The coefficients of $z^n/n!$ in
\[ \frac{z}{\exp(z)-1} \] are defined to be the Bernoulli numbers $B_n$. The expression $z/2 + z/(\exp(z) - 1)$ is an even function, as we easily check by changing the sign of $z$ and rearranging. Hence, apart from $B_1 = -1/2$, all odd numbered Bernoulli numbers are zero. The first few even numbered members of the sequence are found to be
\[ \begin{align*}
B_2 &= \frac{1}{6}, \\
B_4 &= -\frac{1}{30}, \\
B_6 &= \frac{1}{42}, \\
B_8 &= -\frac{1}{30}, \\
B_{10} &= \frac{5}{66}, \\
B_{12} &= -\frac{691}{2730}.
\end{align*} 
\]
I well remember, in about 1957, using a formula based on $1 + z/(1 - 2z^2/12 - z^4/720 + \cdots)$, as an alternative to $1 + z^2/2 + z^3/6 + \cdots$, to compute the exponential function. It mightn’t seem much today, but my subroutine took 8ms to do what otherwise would have taken 15ms per evaluation.

What if we interpret $z$, not as a complex number, but as the operator $d/dx$? We should then interpret $\exp(z) - 1$ as the forward difference operator because the terms in the expansion of $\exp(d/dx)Q(x)$ are formally the same as in the Taylor expansion for $Q(x + 1)$. We can then interpret
\[ Q(x) = \frac{d}{\exp(d/dx) - 1}P(x) \] as being equivalent to the equation $Q(x + 1) - Q(x) = P'(x)$ so that $P(x) = \int_x^{x+1} Q(t) dt$. Expand (2) term by term, rearrange and we find
\[ \frac{1}{2} (Q(x) + Q(x+1)) = \int_x^{x+1} Q(t) dt + \frac{1}{2!} B_2 (Q'(x+1) - Q'(x)) + \cdots. \]

Add this formula for $x = 0, 1, \ldots, n-1$ and we have a formula for the error in the trapezoidal rule approximation for integrals, otherwise known as the Euler-Maclaurin sum formula
\[ \frac{1}{2} (Q(0) + Q(n)) + \sum_{i=1}^{n-1} Q(i) - \int_0^n Q(t) dt = \sum_{i=1}^{\infty} \frac{1}{(2i)!} B_{2i} (Q^{(2i-1)}(n) - Q^{(2i-1)}(0)). \]

Obviously there are convergence questions but they disappear if $Q$ is a polynomial. For example, the well-known formulae for $\sum_{i=1}^{n} i^k$ can easily be derived for $k = 1, 2, \ldots$. Thus
\[ \sum_{i=1}^{n} i^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{24} B_2 (4n^3) + \frac{1}{4!} B_4 (24n) = \frac{1}{30} n (2n + 1)(n + 1)(3n^2 + 3n - 1). \]

If the trapezoidal rule is adapted to the computation of the integral of a periodic function in the form
\[ \int_0^{2\pi} f(\theta) d\theta \approx 2\pi \sum_{k=1}^{n-1} f \left( \frac{2\pi k}{n} \right), \] then the series expansion for the correction is formally zero, if the periodic function $f$ is analytic. This formal result translates into an asymptotic formula for the error not like a power of $n^{-1}$, as in classical quadrature formulae, but like $\exp(-\alpha n)$, where $\alpha$ depends on the integral being evaluated.

The following table shows the computation of $\int_0^{2\pi} (5 + 3 \cos(\theta))^{-1} d\theta$ (for which the exact answer is $\pi/2$), using (3) with a sequence of $n$ values.

<table>
<thead>
<tr>
<th>$n$</th>
<th>approximation</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.78539816339745</td>
<td>-0.78539816339745</td>
</tr>
<tr>
<td>2</td>
<td>1.96349540849362</td>
<td>0.39269908169872</td>
</tr>
<tr>
<td>4</td>
<td>1.61006623496477</td>
<td>0.03926990816987</td>
</tr>
<tr>
<td>8</td>
<td>1.57127522811404</td>
<td>0.00047890131914</td>
</tr>
<tr>
<td>16</td>
<td>1.5707963997590</td>
<td>0.00000007298100</td>
</tr>
<tr>
<td>32</td>
<td>1.57079632679490</td>
<td>0.00000000000000</td>
</tr>
</tbody>
</table>

Another important interpretation of (1) is found by replacing $z$ by the linear operator $X \mapsto [A, X]$, where $[\cdot, \cdot]$ denotes the commutator $[A, X] = AX -XA$. This means that 1 corresponds to the identity operator and $z^2$ corresponds to $X \mapsto [A, [A, X]]$. The derivative of $\exp(A)$ with respect to $A$ is found to be
\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \exp(A + \epsilon X) - \exp(A) \right) = \left( X + \frac{1}{2!} [A, X] + \frac{1}{3!} [A, [A, X]] + \cdots \right) \exp(A). \] (4)

In geometric integration, the inverse of the linear operator represented by the first factor on the right-hand side of (4) is needed. This is found formally as
\[ X \mapsto X - \frac{1}{2} [A, X] + \frac{1}{2!} B_2 [A, [A, X]] + \frac{1}{4!} B_4 [A, [A, [A, X]]] + \cdots. \]

In all these diverse applications, the unifying themes are Bernoulli numbers and the expansion of (1).