Partitions, generating functions and the chain rule

Given $f : Y \to Z$ and $g : X \to Y$ the chain rule gives the formula for the derivative of $f \circ g$. The exact nature of the vector spaces X, Y and Z and the smoothness and differentiability requirements for the functions f and g are a luxury we cannot afford in this miniature. Suffice it to say that for every $y \in Y$ and $x \in X$ and every positive integer n the n-linear mappings $f^{(n)}(y) : Y^n \to Z$ and $g^{(n)}(x) : X^n \to Y$ exist. We will assume that these functions possess the usual meanings of n-fold derivatives with X one-dimensional for simplicity. Let $\mathcal{P}(n)$ denote the set of all partitions of a non-negative integer n. The generating function for the number of members of $\mathcal{P}(n)$, is given by

$$\phi(t) = 1 + t + 2t^2 + 3t^3 + 5t^4 + \cdots$$

and satisfies the wonderful formulae of Euler

$$\phi(t)^{-1} = \prod_{i=1}^{\infty}(1 - t^i) = \sum_{i=-\infty}^{\infty}(-1)^i t^{i(3i+1)}. \tag{1}$$

By contrast the number of partitions of a set with n members has the generating function

$$\psi(t) = 1 + t + 2t^2 + 5t^3 + 15t^4 + \cdots,$$

where $\psi(t)$ can be written in the form

$$\psi(t) = 1 + \frac{t}{1-t} + \frac{t^2}{(1-t)(1-2t)} + \frac{t^3}{(1-t)(1-2t)(1-3t)} + \cdots. \tag{2}$$

The first few derivatives of $f \circ g$ are equal to

$$(f \circ g)'(x) = f'\Big(g(x)\Big)\Big(g'(x)\Big), \qquad (f \circ g)''(x) = f'\Big(g(x)\Big)\Big(g''(x)\Big) + f''\Big(g(x)\Big)\Big(g'(x), g'(x)\Big),$$
$$(f \circ g)'''(x) = f'\Big(g(x)\Big)\Big(g'''(x)\Big) + 3f''\Big(g(x)\Big)\Big(g'(x), g''(x)\Big) + f'''\Big(g(x)\Big)\Big(g'(x), g'(x), g'(x)\Big).$$

The general case of this "Faa di Bruno formula" is

$$(f \circ g)^{(n)}(x) = \sum_{p \in \mathcal{P}(n)} C(p) f^{(r)}\Big(g(x)\Big)\Big(g^{(n_1)}(x), g^{(n_2)}(x), \ldots, g^{(n_r)}(x)\Big),$$

where it is assumed that the partition p of the integer n is given by $n = n_1 + n_2 + \cdots + n_r$. The coefficient $C(p)$ is given making use of a multinomial coefficient

$$C(p) = \binom{n}{n_1, n_2, \ldots, n_r} \frac{1}{m_1! m_2! \ldots}, \tag{3}$$

where m_1 of $\{n_1, n_2, \ldots, n_r\}$ are equal of one kind, m_2 are equal of a second kind, and so on.
A neater form of the Faa di Bruno formula, which is easy to prove and has no coefficient in front of each term, can be written in terms of the partitions not of n but of a set with n members. It is then easy to recover the traditional form of the formula by inserting a factor equal to the number of partitions of a set which correspond to a given partition of n.
Let $\mathcal{P}(S)$ denote the set of partitions of a finite set S. For $\pi \in \mathcal{P}(S)$, suppose the components are π_1, π_2, ... The number of members of each component will be denoted by $\#\pi_1$, $\#\pi_2$, ... and the number of components will be $\#\pi$. We then have the modified form of the Faa di Bruno formula

$$(f \circ g)^{(\#S)}(x) = \sum_{\pi \in \mathcal{P}(S)} f^{(\#\pi)}\Big(g(x)\Big)\Big(g^{(\#\pi_1)}(x), g^{(\#\pi_2)}(x), \ldots\Big). \tag{4}$$

It remains to identify some exercises that you might like to try.

1. Use (1) to obtain a formula for computing the coefficients in $\phi(t)$ recursively.
2. Prove (2).
3. Show that the number of partitions of a set corresponding to a given partition of an integer is given by (3).
4. Prove (4) by using induction on the number of members in S.
5. Remove the restriction that X is one dimensional so that $(f \circ g)^{(n)}(x)$ has to be regarded as an n–linear operator. Does this help to make sense of the modified form of the Faa di Bruno formula?

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