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Vertex-transitive graphs and their local actions II

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Arc-transitive graphs

An arc in a graph is an ordered pair of adjacent vertices.

Γ is G-arc-transitive if $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on arcs of Γ.

Lemma

Γ *is* G*-arc-transitive if and only if* (Γ, G) *is locally-transitive.*

(Vertex-transitivity is embedded in the definition of "locally".)

Tutte's Theorem

Theorem (Tutte 1947)

If Γ is a connected 3-valent G-arc-transitive graph, then $|G_v| = 3 \cdot 2^s$ for some $s \leq 4$.

$|G_v| \leq 48$, so $|G| \leq 48|\mathrm{V}(\Gamma)|$.

Application of Tutte

Each pair (Γ, G) occurs as a finite quotient of an (infinite) group amalgam acting on the (infinite) cubic tree. By Tutte, there are only finitely many amalgams to consider, and the index is linear in the order of the graph.

This allows one (for example Conder) to enumerate these graphs up to "large" order (in this case, 10000).

`https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt`

(There are 3815 such graphs.)

Application of Tutte II

Theorem (Potočnik,Spiga,V 2017)

The number of 3-valent arc-transitive graphs of order at most n is at most

$$n^{5+4b\log n} \sim n^{c\log n}.$$

Proof.

Let Γ be a 3-valent arc-transitive graph of order at most n and let $A = \mathrm{Aut}(\Gamma)$. Note that $|A| \leq 48n < n^2$ and A is 2-generated. By a result of Lubotzky, there exists b such that the number of isomorphism classes for A is at most $(n^2)^{b\log n^2} = n^{4b\log n}$. A_v is 2-generated, so at most $(n^2)^2 = n^4$ choices for A_v. At most n choices for a neighbour of v, and this determines Γ. □

There also exists c' such that the number is at least $n^{c'\log n}$. This also relies on Tutte's Theorem.

Application of Tutte III

Theorem (Conder, Li, Potočnik 2015)

Let k be a positive integer. There are only finitely many locally-$\mathrm{Sym}(3)$ pairs (Γ, G) with Γ of order kp with p a prime.

Proof.

Let $p > 48k$ be prime, (Γ, G) be locally-$\mathrm{Sym}(3)$ with Γ of order kp with p a prime. Then $|G| = kp|G_v| \leq 48kp$. By Sylow, G has a normal Sylow p-subgroup P. Let C be the centraliser of P in G. By Schur-Zassenhaus, $C = P \times J$ for some J. Since $|P|$ and $|J|$ are coprime, J is characteristic in C and normal in G and

$$C_v = C \cap G_v = (P \times J) \cap G_v = (P \cap G_v) \times (J \cap G_v) = P_v \times J_v.$$

Since $P_v = 1$, we have $C_v = J_v$. If $J_v \neq 1$, then J has at most two orbits of the same size, which is divisible by p since $p > 2$. This contradicts the fact that $|J|$ is coprime to p. It follows that $C_v = J_v = 1$, and thus G_v embeds into $\mathrm{Aut}(P)$ which is cyclic. Contradiction. □

Generalisation to 4-valent?

The wreath graph $\mathrm{W}_m = \mathrm{C}_m[\mathrm{K}_2^c]$ is the lexicographic product of a cycle of length m with an edgeless graph on 2 vertices.

We have $G = \mathrm{C}_2 \wr \mathrm{D}_m \leq \mathrm{Aut}(\mathrm{W}_m)$.

So W_m is a 4-valent arc-transitive graph, $|\mathrm{V}(\mathrm{W}_m)| = 2m$, $|G| = m2^{m+1}$, so $|G_v| = 2^m$.

$|G_v|$ is exponential in $|\mathrm{V}(\mathrm{W}_m)|$.

Generalisation to vertex-transitive?

The split wreath graph SW_m is a 3-valent vertex-transitive graph.

$|\mathrm{V}(SW_m)| = 4m$, $|G| = m2^{m+1}$, so $|G_v| = 2^{m-1}$.

Splitting by local action

Theorem (Gardiner 1973)

Let Γ *be* 4*-valent and* (Γ, G) *be locally-*$\mathrm{Alt}(4)$ *or* $\mathrm{Sym}(4)$*. Then* $|G_v| \leq 2^4 \cdot 3^6$*.*

Corollary

Let Γ *be a 4-valent* G*-arc-transitive graph, and let* L *be the local action. The possibilities are:*

L	L_x	$\lvert G_v \rvert$
C_4	1	4
C_2^2	1	4
D_4	C_2	2^x
$\mathrm{Alt}(4)$	C_3	$\leq 2^2 \cdot 3^4$
$\mathrm{Sym}(4)$	$\mathrm{Sym}(3)$	$\leq 2^4 \cdot 3^6$

The only "problem" is the locally-D_4 case. (As in W_m.)

Graph-restrictive

Definition

A permutation group L is graph-restrictive if there exists a constant c such that, for every locally-L pair (Γ, G), we have $|G_v| \leq c$.

Example

$\mathrm{Sym}(3)$ (in its natural action) is graph-restrictive, but D_4 is not.

Many of the previous results can be proved under the assumption that the local group is graph-restrictive.

Primitive groups

A permutation group is primitive if it preserves no non-trivial partition. (The partition with a unique part, or into singletons.)

Conjecture (Weiss 1978)

Primitive groups are graph-restrictive.

Theorem (Weiss, Trofimov 1980-2000)

Transitive groups of prime degree and 2-transitive groups are graph-restrictive.

In particular, Tutte's Theorem generalises to prime valencies.

Theorem (Spiga 2015)

Primitive groups of affine type are graph-restrictive.

Semiprimitive groups

A permutation group is semiprimitive if every normal subgroup is transitive or semiregular.

Examples

1. Primitive groups
2. Dihedral groups of odd degree
3. $\mathrm{GL}(V)$ acting on a vector space V

Theorem (Potočnik, Spiga, V 2012)

Graph-restrictive $\Longrightarrow$ *semiprimitive.*

Conjecture (Potočnik, Spiga, V 2012)

Graph-restrictive $\Longleftrightarrow$ *semiprimitive.*

Theorem (Spiga, V 2014)

Intransitive+graph-restrictive $\Longleftrightarrow$ *semiregular.*

Not graph-restrictive

Theorem (Potočnik, Spiga, V 2015)

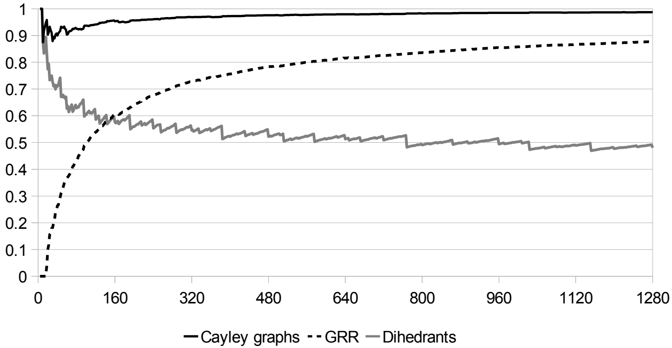
Let (Γ, G) *be a* *locally-*D_4 *pair. Then one of the following occurs:*

1. $\Gamma \cong \mathrm{W}_{m,k}$.
2. $|\mathrm{V}(\Gamma)| \geq 2|G_v|\log_2(|G_v|/2)$.
3. *Finitely other exceptions.*

We can recover some of the results we got in the 3-valent case.
For example, enumeration, both asymptotic and small order.

3-valent vertex-transitive

We get a similar result for 3-valent vertex-transitive graphs. In particular, we get a census up to order 1280.



Other growth type

More generally, for a group L, we can consider the "growth" of locally-L pairs, namely, how fast does $|G_v|$ grow with the order of the graph in such pairs. Fastest possible is exponential.

Graph-restrictive is "constant".

Dihedral group of even degree at least 6 have polynomial growth.

Up to permutation isomorphism, there are 37 transitive groups of degree at most 7.

1. 26 are constant/graph-restrictive
2. 10 are exponential
3. 1 is polynomial

Any other type possible?

Open problems

Conjecture

Are almost all vertex-transitive graphs Cayley?

What about for fixed valency?

Conjecture

Are there only five connected vertex-transitive graphs without Hamiltonian cycles?

Conjecture ("Polycirculant Conjecture" Marušič 1981)

Every vertex-transitive (di)graph admits a non-trivial semiregular automorphism.

Known only for graphs of valency at most 4, arc-transitive graphs of prime valency or twice a prime valency, etc.