The circular restricted three body problem and an interesting orbit

Even though the classical gravitational orbit problem is completely understood for two bodies, it starts getting difficult for three bodies. However, the problem can be tamed in one limiting case; this is when two of the bodies are so massive compared with the lightest of them that it is a fair approximation to assume that they orbit around each other without any effect on their motion due to the light body. The light body, on the other hand moves under the gravitational attraction of the two heavy bodies. Things become even simpler if the motion of the two heavy bodies about their common centre of mass is exactly circular. In this case it is convenient to freeze their motion and to express the motion of the light body relative to a coordinate system fixed to the heavy bodies.

Suppose the ratio of masses of the two heavy bodies are $1-\mu:\mu$ and that the centre of mass is fixed at the origin. The scales can be chosen so that the distance between the bodies is exactly 1 and that they are moving about their common centre of mass with unit angular velocity. Assume that the plane in which they move is the XY plane. At some specific time, the two heavy bodies will be at the positions $(1-\mu)[\cos(t),\sin(t)]^T$ and at $-\mu[\cos(t),\sin(t)]^T$ respectively. Write the position of the light body as X, Y and z the, because of the gravitational attraction of the heavy bodies, its equations of motion, given that units are chosen suitably, are

$$\begin{aligned}
X'' &= -(1-\mu)(X+\mu\cos(t))R^{-3} - \mu(X-(1-\mu)\cos(t))S^{-3},\\
Y'' &= -(1-\mu)(Y+\mu\sin(t))R^{-3} - \mu(Y-(1-\mu)\sin(t))S^{-3},\\
z'' &= -(1-\mu)zR^{-3} - \mu z S^{-3},
\end{aligned}$$

where

$$R^2 = (X+\mu\cos(t))^2 + (Y+\mu\sin(t))^2 + z^2, \quad S^2 = (X-(1-\mu)\cos(t))^2 + (Y-(1-\mu)\sin(t))^2 + z^2.$$

Make the substitution $\mathbf{X} = P(t)\mathbf{x}$, where $\mathbf{X}$, $\mathbf{x}$ and the rotation matrix $P(t)$ are given by

$$\mathbf{X} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad P(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

It is easy to see that $P'(t) = P\left(t+\frac{\pi}{2}\right)$ so that

$$\mathbf{X}''(t) = P(t)\mathbf{x}''(t) + 2P\left(t+\frac{\pi}{2}\right)\mathbf{x}'(t) + P(t+\pi)\mathbf{x}.$$

Solve for $\mathbf{x}''$ by multiplying by $P(-t)$ and we find that

$$\begin{aligned}
x'' &= x + 2y' - (1-\mu)(x+\mu)r^{-3} - \mu(x-1+\mu)s^{-3},\\
y'' &= y - 2x' - (1-\mu)yr^{-3} - \mu y s^{-3},\\
z'' &= -(1-\mu)zr^{-3} - \mu z s^{-3},
\end{aligned}$$

where r and s are equal to R and S respectively, but written in terms of x, y and z

$$r^2 = (x+\mu)^2 + y^2 + z^2, \quad s^2 = (x-1+\mu)^2 + y^2 + z^2.$$

It is interesting to ask whether or not there exist points where the small body can stay without moving and whether there exist periodic orbits. The answer to both of these questions is "yes". There are exactly five "libration points" (or "Lagrange points" to name them after their discoverer) for which $x'' = y'' = z'' = 0$, if $x' = y' = z' = 0$. There is an infinite family of periodic orbits some of which have been utilised by the space programme. Even though the three body problem is notoriously difficult and "exceeds, if I am not mistaken, the force of any human mind" (quoted words of Isaac Newton), quite a lot is known about the restricted version of the problem we have discussed here. The restricted problem yields reasonable approximations for the earth-moon and sun-earth systems. Like many problems in astronomy, predictions from the theory have sometimes preceded observation. Lagrange predicted the presence of asteroids at two of the libration points of Jupiter (the two which are not collinear with the planet and the sun) and these were much later observed for Jupiter and also for Earth.

In principle, it is also possible to obtain what can be described as a "horseshoe orbit", in which a small body more or less shares the same orbit as the lighter of the two heavy bodies and seems to approach it from one side then the other, retreating in between times to wander round the heaviest body. This has now been observed in our own sun-earth system. In fact the asteroid with whom we share this strange relationship has as its orbit a path much more complicated than a simple horseshoe. This asteroid known as 3753 (1986 TO) has had its path carefully identified and understood only this year. To read about it you can start with the web page at `http://www.asteroid.yorku.ca/companion/`.

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