An introduction to Tits buildings

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19th century revolution in geometry and symmetry

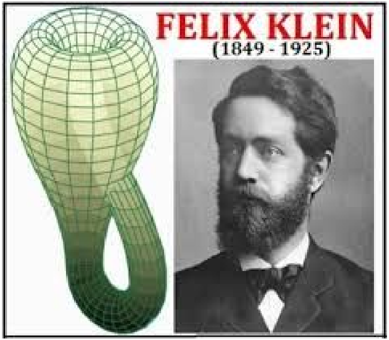


Figure: Klein's Erlanger Programm: study geometries by means of their symmetries (1872), later generalised by Elie Cartan.

Figure: Sophus Lie

Sophus Lie's theory of continuous symmetry

- Lie group: group which is also a differentiable manifold with continuous inversion and multiplication operations.
- They form natural models for continuous symmetry, e.g. rotational symmetries in 3 dimensions are described by $\mathrm{SO}(3)$.
- Lie algebra: linear object that can be canonically attached to a Lie group and contains a lot of information about it.
- Semisimple Lie algebras over an algebraically closed field of characteristic zero are completely classified by their root system (Killing-Cartan), which are in turn classified by Dynkin diagrams.

From Lie groups to algebraic groups

Figure: Claude Chevalley

Theorem

Let k be an algebraically closed field of characteristic zero. Then every almost simple linear algebraic group is isogenous to exactly one of the following

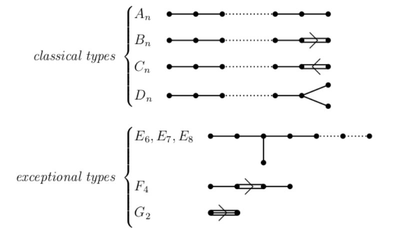
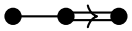


Figure: *A_n isogenous to SL_{n+1}, B_n to $\mathrm{SO}(2n+1)$, C_n to Sp_{2n} and D_n to $\mathrm{SO}(2n)$.*

Buildings: A geometric theory of algebraic groups

Figure: Jacques Tits (Abel prize 2008, Wolf Prize 1993) gave a converse to Klein's Erlanger programm: study groups by means of their geometries

Coxeter groups as abstraction of Weyl groups

- $W = \langle S | (s_i s_j)^{m(s_i,s_j)} = 1 \rangle$ where $m(s_i, s_i) = 1$ (so all of the generators are involutions), $m(s_i, s_j) = m(s_j, s_i)$ and $2 \leq m(s_i, s_j) \leq \infty$ for $i \neq j$. We always assume S is finite.
- *Coxeter Diagram*: Draw one node for each generator s_i and then join s_i to s_j (labeled) if and only if $m(s_i, s_j) \geq 3$. These are intimately linked with the Dynkin diagram.
-  is $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$.
-  is $W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^4 = (su)^2 = 1 \rangle$.
-  is $W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \rangle$.

The Coxeter complex of a Coxeter system (W, S)

- $W_J := \langle J \rangle$ ($J \subseteq S$) is a *standard subgroup*.
- $\Sigma(W, S)$: poset of standard cosets in W, ordered by reverse inclusion. Thus $B \leq A$ in Σ if and only if $A \subseteq B$ as subsets of W, and we call B a *face* of A.
- $\Sigma(W, S)$ is called the *Coxeter complex* associated to (W, S).

Example: $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$. The standard subgroups are $1, \{1, s\}, \{1, t\}$ and W and for example the faces of $\{t\}$ are $\{1, t\}$ and $\{t, ts\} = t\{1, s\}$.

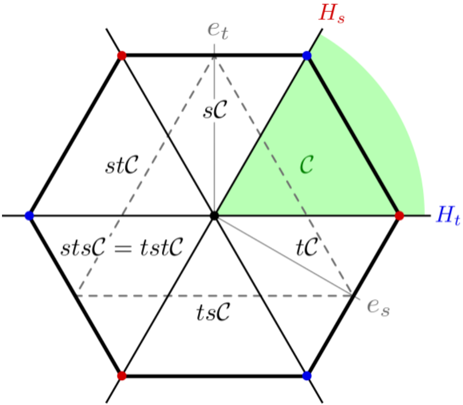
The Tits representation of a Coxeter group

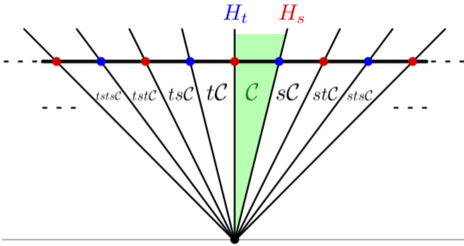
Definition

Let V be a real vector space with basis $\{(e_i)_{i=1}^{n}\}$. Define a symmetric bilinear form on V by $B(e_i, e_j) = -\cos(\pi/m_{ij})$. The *geometric representation of* W on V is defined by $s(v) = v - 2B(v, e_s)e_s$.

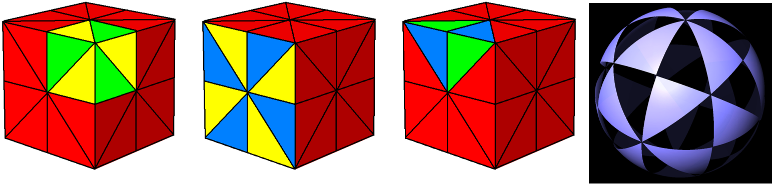
- No information is lost (i.e. representation is faithful Tits).
- B positive definite if and only if W is finite. We say W is *spherical*.
- If B is positive semi-definite of corank 1 we say W is *Euclidean*.
- Coxeter groups are linear over a field of characteristic zero and by our finite generation assumption thus virtually torsion-free (Selberg 1960) and residually finite (Malcev 1940).

Two classic examples of Coxeter complexes



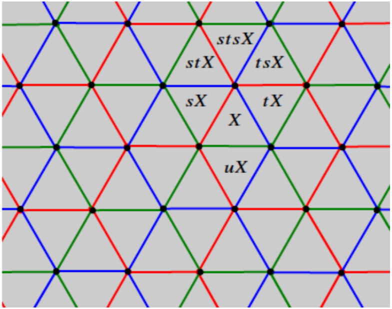


Another spherical example coming from the cube



- $EFE = FEF$, $VEVE = EVEV, VF = FV$
- Spherical Coxeter complex
- Dynkin diagram 

Another Euclidean Coxeter complex



- W: group of isometries of the plane generated by the (affine) reflections with respect to the sides of an equilateral triangle.
- Example of a Euclidean reflection group.
- $W := \langle s, t, u; s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \rangle$
- Coxeter complex: plane tiled by equilateral triangles.

Buildings and groups

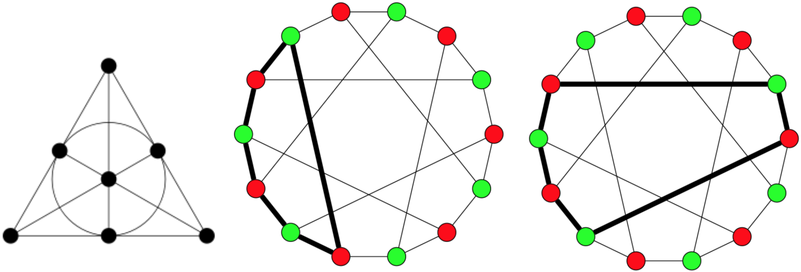
A *building* is a simplicial complex Δ that can be expressed as the union of sub complexes Σ (called *apartments*) satisfying the following axioms

(B0) Each apartment Σ is a Coxeter complex.

(B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing them.

(B2) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

A building Δ associated to a vector space V



- V: $n \geq 2$-dim vector space over an arbitrary field k.
- $\mathbb{P}(V)$: non-zero subspaces of V.
- $\Delta = \Delta(V)$: flag complex of $\mathbb{P}(V)$; thus the simplices are chains $V_1 < V_2 < \cdots < V_k$ of nonzero proper subspaces of V. The maximal simplices (*chambers*) are the chains $V_1 < \cdots < V_{n-1}$ such that $\dim V_i = i$.

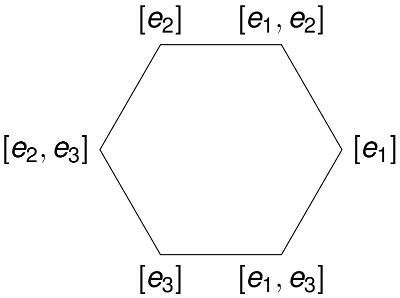
Strongly transitive actions on buildings

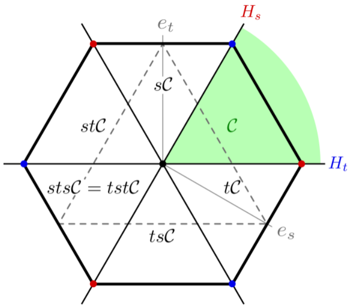
Assume Δ is a simplicial building of type (W, S) with a type preserving action of G on it. Suppose $\mathcal{A}$ is a G-invariant system of apartments. We say the G-action is *strongly transitive* (with respect to $\mathcal{A}$) if G acts transitively on the set of pairs (Σ, C) consisting of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$.

Assume the G-action is strongly transitive, and choose an arbitrary pair (Σ, C) as in the definition, we will refer to Σ as the *fundamental apartment* and to C as the *fundamental chamber*.

The action of special subgroups B, N and T

- Σ: that of standard basis, C: edge joining $[e_1]$ to $[e_1, e_2]$.
- $B := \{g \in G \mid gC = C\}$: upper triangular matrices
- $N := \{g \in G \mid g\Sigma = \Sigma\}$: monomial matrices
- $T := \{g \in G \mid g$ fixes Σ pointwise $\}$: diagonal matrices
- $W = N/T = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$ where $s = (12), t = (23)$.





Abstracting: Groups with BN-pair

A group G has a BN pair of subgroups B and N if the following hold:

- $G = \langle B, N \rangle$
- $T := B \cap N \leq N$
- $W := N/T$ with set of generators S
- For $s \in S$ and $w \in W$ one has $sBw \subset BswB \cup BwB$
- For $s \in S$ one has $sBs^{-1} \not\leq B$

W is called the *Weyl group* and (G, B, N, S) a *Tits system*.

Side note: Bruhat decomposition: $G = \coprod_{w \in W} BwB$.

From groups with BN-pair to buildings and back

Theorem

Given a BN-pair in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta = \Delta(G, B)$ that admits a strongly transitive G-action such that B is the stabiliser of a fundamental chamber and N stabilises a fundamental apartment and is transitive on its chambers.

Theorem

Suppose a group G acts strongly transitively on a thick building Δ with fundamental apartment Σ and fundamental chamber C. Let B be the stabiliser of C, and let N be a subgroup of G that stabilises Σ and is transitive on the chambers of Σ. Then (B, N) is a BN-pair in G and Δ is canonically isomorphic to $\Delta(G, B)$.

The classification of spherical buildings

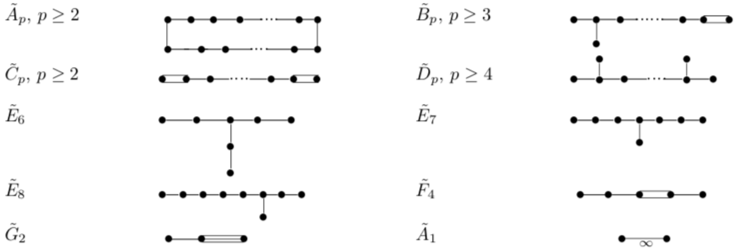
Theorem (Tits ’74)

Thick, irreducible, spherical buildings of rank at least ≥ 3 are either

- *Classical buildings (associated to classical groups), or*
- *Algebraic buildings (associated to algebraic groups), or*
- *Mixed buildings (associated to mixed groups).*

Restriction to rank at least three is needed, as there are free constructions in rank two. Moreover, classifying finite buildings of type A_2 is equivalent to classifying finite projective planes, a well-known problem which is out of reach.

Classification of Euclidean buildings



- Euclidean building of dimension at least three is a Bruhat-Tits building (Tits ’86).
- Building at infinity of Bruhat-Tits building is Moufang.
- Tits-Weiss: Classification of Moufang polygons.
- Artin-Zorn: Every finite alternative division ring is a field.

Bruhat-Tits buildings

- Introduced to study reductive algebraic groups over valued fields with not necessarily discrete valuation.
- Important subclass when valuation is discrete (seen via geometric realization): simplicial Euclidean buildings (only ones known before Bruhat-Tits '72)
- Let $\mathbb{L}$ be a locally compact, non-discrete topological field. Then $\mathbb{L}$ is $\mathbb{R}$, $\mathbb{C}$, or a finite extension of either $\mathbb{Q}_p$ or $K = \mathbb{F}_p((t))$.

Theorem (Martin,JS,Steinke,Struyve)

The Bruhat-Tits building is metrically complete if and only if the associated (skew) field is spherically complete, up to certain cases involving infinite-dimensionality and residue characteristic two.

Discrete valuation (p-adic valuation)

A discrete valuation on $\mathbb{K}$ ($\mathbb{Q}$) is a surjective homomorphism $\nu : \mathbb{K}^\star \to \mathbb{Z}$ satisfying

$$\nu(x+y) \geq \min(\nu(x), \nu(y))$$

$x \in \mathbb{Q}^\star$ written uniquely as $x = p^n u$, p-adic valuation: $\nu(x) = n$.

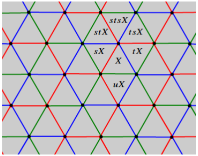
- $A := \{x \in \mathbb{K} \mid \nu(x) \geq 0\}$ is a *discrete valuation ring*.
  fractions a/b with b not divisible by p.
- $\mathbb{K}$ is the field of fractions of A.
- Uniformiser: π such that $\nu(\pi) = 1$. for $\mathbb{Q} : \pi = p$
- Residue field: $k = A \backslash \pi A$, for $\mathbb{Q} : k = \mathbb{F}_p$

The p-adic numbers $\mathbb{Q}_p$.

- from discrete valuation ν define the p-adic absolute value $|x| := p^{-\nu(x)}$ for $x \in \mathbb{K}$.
- Setting $d(x, y) := |x - y|$ yields an ultrametric on $\mathbb{K}$, i.e. $d(x, z) \leq \max(d(x, y), d(y, z))$
- form completion $\hat{\mathbb{K}}$ by formally adjoining limits of Cauchy sequences (similar to how you get $\mathbb{R}$ from $\mathbb{Q}$).
- The completion of $\mathbb{Q}$ with respect to the p-adic valuation is $\mathbb{Q}_p$.
- Can also be defined purely algebraically using inverse limits.

Discrete valuations yield a second BN pair for $\mathrm{SL}_n(\mathbb{K})$

- first observed by Matsumoto and Iwahori then vastly generalised by Bruhat and Tits.
- B: inverse image in $\mathrm{SL}_n(A)$ of upper triangular matrices in $\mathrm{SL}_n(k)$.
- N: monomial subgroup of $\mathrm{SL}_n(\mathbb{K})$.
- $T = B \cap N$ is diagonal subgroup of $\mathrm{SL}_n(A)$, conjugation action of N on T permutes the diagonal entries.
- $1 \to T(\mathbb{K})/T(A) \to W := N(\mathbb{K})/T(A) \to N(\mathbb{K})/T(\mathbb{K}) \to 1$.
- $W \cong (\mathbb{K}^\star/A^\star)^{n-1} \rtimes S_n$.

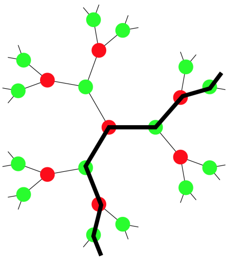


The Bruhat-Tits tree (=BT building of dimension 1)

- lattice: $L = Ae_1 \oplus Ae_2$.
- Call two A-lattices *equivalent* in $\mathbb{K}^2$ if $L = \lambda L'$ for some $\lambda \in \mathbb{K}^*$.
- Type of $[[f_1, f_2]]$ as $v(\mathrm{det}(f_1, f_2)) \mod 2$.
- Distinct lattice classes Λ, Λ' are *incident* if they have representatives that satisfy $\pi L < L' < L$.
- This relationship is symmetric since $\pi L' < \pi L < L'$.
- Graph: Vertices as lattice classes, edges via incidence. This is a tree, called the **Bruhat-Tits tree**.

Relating the Bruhat-Tits tree to the second BN pair

- C: edge given by $[e_1, e_2]$ and $[e_1, \pi e_2]$
- Stabiliser of $[e_1, e_2]$ = $\mathrm{SL}(2, A)$, stabiliser of $[e_1, \pi e_2]$ is $g\mathrm{SL}(2, A)g^{-1}$ where $g = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$, their intersection is B.
- Fundamental apartment obtained by applying N to C, we get $[[\pi^a e_1, \pi^b e_2]], a, b \in \mathbb{Z}$. Arbitrary apartment $g\Sigma$ is similar but with e_1, e_2 replaced by an arbitrary basis of $\mathbb{K}^2$.



The Bruhat-Tits tree was crucial to the construction of Ramanujan graphs by Lubotzky-Philips-Sarnak and Bruhat-Tits buildings are used to construct high-dimensional expanders.

An application of the Bruhat-Tits tree

Theorem (Ihara (Serre))

Every discrete torsion-free subgroup Γ *of* $\mathrm{SL}(2, \mathbb{Q}_p)$ *is free.*

- A group acting freely on a tree (no inversions, trivial point stabilisers) is a free group (Bass-Serre)
- Note that the action of $\mathrm{SL}(2, \mathbb{Q}_p)$ on the Bruhat-Tits tree is type-preserving so there are no edge inversions.

Assume thus $H \leq \Gamma$ fixes a vertex of the Bruhat-Tits tree

- H is bounded and hence relatively compact, hence compact
- H is compact and discrete, hence finite, thus trivial.

Euclidean Buildings are examples of $\mathrm{CAT}(0)$-spaces

- Given x, y, z in X, the triangle inequality implies there is a *comparison triangle* in the Euclidean plane $\mathbb{R}^2$ (unique up to an isometry of $\mathbb{R}^2$).
- Given a geodesic $[x, y]$ and a point $p = p_t \in [x, y]$, there is a corresponding point $\bar{p} = \bar{p}_t$ on the line segment $[\bar{x}, \bar{y}]$ in $\mathbb{R}^2$.
- A metric space is $\mathrm{CAT}(0)$ if for any $x, y \in X$ there is a geodesic $[x, y]$ such that: For all $p \in [x, y]$ and all $z \in X$ one has $d_X(z, p) \leq d_{\mathbb{R}^2}(\bar{z}, \bar{p})$.
- Examples include: Euclidean spaces, Hilbert spaces, Riemannian symmetric spaces of non-positive curvature, Euclidean buildings.
- Let X be a locally compact $\mathrm{CAT}(0)$ space of geometric dimension n. If any two points are contained in a common n-flat, then X is the metric realisation of a Euclidean building (Kleiner).

Group actions on CAT(0) spaces

Elie Cartan: If G is a compact group of isometries of a complete simply connected Riemannian manifold M of nonpositive curvature, then G fixes a point of M.

Theorem (Bruhat-Tits fixed point theorem)

Let G be a group of isometries of a complete CAT(0) space X. If G stabilises a nonempty bounded subset of X, then G fixes a point of X.

Application: Every compact subgroup of $\mathrm{SL}_(n, R)$ is conjugate to a subgroup of $\mathrm{SO}_n(\mathbb{R})$ using symmetric space and can obtain a p-adic analogue from the Bruhat-Tits building.

Serre' proof of Bruhat-Tits fixed point theorem

Let X be a metric space, A a non-empty bounded subset.

- $r(x, A) := \sup_{a \in A} d(x, a)$
- *Circumradius* of A; $r(A) := \inf_{x \in X} r(x, A)$.
- If $r(A) = r(x, A)$ for some $x \in X$, then x is a *circumcenter* of A.

Theorem (Serre)

If X is a complete $\mathrm{CAT}(0)$ space, then every non-empty bounded subset A admits one and only one circumcenter.

BN pair for an algebraic group

- $G(k)$: k-rational points of a (connected) reductive linear algebraic group G, T: maximal k-split torus, N: normaliser in G of G.
- Grothendieck: Any smooth connected affine group G over a field k contains a k-torus T such that $T_{\bar{k}}$ is maximal in $G_{\bar{k}}$.
- B: Borel subgroup B in G, i.e. B is minimal such that G/B is a projective variety.
- Borel-Tits: $(B(k), N(k))$ is a BN-pair for $G(k)$ relying on the crucial result by Grothendieck.
- Tits: uniform proof of the simplicity (modulo center) of the groups of rational points of irreducible isotropic simple groups (over sufficiently large fields).

Spherical buildings from algebraic groups

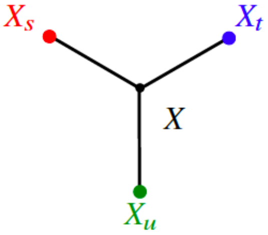
Let $\Delta = \Delta(G)$ be the simplicial complex whose simplexes correspond to proper k-parabolic subgroups of G as follows:

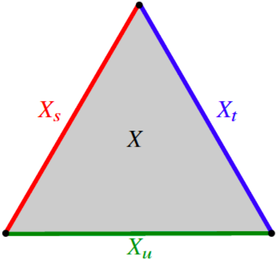
- The vertices of Δ correspond to maximal (proper) k-parabolic subgroups of G and chambers to minimal parabolic subgroups.
- Vertices $Q_1, \cdots, Q_m$ form the vertices of a simplex σ iff $\cap_{i=1}^{m} Q_i$ is a k-parabolic subgroup, which corresponds to the simplex σ.
- For any maximal k-split torus T of G, there are only finitely many k-parabolic subgroups containing T, and their corresponding simplices in Δ form a Coxeter complex (an apartment) whose Coxeter group is $W = N(T)/T$.
- $G(k)$ acts on the set of k-parabolic subgroups by conjugation and hence acts on the building $\Delta(G)$ by simplicial automorphisms.

Mirror structure of a Coxeter system

(W, S) any Coxeter system (with S finite!),
X: connected Hausdorff topological space.

Mirror structure on X over S: Collection $(X_s)_{s \in S}$ all X_s closed and non-empty, call X_s the s-mirror of X





Basic construction of a geometric realisation

For each $x \in X$, define $S(x) \subset S$ by $S(x) = \{s \in S \mid x \in X_s\}$. Define $\sim$ on $W \times X$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{s(x)}$. Then define $\mathcal{U}(W, X) = W \times X/\sim$ equipped with the quotient topology.

(1) Cayley graphs: obtained from the "star"

(2) Coxeter complexes: obtained from the "triangle"

(3) $\mathcal{U}(W, X)$ is connected, Haussdorff and with X as the fundamental domain for the natural action of W, i.e. $\mathcal{U}(W, X)/W = X$.

The nerve and the mirror structure

The *nerve* $L(W, S)$ of (W, S) is the simplicial complex with a simplex σ_T for each $T \subset S$ such that $T \neq \emptyset$ and W_T is finite.

The *chamber* K is the cone on the barycentric subdivision L' of the nerve $L = L(W, S)$. For each $s \in S$, define $K_s \subset K$ to be the closed star in L' of the vertex s.

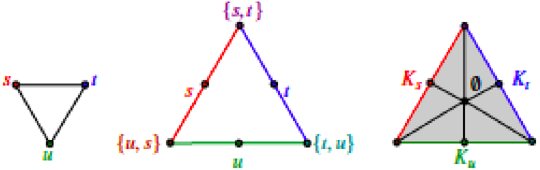


Figure: Example for $W = \langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle$

A geometric realisation of a building: the Davis complex

- connected, Hausdorff, locally finite.
- W-action on Σ is properly discontinuous with quotient K, and all point stabilisers are conjugates of finite special subgroups of W.
- Contractible so in particular simply connected.
- $\mathrm{CAT}(0)$ using the Cartan-Hadamard theorem and the Gromov link condition (see next slide).
- If a group G acts properly discontinuously and co-compactly by isometries on a $\mathrm{CAT}(0)$ space then the word problem and conjugacy problem are both solvable for G.

The Davis complex is a CAT(0) space

- Cartan-Hadamard theorem: Let X be a complete, connected geodesic metric space. If X is locally $\mathrm{CAT}(0)$ then the universal cover of X is $\mathrm{CAT}(0)$.
- Gromov link condition: If X is a piecewise Euclidean polyhedral complex then X is locally $\mathrm{CAT}(0)$ if and only if for every vertex v of X, the link of v in X is $\mathrm{CAT}(1)$.