Hardy's taxi, $x^2 + 3y^2 = p$ and Michael Lennon

Michael Lennon was a member of the academic staff of The University of Auckland, first in the Mathematics Department and then in the Computer Science Department, from 1970 until his untimely death in 1999. Although he was far from being a "publish or perish" scientist, he made his own distinctive contributions. He is, for example, recognised as the teacher of Vaughan Jones who most influenced the early career of that famous mathematician. As a colleague of Michael, I turned to him from time to time for advice on technical matters. There were many areas in which he was the sole authority in Auckland, and probably in New Zealand. An ambition I never achieved was to write something with Michael, but I at least had the privilege of seeing how his brilliant mind worked, as we tried out a few projects together.

The famous anecdote in which Hardy told Ramanaujan, that the number of a taxi he had used was not particularly interesting, was recalled by Mrs Shakuntala Devi, a visiting calculating prodigy, when she spoke in Auckland in 1978. Clever though she was, Mrs Devi was not a mathematician in the usual sense, and both Michael and I were surprised by a mistake she made in quoting Ramanujan's rejoinder. The number 1729 was, she said, the *only* number that could be written as the sum of two cubes in two different ways. Of course the correct statement would have said that the taxi number is the *lowest* such number. Michael and I started to consider the question as to what the other solutions to the Diophantine equation $x^3 + y^3 = u^3 + v^3$ are like. Obviously we exclude as trivial solutions for which x, y, u and v have a common factor greater than 1, but there is still a family of solutions that seemed to go on forever, as we found from computer searches. After poring through the pages of output we generated, Michael found some interesting patterns and was able to prove a formula for an infinite sub-family of solutions. Unfortunately, I cannot reproduce this formula after all these years, so I will do something else with the "Ramanujan Diophantine Equation" in this miniature, which I dedicate to the memory of Michael Lennon.

First a special result which will be used below, although just beneath the surface.

Theorem 1 *Let $p > 3$ be a prime then there exist integers x and y such that $x^2 + 3y^2 = p$ if and only if $p \equiv 1 \bmod 6$.*

Proof. The "only if" part follows from the fact that -3 is a quadratic residue only for the primes referred to in the statement of the theorem. To prove the "if" part, consider the lattice points in $S = \mathbb{Z}_p \times \mathbb{Z}_p$ satisfying $x^2 + 3y^2 \equiv \bmod p$. For convenience, we represent $\mathbb{Z}_p$ as the set of integers reduced mod p, $\{0, 1, 2, \ldots, p-1\}$, although the word "closest" that we use below will refer to the closest distance between a given point and *any* representative of another point. Using the inner product $\langle x, y\rangle = x^2 + 3y^2$ and the associated norm, the area of S is $\sqrt{3}p^2$. Let $P_1 = (x_1, y_1)$ denote the closest lattice point to $P_0 = (0, 0)$ and let $P_2 = (x_2, y_2) = (-3y_1, x_1)$. The vectors P_0P_1 and P_0P_2 are orthogonal and there is no lattice point on the interval P_0P_2, except P_0 and P_2, since such a point would be closer to P_0 than P_1 is. The rectangle with corners P_0, P_1, $P_1 + P_2$ and P_2, has area $\sqrt{3}np$, where $x_1^2 + 3y_1^2 = np$. Exactly p of these rectangles make up an area equal to that of S. Thus, $\sqrt{3}np^2 = \sqrt{3}p^2$, implying that $n = 1$.

A simple corollary is that a square-free positive integer is of the form $x^2 + 3y^2$ if and only if its prime factorisation contains only primes congruent to 1 mod 6. The "if" part of the proof is based on the fact that $(x_1 + y_1\sqrt{-3})(x_2 + y_2\sqrt{-3}) = (x_1x_2 - 3y_1y_2) + (x_1y_2 + y_1x_2)\sqrt{-3}$.

Our main result is given in the following discussion.

Let $N = x^3+y^3 = u^3+v^3$, where the four integers have no common factor. If N is even, define $a' = (x+y)/2$, $b' = (x-y)/2$, $c' = (u+v)/2$, $d' = (u-v)/2$; if N is odd, define $a' = x+y$, $b' = x-y$, $c' = u+v$, $d' = u-v$. In either case $\gcd(x, y, u, v) = 1$ implies $\gcd(a', b', c', d') = 1$. Let $\mu = \gcd(a', b')$, $\nu = \gcd(c', d')$ and let $a = a'/\mu\nu^3$, $b = b'/\mu$, $c = c'/\nu\mu^3$, $d = d'/\nu$, where we note that, because $\gcd(\mu, \nu) = 1$, a and b must be integers. It is found that $a(\nu^6a^2 + 3b^2) = c(\mu^6c^2 + 3d^2)$. It now follows that integers r, X, Y, Z exist such that

$$a = rY, \qquad \nu^6a^2 + 3b^2 = XZ, \tag{1}$$

$$c = rX, \qquad \mu^6c^2 + 3d^2 = YZ. \tag{2}$$

Because $\gcd(a, b) = \gcd(c, d) = 1$, it follows that X, Y and Z can be written in the form $X = |\xi|^2$, $Y = |\eta|^2$, $Z = |\zeta|^2$, where $\xi = \alpha + \beta\sqrt{-3}$, $\eta = \gamma + \delta\sqrt{-3}$, $\zeta = s + t\sqrt{-3}$ and

$$\nu^3a + b\sqrt{-3} = \xi\zeta, \tag{3}$$

$$\mu^3c + d\sqrt{-3} = \eta\zeta. \tag{4}$$

We can now find conditions on r, s and t to ensure that (1), (2), (3) and (4) are consistent. These conditions are $\nu^3(\gamma^2 + 3\delta^2)r = \alpha s - 3\beta t$ and $\mu^3(\alpha^2 + 3\beta^2)r = \gamma s - 3\delta t$. Solve these and we have a representation of the solution which enables us to back-track from given values of α, β, γ, δ, μ and ν to find a', b', c' and d' and finally x, y, u and v. For example, $\xi = 1 + 2\sqrt{-3}$, $\eta = 4 + \sqrt{-3}$, $\mu = \nu = 1$ gives the values $r = 1$, $\zeta = 1 - 3\sqrt{-3}$ and the famous solution of Ramanujan $9^3 + 10^3 = 1^3 + 12^3 = 1729$. Other examples are (i) $\xi = \sqrt{-3}$, $\eta = 1 + \sqrt{-3}$, $\mu = \nu = 1$ which gives $r = 3$, $\zeta = -3 - 4\sqrt{-3}$ and the solution $9^3 + 15^3 = 2^3 + 16^3 = 4104$ and (ii) $\xi = \sqrt{-3}$, $\eta = 1$, $\mu = 1$, $\nu = 2$ which gives $r = 3$, $\zeta = 9 - 8\sqrt{-3}$ and the solution $33^3 + 15^3 = 2^3 + 34^3 = 39312$. Finally, $\xi = \sqrt{-3}$, $\eta = -1$, $\mu = \nu = 1$, gives a solution which rearranges to $3^3 + 4^3 + 5^3 = 6^3$.

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