non-degenerate quadric or ovoid generating ξ.
- The *tangent space at $x \in X$ to X* is the subspace T_x generated by the tangent spaces to quadrics and singular lines.

Definition

We say that the pair (X, Ξ) is an *axiomatic Veronese variety of type d* (or, briefly, an AVV of type d) if it satisfies the following axioms:

- Any pair of points $x_1, x_2 \in X$ lies in at least one element of Ξ;
- if $\xi_1, \xi_2 \in \Xi$ are distinct, then $\xi_1 \cap \xi_2 \subseteq X$;
- for each $x \in X$, $\dim T_x \leq 2d$.

Second row of the Freudenthal-Tits magic square

Theorem (JS-Van Maldeghem, partly De Schepper, Krauss)

An AVV of type d in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:

- *The quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$ ($N = 5$);*
- *the Segre variety $\mathcal{S}_{1,2}(\mathbb{K})$ ($N = 5$), $\mathcal{S}_{1,3}(\mathbb{K})$ ($N = 7$) or $\mathcal{S}_{2,2}(\mathbb{K})$ ($N = 8$);*
- *the line Grassmannian variety $\mathcal{G}_{4,1}(\mathbb{K})$ ($N = 9$) or $\mathcal{G}_{5,1}(\mathbb{K})$ ($N = 14$);*
- *the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$, and then $N = 15$;*
- *the (Cartan) variety $\mathcal{E}_{6,1}(\mathbb{K})$, and then $N = 26$;*
- *the Veronese variety $\mathcal{V}_2(\mathbb{K}, \mathbb{A})$, for some d-dimensional quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$. Moreover, if $\mathrm{char}(\mathbb{K}) \neq 2$, then $d \in \{1, 2, 4, 8\}$. Here, $N = 3d + 2$ where $d = 2^\ell$.*

Severi's theorem

Theorem (Severi (1901))

Every irreducible smooth non-degenerate, secant-defective surface in $\mathbb{P}^5(\mathbb{C})$ is projectively equivalent to the quadric Veronese variety.

Secant defective: $SX = \overline{\cup_{x_1 \neq x_2, x_i \in X} \langle x_1, x_2 \rangle} \neq \mathbb{P}^5(\mathbb{C})$

The quadric Veronese variety is secant-defective since identifying points on it with symmetric matrices A of rank 1, secants can be identified with matrices of rank at most 2, hence in this case SX is the cubic hypersurface given by $\det(A) = 0$.

Dale proved a characteristic p version of Severi's theorem in 1985.

Zak's theorem

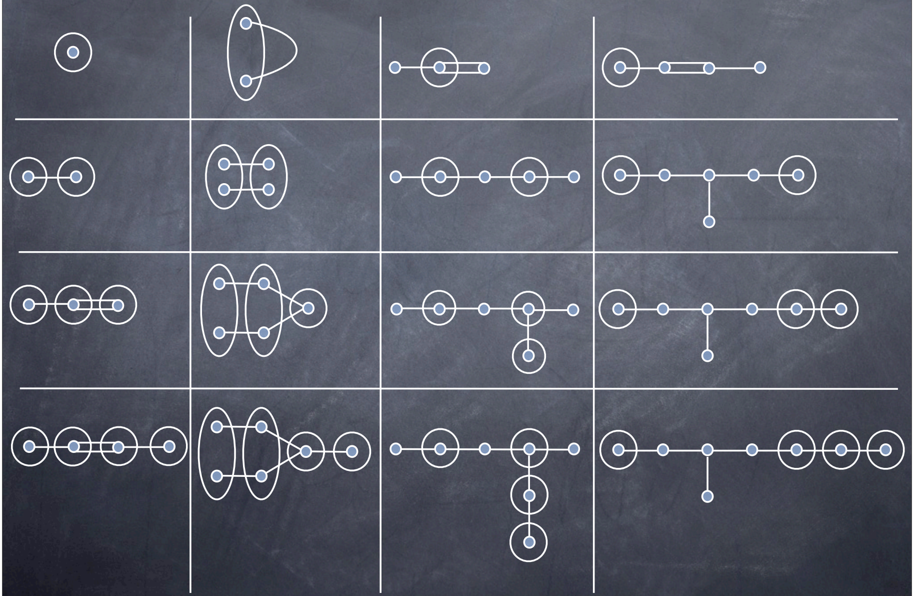
Let X be a smooth irreducible non-degenerate projective variety of dimension d over an algebraically closed field of characteristic zero.

Theorem

If X is secant defective then $N \geq \frac{3}{2}d + 2$. Moreover if equality occurs then X is either the Veronese variety in $\mathbb{P}^5 (d = 1)$, the Segre variety in $\mathbb{P}^8 (d = 2)$, the line Grassmannian in $\mathbb{P}^{14} (d = 4)$ or the Cartan variety in $\mathbb{P}^{26} (d = 8)$.

- $d = 1$: Severi (1901) and $d = 2$: Scorza (1908) and conjectured in (1979) by Griffiths-Harris, Fujita-Roberts.
- Case of equality essentially equivalent to Jacobson's classification of Jordan algebras over algebraically closed fields.
- The Severi varieties correspond to the split composition algebras.
- Zak follows from (the split case of) our theorem.

Freudenthal-Tits Magic Square



$\mathbb{R}$-trees

A metric space T is called a *tree* (or $\mathbb{R}$-tree) if it satisfies

- (T1) For any two points $x, y \in T$, there is a unique geodesic $\gamma : [0, d(x, y)] \to T$ with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. We put $[x, y] = \gamma([0, d(x, y)])$.
- (T2) If $0 < r < s$ and if $\gamma : [0, s] \to T$ is an injection such that $\gamma|_{[0,r]}$ and $\gamma|_{[r,s]}$ are geodesics, then γ is a geodesic.

Theorem (Serre, Morgan-Shalen)

Let G be a finitely group acting on a tree such that every element of G fixes a point. Then G has a global fixed point.

An end of an $\mathbb{R}$-tree T is an equivalence class of rays in T, with two rays identified if their intersection is a ray.

From the building at infinity to the global building

Theorem (Kramer-JS)

Let X be a thick simplicial Euclidean building and let Δ be the spherical building at infinity (with respect to the complete apartment system of X). Let G be a group of type-preserving automorphisms of X. Then the action of G on Δ is strongly transitive if and only if the action of G on X is strongly transitive.

- Special case: Caprace and Ciobotaru assuming in addition that X is locally finite and that G is a closed subgroup of $\mathrm{Aut}(X)$.
- Special case of trees: Burger and Mozes.
- Ciobotaru and Rousseau proved an analogon of this special case in the more general context of hovels.

Klein's criterion aka the Ping-pong Lemma

Theorem

Let G be a group acting on a set X, and let $\Gamma_1, \Gamma_2 \leq G$ with $|\Gamma_1| \geq 3$ and $|\Gamma_2| \geq 2$ and let $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$. Assume there exist non-empty sets $X_1, X_2 \subset X$ with $X_2 \not\subset X_1$ such that $\gamma(X_2) \subset X_1 \ \forall \gamma \in \Gamma_1, \gamma \neq 1$ and $\gamma(X_1) \subset X_2 \ \forall \gamma \in \Gamma_2, \gamma \neq 1$. Then $\Gamma = \Gamma_1 \star \Gamma_2$.

Application: The Sanov subgroup in $\mathrm{SL}(2, \mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is a free group. It has index 12 in $\mathrm{SL}(2, \mathbb{Z})$.

Exercise: Can you find subsets of $\mathbb{R}^2$ to make ping-pong work?

Tits alternatives

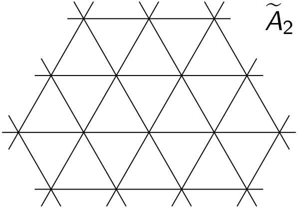
A group G is said to satisfy the Tits alternative if every subgroup of G is either virtually soluble or contains a free subgroup of rank 2.

Theorem (Tits)

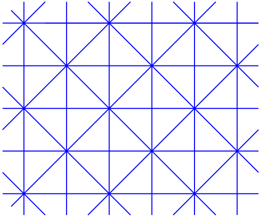
Finitely generated linear groups satisfy the Tits alternative.

- The ping-pong lemma is a crucial ingredient in the proof.
- A finitely generated group has polynomial growth if and only if it is virtually nilpotent (Gromov).
- Open question: Do $\mathrm{CAT}(0)$ groups, i.e. groups which act geometrically on $\mathrm{CAT}(0)$ spaces satisfy the Tits alternative?

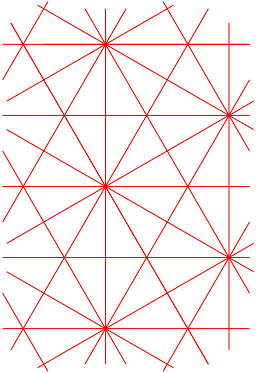
Apartments in euclidean buildings of dimension 2



$\widetilde{A}_2$



$\widetilde{B}_2$



$\widetilde{G}_2$

A local to global result for groups acting on $\mathbb{R}$-buildings

Theorem (JS, Struyve and Thomas)

Let G be a finitely generated group of automorphisms of an affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$. If every element of G fixes a point of X, then G fixes a point of X.

By considering finitely generated subgroups and using a theorem of Caprace and Lytchak we can extend to non-finitely generated groups as follows.

Corollary

Suppose a group G acts on a complete affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$ such that every element of G fixes a point of X. Then G fixes a point in the bordification $\overline{X} = X \cup \partial X$ of X.

Reductions

Let G be finitely generated acting on X which is an $\mathbb{R}$-building of type $\tilde{A}_2$ or $\tilde{C}_2$. Then we may assume the following

- Every point of X is a special vertex.
- X is metrically complete (passing to the ultrapower).
- G is type-preserving (finite-index subgroup + Bruhat-Tits fixed point theorem)

Lemma

Let G be a group acting isometrically on a complete 2-dimensional Euclidean building X. If $A := \mathrm{Fix}(G_A)$ and $B := \mathrm{Fix}(G_B)$ are two nonempty fixed point sets of finitely generated subgroups G_A and G_B both of whose isometries are all elliptic, then there exist points $\alpha^\star \in A$ and $\beta^\star \in B$ such that $d(\alpha^\star, \beta^\star) = d(A, B)$.

The ideas of the proof

Lemma

Suppose G has two proper finitely generated subgroups G_0 and G_1 such that the respective fixed point sets $B_0 := \mathrm{Fix}(G_0)$ and $B_1 := \mathrm{Fix}(G_1)$ are nonempty and disjoint. Then G contains a hyperbolic element.

- Pick $a_0 \in B_0, a_1 \in B_1$ such that $d(a_0, a_1) = d(B_0, B_1)$.
- Find a "good" $g_1 \in G_1$ and define $a_2 = g_1 a_0$, $G_2 = g_1 G_0 g_1^{-1}$ with fixed set $B_2 = g_1 B_0$. Obtain "good" g_2 and define $a_3 = g_2 a_1$.
- Define $g = g_2 g_1$ and inductively $a_i = g a_{i-2}$ and $g_i = g g_{i-2} g^{-1}$.
- For all $i \geq 1$, we have $a_i, a_{i+1} \in \mathcal{A}_i$, $\xi \in \partial\mathcal{A}_i$ and $\angle_{a_i}(\xi, a_{i+1}) \geq \frac{2\pi}{3}$.
- Show g has unbounded orbit using Busemann functions of geodesic rays $b_\gamma(x) := \lim_{t\to\infty}[d(x, \gamma(t)) - t]$.

Hausdorff distance I

Two (nonempty) subsets U, V of a metric space X have *Hausdorff distance at most r* if

$$U \subseteq B_r(V) \quad \text{and} \quad V \subseteq B_r(U).$$

In this case we write $Hd(U, V) < r$. We define for $U, V \subseteq X$ the Hausdorff distance as

$$Hd(U, V) = \inf\{r > 0 \mid Hd(U, V) < r\}$$

For example, a nonempty subset is bounded if and only if it has finite Hausdorff distance from some point.

Hausdorff distance II

More generally, we say that V *dominates* U if $U \subseteq B_r(V)$ for some $r > 0$, and we write then

$$U \subseteq_{Hd} V.$$

This defines a preorder on the subsets of X.

We call two Weyl simplices $a, a' \subseteq X$ *Hausdorff equivalent* if they have finite Hausdorff distance. The equivalence class of a is denoted ∂a. The preorder $\subseteq_{Hd}$ induces a partial order on these equivalence classes.

Related work I

- Parreau: similar result for subgroups Γ of connected reductive groups $\mathcal{G}$ over certain fields F, where Γ is generated by a bounded subset of $\mathcal{G}(F)$ and the action is on the completion of the associated Bruhat–Tits building.
- Breuillard and Fujiwara: quantitative version of Parreau's result for discrete Bruhat–Tits buildings and asked whether their result holds for the isometry group of an arbitrary affine building.
- Leder and Varghese (using work of Sageev): similar result for groups acting on finite-dimensional $\mathsf{CAT}(0)$ cube complexes.
- False for infinite-dimensional $\mathsf{CAT}(0)$ cubical complexes: Osajda using actions of infinite free Burnside groups.

Related work II

Recently, Norin, Osajda and Przytycki proved:

Theorem

Let X be a $\mathrm{CAT}(0)$ *triangle complex and let G be a finitely generated group acting on X with no global fixed point. Assume that either each element of G fixing a point of X has finite order, or X is locally finite, or X has rational angles. Then G has an element with no fixed point in X.*

Remark

Note that $\mathrm{CAT}(0)$ *triangle complexes include discrete buildings of type $\tilde{G}_2$. They use Helly's theorem together with sophisticated results including Masur's theorem on periodic trajectories in rational billiards, and Ballmann and Brin's methods for finding closed geodesics in 2-dimensional locally* $\mathrm{CAT}(0)$ *complexes.*

Non-discrete Euclidean buildings

A metric space X with a collection $\mathcal{F}$ of charts (isometric injections of a Euclidean model space $\mathbb{A}$ into X) is a *Euclidean building* if

(EB1) For all $\varphi \in \mathcal{F}$ and $w \in W\mathbb{R}^n$, $\varphi \circ w$ is in $\mathcal{F}$.

(EB2) The charts are W-compatible, more precisely If $f, f' \in \mathcal{F}$, then $X = f^{-1}(f'(\mathbb{A}))$ is a closed and convex subset of $\mathbb{A}$, and $f_{\restriction}X = f' \circ w_{\restriction}X$ for some $w \in W$.

(EB3) Any two points $x, y \in X$ are contained in some affine apartment.

(EB4) If $a, b \subseteq X$ are Weyl chambers, then there is an affine apartment A such that the intersections $A \cap a$ and $A \cap b$ contain Weyl chambers.

(EB5) If A_1, A_2, A_3 are affine apartments which intersect pairwise in half spaces, then $A_1 \cap A_2 \cap A_3 \neq \emptyset$.

The spherical building at infinity

Let $\partial_{\mathcal{A}} X$ denote the set of all equivalence classes of Weyl simplices, partially ordered by domination $\subseteq_{Hd}$. For every affine apartment A, the poset ∂A consisting of the Weyl simplices contained in A may be viewed as a sub-poset of $\partial_{\mathcal{A}} X$.

Proposition The poset $\partial_{\mathcal{A}} X$ is a spherical building. The map $A \mapsto \partial A$ is a one-to-one correspondence between the affine apartments in X and the apartments of the spherical building $\partial_{\mathcal{A}} X$.