Some applications of Bernoulli numbers

The coefficients of $z^n/n!$ in

$$\frac{z}{\exp(z)-1} \tag{1}$$

are defined to be the Bernoulli numbers B_n. The expression $z/2 + z/(\exp(z)-1)$ is an even function, as we easily check by changing the sign of z and rearranging. Hence, apart from $B_1 = -\frac{1}{2}$, all odd numbered Bernoulli numbers are zero. The first few even numbered members of the sequence are found to be

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

I well remember, in about 1957, using a formula based on $1+z/(1-z/2+z^2/12-z^4/720+\cdots)$, as an alternative to $1+z+z^2/2+z^3/6+\cdots$, to compute the exponential function. It mightn't seem much today, but my subroutine took 8ms to do what otherwise would have taken 15ms per evaluation.

What if we interpret z, not as a complex number, but as the operator d/dx? We should then interpret $\exp(z)-1$ as the forward difference operator because the terms in the expansion of $\exp(d/dx)Q(x)$ are formally the same as in the Taylor expansion for $Q(x+1)$. We can then interpret

$$Q(x) = \frac{\frac{d}{dx}}{\exp(\frac{d}{dx})-1} P(x) \tag{2}$$

as being equivalent to the equation $Q(x+1) - Q(x) = P'(x)$ so that $P(x) = \int_x^{x+1} Q(t)dt$. Expand (2) term by term, rearrange and we find

$$\frac{1}{2}\big(Q(x) + Q(x+1)\big) = \int_x^{x+1} Q(t)dt + \frac{1}{2!}B_2\big(Q'(x+1) - Q'(x)\big) + \cdots .$$

Add this formula for $x = 0, 1, \ldots, n-1$ and we have a formula for the error in the trapezoidal rule approximation for integrals, otherwise known as the Euler-Maclaurin sum formula

$$\frac{1}{2}\big(Q(0) + Q(n)\big) + \sum_{i=1}^{n-1} Q(i) - \int_0^n Q(t)dt = \sum_{i=1}^{\infty} \frac{1}{(2i)!} B_{2i}\big(Q^{(2i-1)}(n) - Q^{(2i-1)}(0)\big).$$

Obviously there are convergence questions but they disappear if Q is a polynomial. For example, the well-known formulae for $\sum_{i=1}^n i^k$ can easily be derived for $k = 1, 2, \ldots$. Thus

$$\sum_{i=1}^n i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{2!}B_2(4n^3) + \frac{1}{4!}B_4(24n) = \frac{1}{30}n(2n+1)(n+1)(3n^2+3n-1).$$

If the trapezoidal rule is adapted to the computation of the integral of a periodic function in the form

$$\int_0^{2\pi} f(\theta)d\theta \approx \frac{2\pi}{n}\sum_{k=0}^{n-1} f\left(\frac{2\pi k}{n}\right), \tag{3}$$

then the series expansion for the correction is formally zero, if the periodic function f is analytic. This formal result translates into an asymptotic formula for the error not like a power of n^{-1}, as in classical quadrature formulae, but like $\exp(-\alpha n)$, where α depends on the integral being evaluated.

The following table shows the computation of $\int_0^{2\pi}\big(5+3\cos(\theta)\big)^{-1}d\theta$ (for which the exact answer is $\pi/2$), using (3) with a sequence of n values.

n	approximation	error
1	0.78539816339745	−0.78539816339745
2	1.96349540849362	0.39269908169872
4	1.61006623496477	0.03926990816987
8	1.57127522811404	0.00047890131914
16	1.57079639977590	0.00000007298100
32	1.57079632679490	0.00000000000000

Another important interpretation of (1) is found by replacing z by the linear operator $X \mapsto [A, X]$, where $[\cdot,\cdot]$ denotes the commutator $[A, X] = AX - XA$. This means that 1 corresponds to the identity operator and z^2 corresponds to $X \mapsto [A, [A, X]]$. The derivative of $\exp(A)$ with respect to A is found to be

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\exp(A+\epsilon X) - \exp(A)\big) = \Big(X + \frac{1}{2!}[A, X] + \frac{1}{3!}[A, [A, X]] + \cdots\Big)\exp(A). \tag{4}$$

In geometric integration, the inverse of the linear operator represented by the first factor on the right-hand side of (4) is needed. This is found formally as

$$X \mapsto X - \frac{1}{2}[A, X] + \frac{1}{2!}B_2[A, [A, X]] + \frac{1}{4!}B_4[A, [A, [A, [A, X]]]] + \cdots .$$

In all these diverse applications, the unifying themes are Bernoulli numbers and the expansion of (1).

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