Algorithmic Randomness – Part 2

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U and Ω

There is a partial computable, prefix-free function U such that for every g which is computable from above and has $\sum_\sigma 2^{-g(\sigma)} < \infty$, there is a b with

$$K_U(\sigma) \leq g(\sigma) + b.$$

$$\Omega = \sum_{\tau \in \mathrm{dom}\, U} 2^{-|\tau|}.$$

Ω is computable from below: we can build a computable increasing sequence of rationals which converges to Ω by searching for inputs that make U halt.

Ω is a fixed real... but we can influence it?

Fix $q_0, q_1, \ldots$ computable, increasing, converging to Ω.

We build a g based on this sequence which is computable from above and summable. So there is b with $K_U(\sigma) \leq g(\sigma) + b$.

At some stage s, we pick σ not yet in the range of U and define $g(\sigma) = n$. U must eventually reveal a new string of length at most $n + b$ in its domain.

So there is $t \geq s$ with $q_{t+1} - q_t \geq 2^{-n-b}$.

Another take

We have a computable sequence $q_0 < q_1 < \ldots$ converging to Ω.

We are given $\epsilon < 1$. (It's 2^{-b}.)

At any stage s, we can request that the sequence increase by some δ. In response, there will be a $t \geq s$ with $q_{t+1} - q_t \geq \delta$.

This works as long as the requested δs sum to at most ϵ.

Ω is random

Theorem (Chaitin)

Ω is Martin-Löf random.

Proof.

Fix $q_0, q_1, \ldots$ computable, increasing, converging to Ω. Fix $V_0, V_1, \ldots$ the universal Martin-Löf test.

Let $\epsilon = 2^{-b}$ be as in the previous discussion.

When we see some $[\tau] \subseteq V_{b+1}$ containing the current q_s, we trigger an increase of at least $2^{-|\tau|+1}$. This moves some q_t beyond $[\tau]$.

By topological considerations, $\Omega \notin V_b$.

The total requests are at most $2\mu(V_{b+1}) \leq 2 \cdot 2^{-(b+1)} = \epsilon$. □

c.e. sets

Recall: a c.e. set is the range of a partial computable function.

Lemma

If A is c.e., then there is a computable sequence of finite sets $A_0 \subseteq A_1 \subseteq \dots$ with $A = \bigcup_n A_n$.

Proof.

Use dovetailing to search for inputs that give an output, and build your finite sets out of these. $\square$

It's powerful

Theorem (Calude and Nies)

Ω *computes every c.e. set.*

Proof.

Fix a c.e. set A.

If we see $n \in A_{s+1} \setminus A_s$, trigger an increase of at least $\epsilon 2^{-n}$.

With oracle Ω, to decide if $n \in A$, find an s with $\Omega - q_s < \epsilon 2^{-n}$. Then $n \in A \iff n \in A_s$. □

In particular, Ω computes the halting set (and the halting set computes it).

Joining and splitting reals

For $X, Y \in \{0,1\}^{\mathbb{N}}$, $X \oplus Y$ is made by interleaving X and Y.

$$\begin{aligned} X &= x_0 x_1 x_2 \dots \\ Y &= y_0 y_1 y_2 \dots \\ X \oplus Y &= x_0 y_0 x_1 y_1 x_2 y_2 \dots \end{aligned}$$

For any $Z \in \{0,1\}^{\mathbb{N}}$, there are unique X and Y with $Z = X \oplus Y$.

Moreover, $\oplus : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is a measure-preserving isometry.

Independent parts of a random

Intuitively, a random looks like it was created by flipping a fair coin for each bit.

Each toss of the coin is independent.

So the bits in even positions and those in odd positions are independent of each other.

So if $Z = X \oplus Y$ is random, then X and Y should be random relative to each other.

Relative randomness

A Martin-Löf test relative to X is an appropriate sequence of open sets, where the computable process that defines the sets has access to X.

There is a universal Martin-Löf test relative to X: $V_0^X, V_1^X, \ldots$.

Y is Martin-Löf random relative to X if it avoids all Martin-Löf tests relative to X (equivalently, it avoids the universal one).

Formalizing the previous intuition

Theorem (van Lambalgen's theorem)

For $Z = X \oplus Y$, TFAE:

1. *Z is Martin-Löf random;*
2. *X is Martin-Löf random and Y is Martin-Löf random relative to X.*
3. *Y is Martin-Löf random and X is Martin-Löf random relative to Y.*

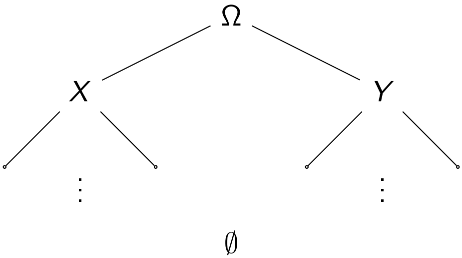
Proof.

Basically Fubini's theorem. □

Splitting a random

So if $Z = X \oplus Y$ is Martin-Löf random, then neither X nor Y computes the other.

For $\Omega = X \oplus Y$:



A motivating question

Question

If X is Martin-Löf random and strictly below Ω, what kinds of c.e. sets can X compute? And how does this relate to these fragments of Ω?

K-trivials

Definition

A is K-trivial if there is a b such that for all n, $K(A\restriction_n) \leq K(0^n) + b$.

Definition

A is low for Martin-Löf randomness if for every Martin-Löf random X, X is Martin-Löf random relative to A.

Theorem (Nies & various coauthors)

For a real A, TFAE:

1. *A is K-trivial;*
2. *A is low for Martin-Löf randomness;*

   (dozens more)

Noncomputable K-trivials exist.

K-trivials and the motivating question

Theorem (Hirschfeldt, Nies & Stephan)

If A is c.e. and computable from a Martin-Löf random strictly below Ω, then A is K-trivial.

Question (The Covering Problem)

If A is K-trivial, is there a Martin-Löf random strictly below Ω that computes A?

The Covering Problem

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Theorem (Bienvenu, Day, Greenberg, Kučera, Miller, Nies, T)

Yes.

Proof.

Goes via the Lebesgue density theorem. □

Cost functions

We said that a c.e. set is the union of a computable increasing sequence of finite sets.

Can we find finer measures for how that happens?

Definition

A cost function is a computable function $c : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfying:

- For all n and s, $c(n+1, s) \leq c(n, s) \leq c(n, s+1)$;
- For all n, $c(n) = \lim_{s\to\infty} c(n, s) < \infty$;
- $\lim_{n\to\infty} c(n) = 0$.

The most important cost function

Fix $q_0, q_1, \ldots$ a computable increasing sequence converging to Ω.

Let $c_\Omega(n,s) = \max\{q_s - q_n, 0\}$.

c_Ω is a cost function. $c_\Omega(n) = \lim_{s\to\infty} c_\Omega(n,s) = \Omega - q_n$.

Obeying a cost function

Definition

If c is a cost function, and A is a c.e. set with a computable sequence of finite sets $A_0, A_1, \ldots$, then A obeys c, written $A \models c$, if

$$\sum_s c(n_s, s) < \infty,$$

where $n_s = \min(A_{s+1} \setminus A_s)$.

Lemma (Nies)

A c.e. set is K-trivial iff it obeys c_Ω.

Picking out subsequences

Definition

For $f : \mathbb{N} \to \mathbb{N}$ a strictly increasing function, f has density δ if $\sup_n |\delta f(n) - n| < \infty$.

Definition

For $X \in \{0,1\}^{\mathbb{N}}$ with $X = x_0 x_1 x_2 \ldots$, and $f : \mathbb{N} \to \mathbb{N}$ a strictly increasing function, $X_f = x_{f(0)} x_{f(1)} x_{f(2)} \ldots$.

By a modification of van Lambalgen's theorem, if f is strictly increasing and computable, and its range is co-infinite, and X is Martin-Löf random, then X_f is strictly below X.

Relating these all

Define c_Ω^δ by $c_\Omega^\delta(n,s) = (c_\Omega(n,s))^\delta$. This is a cost function.

Theorem (Greenberg, Miller, Nies, T)

If $f : \mathbb{N} \to \mathbb{N}$ is a computable strictly increasing function of density $\delta < 1$, then the c.e. sets computable from Ω_f are precisely the c.e. sets which obey c_Ω^δ.

Using cost functions in a new way

Definition

If c is a cost function, a c-test is a Martin-Löf test, except that the condition of $\mu(V_n) \leq 2^{-n}$ is changed to
$\mu(V_n) \leq c(n) = \lim_s c(n, s)$.

X is c-random if it avoids all c-tests.

For the cost functions we care about, this is a relaxing ($2^{-n} < c(n)$). Hence there are more tests and fewer randoms.

For $\delta < \epsilon$, $c_\Omega^\epsilon < c_\Omega^\delta$, so there are more c_Ω^ϵ-randoms than c_Ω^δ-randoms.

Relating the two uses

Theorem (Greenberg, Miller, Nies, T)

For X Martin-Löf random and $\delta < 1$, TFAE:

1. *X computes every c.e. set obeying c_Ω^δ;*
2. *X is not c_Ω^δ-random*

Corollary

If f has density δ, then Ω_f is c_Ω^ϵ-random for $\epsilon > \delta$, but not c_Ω^δ-random.

Moral: thinner subsequences of Ω are computationally weaker, and thus more random.

References

- Downey & Hirschfeldt, *Algorithmic Randomness and Complexity*, Springer, 2010.
- Franklin & Porter, *Algorithmic Randomness: Progress and Prospects*, Cambridge, 2020.