Topological Properties Defined By Games And Their Applications

Jiling Cao †, Warren Moors ‡ and Ivan Reilly †

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Abstract

In this paper, we study some two person games and some topological properties defined by them. As an application of our results we refine the Choquet-Dolecki theorem on multifunctions and the classical Vaĭnšteĭn lemma on closed maps.

1 Introduction

Let $\Phi : T \to X$ be a multifunction from a topological space T into a topological space X. We say that Φ is *upper semicontinuous*, abbreviated as *usc*, at $t_0 \in T$ if for each open subset V with $\Phi(t_0) \subseteq V$ there exists an open neighbourhood U of t_0 such that $\Phi(U) \subseteq V$. A subset $K \subseteq \Phi(t_0)$ is said to be a *kernel* [4] for Φ at t_0, if the multifunction $\Phi_K : T \to X$ defined by:

$$\Phi_K(t) = \begin{cases} K, & \text{if } t = t_0; \\ \Phi(t) \setminus \Phi(t_0), & \text{otherwise.} \end{cases}$$

is usc at t_0. In [4, p.70], Choquet stated, without proof, that if T and X are metric spaces, then any multifunction $\Phi : T \to X$ which is usc at $t_0 \in T$ possesses a compact kernel at t_0. This gives rise to the following natural problem.

Problem 1.1. *Can we determine when an usc multifunction $\Phi : T \to X$ possesses a compact kernel ?*

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A significant amount of work has been afforded to this problem. Some of this effort maybe witnessed in the papers [6], [8] and [9].
One important advance was made by Dolecki in [7] when he identified the following set as a natural candidate for a kernel of any usc multifunction. The *active boundary* of Φ at $t_0 \in T$, denoted by $Frac\Phi(t_0)$, is defined by:

$$Frac\Phi(t_0) \equiv \bigcap_{U \in \mathcal{U}(t_0)} \overline{\Phi(U) \setminus \Phi(t_0)},$$

where $\mathcal{U}(t_0)$ is the set of all neighbourhoods of t_0. Perhaps the best result obtained thus far, which first appeared in [11] (and later refined in [3]) is the following.

Theorem 1.2. *Let T be a first countable space and let X be a Hausdorff angelic space. Suppose that $\Phi : T \to X$ is usc at t_0. Then $Frac\Phi(t_0)$ is compact and is the smallest kernel for Φ at t_0.*

Recall that a space X is *angelic* if *(i)* each relatively countably compact subset A of X is relatively compact; *(ii)* each point in the closure of a relatively compact subset A of X is the limit of some sequence in A, [11].

The main purpose of the present paper is to study Problem 1.1 from the perspective of topological games. Throughout this paper, all spaces are assumed to be at least T_1.

2 $\mathcal{G}(\mathcal{F})$-game and property $(**)$

Let X be a topological space and let $\mathcal{F}$ be a proper filter (filterbase) in X. We shall consider the following $\mathcal{G}(\mathcal{F})$*-game* played in X between players A and B. Player A goes first (always!) and chooses a point $x_1 \in X$. Player B must then respond by choosing a member $F_1 \in \mathcal{F}$. Following this, player A must select another (possibly the same) point $x_2 \in F_1$ and in turn player B must again respond to this by choosing a member $F_2 \in \mathcal{F}$. Repeating this proceedure indefinitely, the players A and B produce a sequence $p \equiv \{(x_n, F_n) : n \in \mathbf{N}\}$ with $x_{n+1} \in F_n$ for all $n \in \mathbf{N}$, called a *play* of the $\mathcal{G}(\mathcal{F})$-game. We shall say that B *wins* a play of the $\mathcal{G}(\mathcal{F})$-game if the sequence $(x_n : n \in \mathbf{N})$ has a cluster point in X. Otherwise, the player A is said to have *won* this play. By a *strategy* σ for the player B we mean a 'rule' that specifies each move of the player B in every possible situation. More precisely, a strategy $\sigma \equiv (\sigma_n : n \in \mathbf{N})$ for B is a sequence of $\mathcal{F}$-valued functions. The domain of each function σ_n is precisely the set of all the finite sequences $(x_1, x_2, ..., x_n)$, of length n, in X with $x_{i+1} \in \sigma_i(x_1, ..., x_i)$ for all $1 \leq i \leq n-1$. Such a finite sequence $(x_1, x_2, \ . \ . \ . \ , x_n)$ or infinite sequence $(x_1, x_2, \ . \ . \ . \ )$ is called a *σ-sequence*. A strategy $\sigma \equiv (\sigma_n : n \in \mathbf{N})$ for player B is called a *winning strategy* if each σ-sequence has a cluster point in X, that is, if player B wins every play of the $\mathcal{G}(\mathcal{F})$-game in which the player B 'applies' the strategy σ.

We shall call a pair $(\mathcal{F}, \sigma)$ a *σ-filter* (*σ-filterbase*) if $\mathcal{F}$ is a proper filter (filterbase) in X and σ is a winning strategy for player B in the $\mathcal{G}(\mathcal{F})$-game played in X. Finally we say that a space X has *property* $(**)$ if $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$ for each σ-filterbase $(\mathcal{F}, \sigma)$ in X. It is easy to see that property $(**)$ is hereditary with respect to closed sets and preserved by perfect maps.

In order to expedite the remainder of this paper we will introduce the following binary relation. Let X be a topological space and let $(\mathcal{F}, \sigma)$ and $(\mathcal{F}', \sigma')$ be two ordered pairs with $\mathcal{F}$ and $\mathcal{F}'$ filterbases, and σ and σ' strategies for the player B in the $\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F}')$-games respectively. Then we say that $(\mathcal{F}', \sigma')$ is *finer* than $(\mathcal{F}, \sigma)$, and write $(\mathcal{F}, \sigma) \preceq (\mathcal{F}', \sigma')$ if:

(i) For each $F \in \mathcal{F}$ there exists a $F' \in \mathcal{F}'$ so that $F' \subseteq F$;

(ii) each σ'-sequence is a σ-sequence.

We note then that if $(\mathcal{F}, \sigma) \preceq (\mathcal{F}', \sigma')$ and σ is a winning strategy for B in the $\mathcal{G}(\mathcal{F})$-game, then σ' is a winning strategy for B is the $\mathcal{G}(\mathcal{F}')$-game, that is, if $(\mathcal{F}, \sigma)$ is a σ-filterbase then so is $(\mathcal{F}', \sigma')$. An important fact for us is the following.

Proposition 2.1. *If $\mathcal{F}$ is a filterbase in X, σ a strategy for the player B in the $\mathcal{G}(\mathcal{F})$-game and $\mathcal{F}'$ a filterbase finer than $\mathcal{F}$, then there exists a strategy σ' for B so that $(\mathcal{F}, \sigma) \preceq (\mathcal{F}', \sigma')$.*

Proof. We define $\sigma' \equiv (\sigma'_n : n \in \mathbf{N})$ inductively. First we define, for each $x_1 \in X$, $\sigma'_1(x_1)$ to be any elements of $\mathcal{F}'$ contained in $\sigma_1(x_1) \in \mathcal{F}$. Note this is possible, since $\mathcal{F}'$ is finer than $\mathcal{F}$. Now, suppose that we have defined $(\sigma'_1, \sigma'_2, ..., \sigma'_n)$ so that each σ'-sequence $(x_1, x_2, ..., x_i)$, $1 \leq i \leq n+1$ is a σ-sequence. Then we define (for each σ'-sequence $(x_1, ..., x_{n+1})$), $\sigma'_{n+1}(x_1, x_2, ..., x_{n+1})$ to be any element of $\mathcal{F}'$ contained in $\sigma_{n+1}(x_1, ..., x_{n+1}) \in \mathcal{F}$. Note this definition is well-defined since $(x_1, ..., x_{n+1})$ is a σ-sequence and $\mathcal{F}'$ is finer than $\mathcal{F}$. Moreover, with this definition we have that each σ'-sequence $(x_1, ..., x_i)$, $1 \leq i \leq n+2$ is σ-sequence. This completes the definition of $\sigma' \equiv (\sigma'_n : n \in N)$. □

Remark 2.2. *The above proposition has two significant implications:*

(i) If $(\mathcal{F}, \sigma)$ *is a* σ*-filterbase then there exists an ultra-filter* $\mathcal{F}^*$ *and a strategy* σ^* *so that* $(\mathcal{F}, \sigma) \preceq (\mathcal{F}^*, \sigma^*)$*;*
(ii) if $\mathcal{F}'$ *is a finer filterbase than* $\mathcal{F}$ *and* $(\mathcal{F}, \sigma)$ *is a* σ*-filterbase in a space with property* $(**)$ *then* $\mathcal{F}'$ *has a cluster point.*

Let Σ_X denote the family of all σ-filters on a topological space X. Then the binary relation "$\preceq$" defines a partial ordering on Σ_X. The only difficulty is in showing that "$\preceq$" is anti-symmetric.

Recall that a family $\mathcal{U}$ of subsets of a topological space X is a *proximate cover* (not necessarily a cover) of X if $X = \bigcup\{\overline{U} : U \in \mathcal{U}\}$.

Theorem 2.3. *Let* X *be a regular space with property* $(**)$*. If* $(\mathcal{F}, \sigma)$ *is a* σ*-filterbase, then* $K \equiv \bigcap_{F \in \mathcal{F}} \overline{F}$ *is compact and for each open set* W *containing* K *there exists an* $F \in \mathcal{F}$ *so that* $F \subseteq W$*.*

Proof. We first show that K is compact. To this end let $\mathcal{U} \equiv \{O_\gamma : \gamma \in \Gamma\}$ be an open cover of K. We shall show that $\mathcal{U}$ admits a finite proximate cover of K. To the contrary let us suppose, in order to obtain a contradiction, that $\mathcal{U}$ does not admit a finite proximate cover of K. Let $\mathcal{F}^* \equiv \{F \setminus \bigcup_{i=1}^{n} \overline{O_{\gamma_i}} : F \in \mathcal{F}$ and $\{\gamma_1, ..., \gamma_n\} \subseteq \Gamma$ is a finite subset$\}$. By our assumption, $\mathcal{F}^*$ is a filterbase in X which is finer than $\mathcal{F}$. Hence, by fact (ii) in Remark 2.2, $\emptyset \neq \bigcap\{\overline{F^*} : F^* \in \mathcal{F}^*\} \subseteq \bigcap\{\overline{F} : F \in \mathcal{F}\} = K$. On the other hand, we have $\emptyset \neq \bigcap\{\overline{F^*} : F^* \in \mathcal{F}^*\} \subseteq X \setminus \bigcup_{\gamma \in \Gamma} O_\gamma \subseteq X \setminus K$; which is impossible. Hence $\mathcal{U}$ admits a finite proximate cover of K. Since X is regular, this implies that K is compact.

We now show that if W is an open set containing K then there exists an $F \in \mathcal{F}$ so that $F \subseteq W$. Indeed, let us suppose in order to obtain a contradiction, that $F \setminus W \neq \emptyset$ for each $F \in \mathcal{F}$.

Let $\mathcal{F}_0 \equiv \{F \setminus W : F \in \mathcal{F}\}$. Then $\mathcal{F}_0$ is a filterbase in X finer than $\mathcal{F}$. Therefore, by fact (ii) in Remark 2.2, $\emptyset \neq \bigcap\{\overline{F_0} : F_0 \in \mathcal{F}_0\} \subseteq \bigcap\{\overline{F} : F \in \mathcal{F}\} = K$. On the other hand, we have $\bigcap\{\overline{F_0} : F_0 \in \mathcal{F}_0\} \subseteq X \setminus W \subseteq X \setminus K$; which is impossible. Hence, $F \subseteq W$ for some $F \in \mathcal{F}$. (Note that since K is compact and X is regular one can actually show that $\overline{F} \subseteq W$ for some $F \in \mathcal{F}$.) $\square$

Proposition 2.4. *Property* $(**)$ *is productive.*

Proof. Let $\{X_\gamma\colon \gamma \in \Gamma\}$ be a family of spaces having property $(**)$ and let $X \equiv \Pi_{\gamma\in\Gamma} X_\gamma$ be endowed with the product topology. Consider $(\mathcal{F}, \sigma)$ a σ-filter in X. By fact (i) in Remark 2.2 we may assume that $\mathcal{F}$ is an ultra-filter. So to show that $\mathcal{F}$ has a cluster point we need only show that for each $\gamma \in \Gamma$

$$\bigcap\{\overline{\pi_\gamma(F)} : F \in \mathcal{F}\} \neq \emptyset, \text{ where } \pi_\gamma : X \to X_\gamma \text{ is the } \gamma\text{-th projection.}$$

To this end, fix $\gamma \in \Gamma$. Since X_γ has property $(**)$ we need only show that the filterbase $\mathcal{F}_\gamma \equiv \{\pi_\gamma(F) : F \in \mathcal{F}\}$ admits a winning strategy $\sigma^\gamma \equiv (\sigma_n^\gamma : n \in \mathbf{N})$ for the player B in the $\mathcal{G}(\mathcal{F}_\gamma)$-game played in X_γ. We use the strategy σ to inductively define σ^γ.

First, suppose that player A chooses $x_1^\gamma \in X_\gamma$. Player B then chooses $x_1 \in X$ so that $\pi_\gamma(x_1) = x_1^\gamma$ and defines $\sigma_1^\gamma(x_1^\gamma) \equiv \pi_\gamma(\sigma_1(x_1))$. Inductively, suppose A has chosen $\{x_1^\gamma, x_2^\gamma, ..., x_{n+1}^\gamma\}$ in X_γ so that $x_{i+1}^\gamma \in \sigma_i^\gamma(x_1^\gamma, ..., x_i^\gamma)$ for each $1 \leq i \leq n$.

Correspondingly, at each stage i, $1 \leq i \leq n$, player B will have constructed a σ-sequence $(x_1, ..., x_i)$ in X so that $\pi_\gamma(x_i) = x_i^\gamma$ and defined $\sigma_i^\gamma(x_1^\gamma, ..., x_i^\gamma) \equiv \pi_\gamma(\sigma_i(x_1, ..., x_i))$.

At stage $n+1$, B chooses $x_{n+1} \in \sigma_n(x_1, ..., x_n)$ so that $x_{n+1}^\gamma = \pi_\gamma(x_{n+1})$ and defines $\sigma_{n+1}^\gamma(x_1^\gamma, ..., x_{n+1}^\gamma) \equiv \pi_\gamma(\sigma_{n+1}(x_1, ..., x_{n+1}))$. This completes the definition of σ^γ.

We claim that σ^γ is a winning strategy for B in the $\mathcal{G}(\mathcal{F}_\gamma)$ played in X_γ. To see this, note that if $(x_i^\gamma : i \in \mathbf{N})$ is a σ^γ-sequence then there exists a σ-sequence $(x_i : i \in \mathbf{N})$ in X so that $\pi_\gamma(x_i) = x_i^\gamma$. Now σ is a winning strategy for B in the $\mathcal{G}(\mathcal{F})$-game played in X and so the sequence $(x_i : i \in \mathbf{N})$ has a cluster point $x_\infty \in X$. It now only remains to observe that $\pi_\gamma(x_\infty)$ is a cluster point of $(x_i^\gamma : i \in \mathbf{N})$. This completes the proof. □

In a metric space (X, d), the *Kuratowski index of non-compactness* of a bounded set F is:

$$\alpha(F) \equiv \inf\{\varepsilon > 0 : F \text{ is a finite union of sets of diameter less than } \varepsilon\}$$

and $\alpha(F) \equiv \infty$ when F is unbounded. It can be shown that a metric space (X, d) is complete if and only if each filterbase $\mathcal{F}$ in X with $\inf\{\alpha(F) : F \in \mathcal{F}\} = 0$ has a cluster point in X, see ([12], p.12).

Theorem 2.5. *Every complete metric space (X, d) has property $(**)$.*

Proof. Let $(\mathcal{F}, \sigma)$ be a σ-filter in X. By the above comments, it suffices to show that $\inf\{\alpha(F) : F \in \mathcal{F}\} = 0$. So to the contrary, suppose that
$\inf\{\alpha(F) : F \in \mathcal{F}\} > 2\varepsilon > 0$. Then we may construct a σ-sequence $(x_n : n \in \mathbf{N})$ in X so that: $x_{n+1} \in \sigma_n(x_1, x_2, ..., x_n) \setminus \bigcup_{i=1}^n B(x_i, \varepsilon)$.
However this implies that $(x_n : n \in \mathbf{N})$ is discrete, which contradicts the fact that σ is a winning strategy for the player B in the $\mathcal{G}(\mathcal{F})$-game. $\square$

Corollary 2.6. *Every Dieudonné-complete space X has property $(**)$.*

Proof. The result follows directly from Proposition 2.4 and Theorem 2.5 and the fact that a space X is Dieudonné-complete if and only if it is homeomorphic to a closed subspace of some cartesian product of complete metric spaces. $\square$

For a topological space X, we shall denote by $C_p(X)$ the space of all continuous real-valued functions defined on X, endowed with the topology of pointwise convergence on X.

Proposition 2.7. *Let X be a compact Hausdorff space, then $C_p(X)$ has property $(**)$.*

Proof. Let $(\mathcal{F}, \sigma)$ be a σ-filter in $C_p(X)$. Since $C_p(X)$ is a subspace of $\mathbf{R}^X$, endowed with the product topology we have by Proposition 2.4 and Theorem 2.5 that $K \equiv \bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$, where the closure is taken in $\mathbf{R}^X$. So we need only show that K is contained in $C_p(X)$. To this end, consider $f \in K$. If f is not continuous on X, then there must be some point $x_0 \in X$ and $\varepsilon > 0$ so that for each $U \in \mathcal{U}(x_0)$:

$$f(U) \not\subseteq N(\varepsilon) \equiv (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

We will use this to obtain a contradiction. We inductively define:
$(x_n, U_n, f_n) \in X \times \mathcal{U}(x_0) \times C(X)$ so that:

(a) $x_m \in U_n$ for all $m > n$ and $f(x_m) \notin N(\varepsilon)$;
(b) $U_n \equiv \bigcap_{j=1}^n \{x \in X : |f_j(x) - f_j(x_0)| \leq 1/n\}$;
(c) $(f_n : n \in \mathbf{N})$ is a σ-sequence and $|f_n(x_j) - f(x_j)| \leq 1/n$ for $0 \leq j \leq n$.

Let x_∞ be any cluster point of $(x_n : n \in \mathbf{N})$ and let f_∞ be any cluster point of $(f_n : n \in \mathbf{N})$. Then $x_0, x_\infty \in \bigcap_{n=1}^\infty U_n$, and for each fixed (n, j), $1 \leq j \leq n$ $|f_j(x_\infty) - f_j(x_0)| \leq 1/n$.

Letting $n \to \infty$ we see that $f_j(x_\infty) = f_j(x_0)$ for all $j \in \mathbf{N}$. Therefore, $f_\infty(x_\infty) = f_\infty(x_0)$. On the other hand, $|f_n(x_j) - f(x_j)| \leq 1/n$ for $0 \leq j \leq n$ and so $f_\infty(x_j) = f(x_j)$ for all $j \in \omega$. In particular, $f_\infty(x_0) = f(x_0)$. Hence,

$$f(x_0) = f_\infty(x_0) = f_\infty(x_\infty) \in \overline{\{f_\infty(x_n) : n \in \mathbf{N}\}} \subseteq \mathbf{R} \backslash N(\varepsilon)$$

which is not possible, since $f(x_0) \in N(\varepsilon)$. Therefore, f must be continuous at x_0 and so $f \in C(X)$. Thus we have shown that $\emptyset \neq K \subseteq C_p(X)$. □

We next show that there are some property $(**)$ spaces that are not Dieudonné-complete, but to do this, we first need to make a small observation. Let X be a Banach space and let $B(X^*)$ denote its dual ball. Then $(X,\ \text{weak})$ is homeomorphic to a closed subspace of $C_p(B(X^*),\ \text{weak}^*)$.

Example 2.8. *By Proposition 2.7 and the above observation the Banach space $C(\beta\mathbf{N} \setminus \mathbf{N})$ with the weak topology has property $(**)$. However, Corson [5] showed that this space is not Dieudonné-complete. Another example of a space with property $(**)$ but which is not Dieudonné-complete is the Banach space $C[0, \omega_1]$ taken with its weak topology, [10].*

3 Applications to multifunctions

In this section, we will discuss some applications.

Lemma 3.1. *[1] Let Φ be a multifunction from a space T into a space X which is usc at t_0. If t_0 is a cluster point of $(t_n : n \in \mathbf{N})$ and $x_n \in \Phi(t_n) \backslash \Phi(t_0)$ for all $n \in \mathbf{N}$, then $(x_n : n \in \mathbf{N})$ has a cluster point in X.*

Let X be a topological space, we say that a point $a \in X$ is a *regular point* of X if for each neighbourhood $V \in \mathcal{U}(a)$ there exists some $U \in \mathcal{U}(a)$ such that $\overline{U} \subseteq V$. Of course, X is regular if and only if each point of X is a regular point. Following A. Bouziad [2] we will say that a point a in a topological space X is a $\mathcal{G}(a)$*-point* if there exists a strategy s for the player B so that $(\mathcal{U}(a), s)$ is a σ-filter and we will call a topological space X a $\mathcal{G}$*-space* if every point of X is a $\mathcal{G}(a)$-point.

Proposition 3.2. *Let Φ be a usc multifunction from a space T into a space X. If t_0 is a regular $\mathcal{G}(t_0)$-point and $\mathcal{F} \equiv \{\Phi(U) \setminus \Phi(t_0) : U \in \mathcal{U}(t_0)\}$ has non-empty elements, then there exists a strategy σ for the player B so that $(\mathcal{F}, \sigma)$ is a σ-filterbase in X.*

Proof. Let $s \equiv (s_n : n \in \mathbf{N})$ be a winning strategy for the player B in the $\mathcal{G}(\mathcal{U}(t_0))$-game played in T. We use the strategy s to inductively define σ. First, suppose that player A chooses $x_1 \in X$. Player B then chooses $t_1 \in T$, sets $W_1 \equiv T$ and defines $\sigma_1(x_1) \equiv \Phi(s_1(t_1) \cap W_1) \setminus \Phi(t_0) \in \mathcal{F}$.

Player A responds to this 'move' by choosing $x_2 \in \sigma_1(x_1)$. In return the player B chooses $t_2 \in s_1(t_1) \cap W_1$ and $W_2 \in \mathcal{U}(t_0)$ so that $x_2 \in \Phi(t_2)$ and $\overline{W_2} \subseteq W_1 \cap \{t \in T : \Phi(t) \cap \{x_2\} = \emptyset\}$ and then defines $\sigma_2(x_1, x_2) \equiv \Phi(s_2(t_1, t_2) \cap W_2) \setminus \Phi(t_0) \in \mathcal{F}$.

Inductively, suppose A has chosen $\{x_1, x_2, \ . \ . \ ., x_{n+1}\}$ in X so that $x_{i+1} \in \sigma_i(x_1, ..., x_i)$ for each $1 \leq i \leq n$.

Correspondingly, at each stage i, $2 \leq i \leq n$, player B will have constructed a sequence $(t_1, t_2, \ . \ . \ . \ , t_i)$ in T, a sequence $(W_1, W_2, \ . \ . \ . \ , W_i)$ of neighbourhoods of t_0 with $W_i \subseteq \overline{W_i} \subseteq W_{i-1} \subseteq . \ . \ .W_1$ so that $t_i \in s_{i-1}(t_1, ..., t_{i-1}) \cap W_i$, $x_i \in \Phi(t_i)$ and $\overline{W_i} \subseteq W_{i-1} \cap \{t \in T : \Phi(t) \cap \{x_2, ..., x_i\} = \emptyset\}$ and defined $\sigma_i(x_1, ..., x_i) \equiv \Phi(s_i(t_1, ..., t_i) \cap W_i) \setminus \Phi(t_0) \in \mathcal{F}$.

At stage $n+1$, player B chooses $t_{n+1} \in s_n(t_1, ..., t_n) \cap W_n$ and $W_{n+1} \in \mathcal{U}(t_0)$ so that $x_{n+1} \in \Phi(t_{n+1})$ and $\overline{W}_{n+1} \subseteq W_n \cap \{t \in T : \Phi(t) \cap \{x_2, ..., x_{n+1}\} = \emptyset\}$. This completes the definition of σ.

We claim that σ is a winning strategy for B in the $\mathcal{G}(\mathcal{F})$-game played in X. Indeed, if $(x_n : n \in \mathbf{N})$ is a σ-sequence in X then by the definition of σ there is a s-sequence $(t_n : n \in \mathbf{N})$ in T and a sequence of neighbourhoods $(W_n : n \in \mathbf{N})$ of t_0 so that $x_n \in \Phi(t_n)$, $t_{n+1} \in s_n(t_1, ..., t_n) \cap W_n$ and $\overline{W}_{n+1} \subseteq W_n \cap \{t \in T : \Phi(t) \cap \{x_2, ...x_{n+1}\} = \emptyset\}$.

Now s is a winning strategy for the player B in the $\mathcal{G}(\mathcal{U}(t_0))$-game played in T, and so the sequence $(t_n : n \in \mathbf{N})$ has a cluster point $t_\infty \in \bigcap_{n \in \mathbf{N}} \overline{W_n} \subseteq \{t \in T : \Phi(t) \cap \{x_n : n \geq 2\} = \emptyset\} \subseteq T$.

By Lemma 3.1, $(x_n : n \in \mathbf{N})$ has a cluster point in X. This shows that σ is a winning strategy for the player B in the $\mathcal{G}(\mathcal{F})$-game played in X. $\square$

If X is a topological space, then the G_δ*-topology* on X is the topology on X generated by taking all the G_δ-sets in X as a base. The following result improves ([11], Theorem 2) and ([9], Theorem 7.6).

Theorem 3.3. *Let Φ be an usc multifunction from a space T into a regular property $(**)$ space X. If t_0 is a regular $\mathcal{G}(t_0)$-point and $\Phi(t_0)$ is closed in the G_δ-topology on X then $Frac\Phi(t_0)$ is a compact kernel for Φ at t_0.*

Proof. It follows in a similar manner to ([11], Lemma 11) that $Frac\Phi(t_0) \subseteq \Phi(t_0)$. The result now follows from Proposition 3.2 and Theorem 2.3. $\square$

For a set A in a topological space X we shall denote by ∂A the boundary of A in X. It was shown in [1] that if a multifunction $\Phi : T \to X$ has a graph-closed at $t_0 \in T$, then $Frac\Phi(t_0) \subseteq \partial\Phi(t_0)$. On the other hand, Dolecki [9] indicated that if Φ is open at t_0, that is, if for each neighbourhood U of t_0 and $x_0 \in \overline{\Phi(t_0)}$ there is a neighbourhood V of x_0 so that $V \subseteq \Phi(U)$, then the whole boundary of $\Phi(t_0)$ is active. By applying these observations and Theorem 3.3, we can present a generalization of the classical Vaǐnšteǐn Lemma.

Proposition 3.4. *Let f be a closed map from a space X with property $(**)$ onto a Hausdorff $\mathcal{G}$-space T. Then $\partial(f^{-1}(t))$ is compact for each $t \in T$.*

Proof. Let Φ be the multifunction from T onto X which is defined by $\Phi(t) \equiv f^{-1}(t)$ for all $t \in T$. Since f is continuous and closed, Φ is both usc and open at each point $t \in T$, which implies that $\partial\Phi(t) = Frac\Phi(t)$ for all $t \in T$. Therefore by Theorem 3.3, $\partial\Phi(t)$ is compact for all $t \in T$. $\square$

We end the paper with a few more observations concerning property $(**)$. First, if X is a G_δ-closed subspace of a regular property $(**)$ space Y, then it is easy to show that X also has property $(**)$. Secondly, with only a small amount of extra effort we could have shown in Proposition 2.7 that $C_p(X)$ has property $(**)$ whenever X is a $\mathcal{G}$-space.

References

[1] G. Beer, *Topologies on closed and closed convex sets*, Kluwer Academic Publishers, 1993.

[2] A. Bouziad, *The Ellis theorem and continuity in groups*, Topology Appl., **50** (1993), 73-80.

[3] J. Cao, W. Moors and I. Reilly, *On the Choquet-Dolecki theorem*, Preprint.

[4] G. Choquet, *Convergences*, Ann. Univ. Grenoble **23** (1947-48), 57-112.

[5] H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc., **101** (1961), 1-15.

[6] S. Dolecki, *Constraints, stability and moduli of semicontinuity*, Prepint prepared for the 2nd IFAC Symposium, Warwick, 1977.

[7] S. Dolecki, *Remarks on semicontinuity*, Bull. Acad. Polon. Sci. Ser. Sci. Math., **25** (1977), 863-867.

[8] S. Dolecki and S. Rolewicz, *Metric characterizations of upper semicontinuity*, J. Math. Anal. Appl., **69** (1979), 146-182.

[9] S. Dolecki and A. Lechicki, *On the structure of upper semicontinuity*, J. Math. Anal. Appl., **88** (1982), 547-554.

[10] G. Edgar, *Measurability in a Banach space I*, Indiana Univ. Math. J., **26** (1977), 663-677.

[11] R. Hansell, J. Jayne, I. Labuda and C. Rogers, *Boundaries of and selectors for upper semicontiunous multi-valued maps*, Math. Z., **189** (1985), 297-318.

[12] Warren B. Moors, *Continuity conditions defined by indices of noncompactness and applications to differentiability*, Ph.D. thesis, The University of Newcastle, Australia, 1992.

[13] I. A. Vaǐnšteǐn, *On closed mappings of metric spaces*, (Russian) Dokl. Akad. Nauk. SSSR., **57** (1947), 319-321.

† Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1, New Zealand

‡ School of Mathematical and Computing Sciences, Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand