Bradman, Beethoven, Brown and Bolt

In 1948, or thereabouts, Mr Cave, a science teacher at my school in Taumarunui, was talking to his class about cricket, which he must have found kept the attention better than science. He described, as bad batting practice, a certain dangerous stroke. A boy piped up "But Sir, Donald Bradman plays that shot". "Oh yes," Mr Cave replied, "but he is a master".

In 1954, I was taught, in a harmony class, that it is best to begin and end a piece of music with a tonic chord in the root position. In A minor, this chord would contain the notes A, C and E, with A in the bass part. The worst possible alternative would be to have E in the bass part. The second movement of the Beethoven seventh symphony was put up as a counterexample. Almost repeating Mr Cave's words, the lecturer reminded us that Beethoven was a master.

In mathematics, and sciences that use mathematics, there is sometimes the opportunity and the temptation to act like masters, substituting intuition for careful reasoning. Also in the 50s, I had the benefit of learning Physics from Professor Dennis Brown. In the arrogance of youth, I questioned an approximation he used for $n!$, as a step in a more extended argument. Technically I was right but sometimes physicists can use mathematics intuitively and informally and get away with it. It is like playing a bad cricket shot but scoring from it or composing unconventional music that sounds good. It is, however, also like faking the evidence in a criminal trial as the best means of putting a presumably guilty person behind bars.

In about 1959, during my first few years as a lecturer in Applied Mathematics, I criticised as pedantic the work of a very good student. The student was analysing the possible stability of a sequence of approximations to the solution of a differential equation. His work was perfect but I advocated instead, a clumsy argument which destroyed the elegance and the subtlety of the student's result. My older colleague, Bruce Bolt, gently taught me that in both Pure and Applied Mathematics, as well as in cricket and music, sloppiness is not, in itself, a virtue.

Deriving a good approximation to $n!$ is no harder, and much more beautiful, than finding a bad approximation. Here is one way it could be done. Let n be a positive integer and calculate $\int_{-1/2}^{1/2}\log(n+t)dt$ using first integration by parts and secondly by term by term integration of the series expansion of $\log(1+t/n)$. We find

$$(n+\tfrac{1}{2})\log(n+\tfrac{1}{2})-(n-\tfrac{1}{2})\log(n-\tfrac{1}{2})-1=\log(n)-T,$$

where

$$T=\frac{1}{2\cdot 3\cdot(2n)^2}+\frac{1}{4\cdot 5\cdot(2n)^4}+\frac{1}{6\cdot 7\cdot(2n)^6}+\cdots.$$

I now claim that

$$\frac{1}{24n-12+4/(2n-1)}-\frac{1}{24n+12+4/(2n+1)}<T<\frac{1}{24n-12}-\frac{1}{24n+12}.$$

The first of these inequalities holds because the expresssion on the left is less than the the first term of T. The second inequality follows by writing the right-hand-side as $1/24(n^2-1/4)$ and expanding in a geometric series; the terms can be compared in turn with the terms in T. Let

$$\begin{aligned}a_n&=\log(n!)-(n+\tfrac{1}{2})\log(n+\tfrac{1}{2})+(n+\tfrac{1}{2})+(24n+12+4/(2n+1))^{-1},\\ b_n&=\log(n!)-(n+\tfrac{1}{2})\log(n+\tfrac{1}{2})+(n+\tfrac{1}{2})+(24n+12)^{-1}.\end{aligned}$$

From our already established inequalities, it follows that

$$a_{n-1}<a_n<b_n<b_{n-1}.$$

Furthermore, the limits as $n\to\infty$ of a_n and b_n are each equal to $\log\sqrt{2\pi}$. Hence, for any fixed n, we have the approximation

$$n!\approx\sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\exp\left(\Big(24n+12+4\theta/(2n+1)\Big)^{-1}\right),$$

where $\theta\in(0,1)$. The standard form of Stirling's approximation

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

easily follows. The proof that a_n and b_n have limiting values $\log\sqrt{2\pi}$ is left as an exercise with several alternative answers. If interest is expressed, I will go over this detail in a later MINIATURE. In the meantime a moral: "Important though it is to think outside the square, it is also important to think inside the square".

Professor Brown dies this month, aged 100. This MINIATURE is respectfully dedicated to the memory of this inspiring Physics teacher.

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