

On Ramanujan, continued fractions and an interesting street number

On finding his compatriot P. C. Mahalanobis working on a newspaper puzzle, Ramanujan is said to have remarked that he knew not only how to find the particular solution but how to find *all* solutions and that they could be written as a continued fraction. I wonder if it is possible to guess what he had in mind. The puzzle itself was about a street in the town of Louvain in Belgium. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500. If m is the number of houses in the street and n is the number of the particular house, then we can write down the equation connecting these as

$$1 + 2 + \dots + (n - 1) = (n + 1) + (n + 2) + \dots + m$$

or, what is equivalent,

$$\frac{n(n - 1)}{2} = \frac{m(m + 1)}{2} - \frac{n(n + 1)}{2},$$

which can be written as

$$(2m + 1)^2 - 2(2n)^2 = 1.$$

Write $2m + 1 = x$ and $2n = y$ and we have the famous ‘‘Pell’s equation’’ $x^2 - 2y^2 = 1$, with solution equal to the numerator and denominator of every second convergent of the continued fraction

$$\sqrt{2} = [1, 2, 2, 2, \dots].$$

For example, the convergents, formed by truncating the continued fractions at successive terms, are

$$1 = \frac{1}{1}, \quad 1 + \frac{1}{2} = \frac{3}{2}, \quad 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}, \quad 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{17}{12}, \quad \dots,$$

giving the solution pairs $(x, y) = (1, 1), (3, 2), (7, 5), (17, 12), \dots$, which satisfy, alternately, $x^2 - 2y^2 = -1$ and $x^2 - 2y^2 = 1$. It is easy to see that the numerators are always odd (enabling us to determine m from $2m + 1 = x$). Also for the solutions that give the correct sign for $x^2 - 2y^2$, the denominator is even, enabling us to solve for n from $2n = y$. If the solution pairs are $V_i = [x_i, y_i]^T$, $i = 1, 2, \dots$ and conventionally $x_{-1} = y_0 = 0$, $y_{-1} = x_0 = 1$, then the solutions are related by

$$V_1 = \boxed{1}V_0 + V_{-1}, \quad V_2 = \boxed{2}V_1 + V_0, \quad V_3 = \boxed{2}V_2 + V_1, \quad \dots,$$

where the numbers in boxes are found from the continued fraction itself. The difference equation satisfied by the numerators and denominators, once we have got past V_2 , has the characteristic polynomial $z^2 - 2z - 1$. This means that the characteristic polynomial for the difference equation for every *second* term will be $z^2 - 6z + 1$ (because its zeros are the squares of those of the former difference equation). If we define

$$W_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} x_2 \\ \frac{1}{2}y_2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} x_4 \\ \frac{1}{2}y_4 \end{bmatrix}, \quad W_3 = \begin{bmatrix} x_6 \\ \frac{1}{2}y_6 \end{bmatrix}, \quad \dots,$$

then

$$W_1 = \boxed{3}W_0 + W_{-1}, \quad W_2 = \boxed{6}W_1 \boxed{-} W_0, \quad W_3 = \boxed{6}W_2 \boxed{-} W_1, \quad \dots,$$

where the boxed coefficients and signs show that the solutions are now formed from a modified type of continued fraction

$$3 - \frac{1}{6 - \frac{1}{6 - \frac{1}{6 - \dots}}}$$

My guess is that Ramanujan was thinking of *this* continued fraction. The solution to the original puzzle? Calculate W_4 and find m from $2m + 1 = 577$ and n from the second component of this vector. I have made a number of unsubstantiated statements in this *MINIATURE*. Regard these as homework. You might want also to answer some other questions: (a) How good is x_i/y_i as an approximation to $\sqrt{2}$? (b) For which square-free integers d does the Pell equation $x^2 - dy^2 = 1$ have a solution and for which d does $x^2 - dy^2 = -1$ have a solution? (c) Find another way of representing all solutions to $x^2 - 2y^2 = \pm 1$ by observing that $(1 + \sqrt{2})(1 - \sqrt{2}) = -1$.