NZMRI Summer Meeting 2021
Vertex-transitive graphs and their local actions I

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Automorphisms of graphs

A (simple) graph Γ is a pair (V, E) with $E \subseteq \binom{V}{2}$. Elements of V are vertices, elements of E are edges.

An automorphism of Γ is a permutation of V that preserves E.

Automorphisms of Γ form $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ.

(Graphs and groups will generally be finite.)

Permutation groups

A permutation group G on a set Ω is:

transitive if, for every $x, y \in \Omega$, there exists $g \in G$ mapping x to y.

In this case, by the Orbit-Stabiliser Theorem, we have $|G| = |\Omega||G_x|$, where G_x is a point-stabiliser.

semiregular if, for every $x, y \in \Omega$, there exists at most one g mapping x to y. (Equivalently, for every $x \in \Omega$, $G_x = 1$.)

In this case, each orbit has size $|G|$.

regular if it is transitive and semiregular. (Equivalently, there exists exactly one g mapping x to y.)

In this case, $|G| = |\Omega|$.

Regular representation

Let G be a group. For $g \in G$, let $\tilde{g} : G \to G, h \mapsto hg$.
(In other words, $\tilde{g}$ is (right) multiplication by g.)

Let $\tilde{G} = \{\tilde{g} \mid g \in G\} \leq \mathrm{Sym}(G)$.

This regular group is called the (right) regular representation of G.

(In some sense, this is the only example.)

Vertex-transitive graphs

Γ is vertex-transitive if $\mathrm{Aut}(\Gamma)$ is transitive (on vertices).

If G is a transitive subgroup of $\mathrm{Aut}(\Gamma)$, Γ is G-vertex-transitive.

(All vertices identical with respect to the structure of the graph.
For examples all vertices have the same valency, etc.)

Connectedness usually a very mild assumption.

Examples of vertex-transitive graphs

Γ	Name	$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$
K_n	Complete	$\mathrm{Sym}(n)$	$\mathrm{Sym}(n-1)$
K_n^c	Edgeless	$\mathrm{Sym}(n)$	$\mathrm{Sym}(n-1)$
C_n	Cycle	D_n	C_2
$\mathrm{K}_{n,n}$	Complete bip.	$\mathrm{Sym}(n)^2 \rtimes \mathrm{Sym}(2)$	$\mathrm{Sym}(n-1) \times \mathrm{Sym}(n)$
$\mathrm{C}_n \square \mathrm{K}_2$ $n \neq 4$	Prism	$\mathrm{D}_n \times \mathrm{C}_2$	C_2
Q_3	Cube	$\mathrm{C}_2^3 \rtimes \mathrm{Sym}(3)$	$\mathrm{Sym}(3)$
Pet	Petersen	$\mathrm{Sym}(5)$	$\mathrm{Sym}(2) \times \mathrm{Sym}(3)$

Cayley graphs

Definition

Let G be a group, $S \subseteq G$. The Cayley graph $\mathrm{Cay}(G, S)$ on G with connection set S has vertex-set G and edge-set will be $\{\{g, sg\} \mid g \in G, s \in S\}$.

For this to really be a simple graph, we need $1 \notin S$ and

$$S = S^{-1} := \{s^{-1} \mid s \in S\}.$$

$\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$. (Exercise)

$\tilde{G} \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$. (Exercise)

In particular, Cayley graphs are vertex-transitive.

Sabidussi's Theorem

Lemma (Sabidussi, 1958)

If Γ is a graph and G is a regular subgroup of $\mathrm{Aut}(\Gamma)$, then $\Gamma \cong \mathrm{Cay}(G, S)$ for some S.

Proof.

Pick a vertex v of Γ, label it with $1 \in G$. For every vertex u of Γ, there is a unique $g \in G$ such that $v^g = u$. Label u with g. Let S be the labels of the neighbours of v. Check this works. □

So Γ is isomorphic to a Cayley graph on a group G if and only if $\mathrm{Aut}(\Gamma)$ has a regular subgroup isomorphic to G.

Examples of Cayley graphs

Γ	G	S
C_n	$\mathbb{Z}_n$	$\{-1, +1\}$
K_n	G_n	G_n^*
K_n^c	G_n	$\emptyset$
$\mathrm{K}_{n,n}$	$G_n \times \mathbb{Z}_2$	$G_n \times \{1\}$
$\mathrm{C}_n \Box \mathrm{K}_2$	$\mathbb{Z}_n \times \mathbb{Z}_2$	$\{\pm(1,0),(0,1)\}$
Q_3	$\mathbb{Z}_2^3$	$\{(1,0,0),(0,1,0),(0,0,1)\}$
Pet	–	–

Graphical regular representation

If $\mathrm{Aut}(\Gamma) \cong G$ is regular, then Γ is a Cayley graph for G, and is called a GRR (graphical regular representation) for G.

Theorem (Godsil, 1978)

Apart from abelian groups, generalised dicyclic groups and a finite list of exceptions, all groups admit GRRs.

Conjecture (Babai, Godsil, Imrich, Lovász, 1982)

Almost all Cayley graphs (on groups that are not abelian or generalised dicyclic) are GRRs.

Theorem (Morris, Spiga, 2018)

Almost all Cayley digraphs are DRRs.

Cayley graphs on abelian groups

If $\alpha \in \mathrm{Aut}(G)$ and $\alpha(S) = S$, then α is an automorphism of $\mathrm{Cay}(G, S)$ fixing the identity. (Exercise)

If G is abelian, then ι (inversion) is an automorphism of G (exercise) and it fixes S, since $S = S^{-1}$.

If Γ is a Cayley graph on an abelian group, then $\tilde{G} \rtimes \langle \iota \rangle \leq \mathrm{Aut}(\Gamma)$, so Γ is not a GRR. (Unless $G \cong \mathrm{C}_2^n$)

Babai and Goldsil (1982) conjectured that, almost always, this is the full automorphism group.

Theorem (Dobson, Spiga, V., 2016)

Almost all Cayley graphs on abelian groups have automorphism group $\tilde{G} \rtimes \langle \iota \rangle$.

There is also a similar result (Morris, Spiga, V., 2015) for generalised dicyclic groups.

Local action

Let Γ be a connected G-vertex-transitive graph.

Let $L = G_v^{\Gamma(v)}$, the permutation group induced by G_v on the neighbourhood $\Gamma(v)$.

We say that (Γ, G) is locally-L.

$G_v^{\Gamma(v)}$ is a permutation group of degree the valency of Γ and does not depend on v.

Let $G_v^{[1]}$ be the subgroup of G consisting of elements fixing v and all its neighbours.

$G_v^{\Gamma(v)} \cong G_v / G_v^{[1]}$.

Examples

Γ	$\mathrm{Aut}(\Gamma)_v$	$\mathrm{Aut}(\Gamma)_v^{\Gamma(v)}$
C_n	C_2	C_2
K_n	$\mathrm{Sym}(n-1)$	$\mathrm{Sym}(n-1)$
$\mathrm{K}_{n,n}$	$\mathrm{Sym}(n-1) \times \mathrm{Sym}(n)$	$\mathrm{Sym}(n)$
$\mathrm{C}_n \square \mathrm{K}_2$	C_2	C_2
$n \neq 4$		
Q_3	$\mathrm{Sym}(3)$	$\mathrm{Sym}(3)$
Pet	$\mathrm{Sym}(2) \times \mathrm{Sym}(3)$	$\mathrm{Sym}(3)$

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Lemma

Let (Γ, G) be a locally-L pair and v be a vertex of Γ. There is a subnormal series for G_v

$$1 = G_n \trianglelefteqslant G_{n-1} \trianglelefteqslant \cdots \trianglelefteqslant G_1 \trianglelefteqslant G_0 = G_v$$

such that $G_0/G_1 \cong L$ and, for $i \geq 1$, $G_i/G_{i+1} \preceq L_x$.

Proof.

Let $(v = v_1, \ldots, v_n)$ be a walk including all vertices of Γ (possibly with repetition). Let $G_0 = G_{v_1}$ and for $i \geq 1$, let $G_i = G_{v_1}^{[1]} \cap \cdots \cap G_{v_i}^{[1]}$. □

Example

If (Γ, G) is locally-$\mathrm{Alt}(4)$, then $|G_v| = 4 \cdot 3^s$.