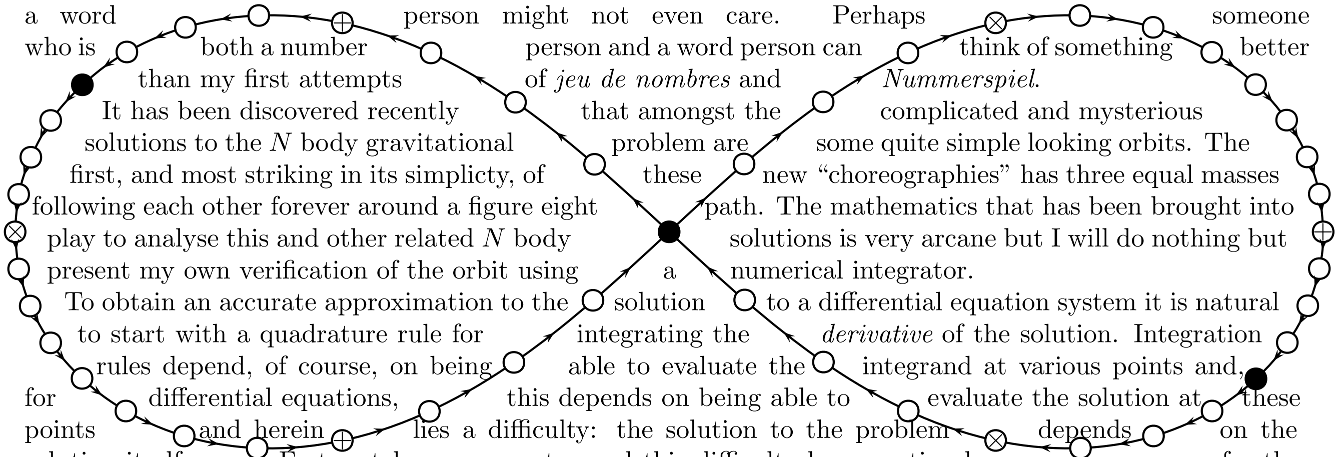
MATHEMATICAL MINIATURE 17

The three-eighths rule and a figure eight three body orbit



If a pun is a play on words, what is a play on numbers? If words and numbers belong to different halves of the brain it will not be easy for a number person to come up with a good suggestion. On the other hand, a word person might not even care. Perhaps someone who is both a number person and a word person can think of something better than my first attempts of *jeu de nombres* and *Nummerspiel*.

It has been discovered recently that amongst the complicated and mysterious solutions to the N body gravitational problem are some quite simple looking orbits. The first, and most striking in its simplicty, of these new "choreographies" has three equal masses following each other forever around a figure eight path. The mathematics that has been brought into play to analyse this and other related N body solutions is very arcane but I will do nothing but present my own verification of the orbit using a numerical integrator.

To obtain an accurate approximation to the solution to a differential equation system it is natural to start with a quadrature rule for integrating the *derivative* of the solution. Integration rules depend, of course, on being able to evaluate the integrand at various points and, for differential equations, this depends on being able to evaluate the solution at these points and herein lies a difficulty: the solution to the problem depends on the solution itself. Fortunately, we can get round this difficulty by accepting lower accuracy for the solution evaluations at the integration points than we aim for in the quadrature formula.

The name Runge, whose famous paper appeared just over 100 years ago, is forever associated with the numerical methods based on this principle of nesting quadrature formulae within quadrature formulae. We want step by step methods in which the approximate solution is advanced from one point to the next in steps that can be made small with the hope of steadily increasing accuracy. Thus if an approximate solution y_{n-1} is known at x_{n-1} then the approximation at $x_n = x_{n-1} + h$ is given by

$$y_n = y_{n-1} + h(b_1 F_1 + b_2 F_2 + \cdots + b_s F_s),$$

where, for $i = 1, 2, \ldots, s$, F_i is an approximation to the derivative of $y(x)$ evaluated at $x_{n-1} + hc_i$. These approximations to $y'(x_{n-1} + hc_i)$ are found from the given differential equation using internal approximations $Y_i \approx y(x_{n-1} + hc_i)$. In the spirit of Nummerspiel I will focus on one of the classical Newton-Cotes formulae known as the "three-eighths rule". This is slightly more accurate than the popular Simpson's rule, but more expensive to use because it requires one more evaluation of the integrand. For the three-eighths rule the b_i and c_i are given by the vectors

$$b^T = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}, \qquad c^T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}.$$

To turn this quadrature rule into a Runge-Kutta method, so that it can be used to solve differential equations, we need four internal quadrature rules which are chosen so that rule number $i = 1, 2, 3, 4$ uses only the abscissa number j if F_j has already been evaluated. The internal rules also have to be cunningly chosen so that the errors committed in the stages cancel each other out as much as possible when combined into the overall quadrature rule. It turns out that the internal rules must be those given in turn by the vectors

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -\frac{1}{3} & 1 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}.$$

The numerical method based on these quadrature rules was discovered by Kutta as an example from his complete classification of fourth order Runge-Kutta methods.

My calculation of the figure eight three body orbit, using this method, is shown on this page. It uses initial data computed by Carles Simo and just follows each of the three bodies far enough for them to reach where the one they are following started from. I have indicated the initial point I have used in my computations by representing the masses by filled discs ●. When they reach the position shown as $\otimes$, and again at $\oplus$, the three masses form an isosceles triangle.

To learn more about this and other interesting orbits, good places to start are

- http://www.ams.org/new-in-math/cover/orbits1.html
- Alain Chenciner and Richard Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math. (2)* **152** (2000), 881–901.

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