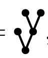
Numerical Analysis and Hopf Algebras

H. G. Forder, who was Professor of Mathematics at Auckland University College from 1934 until his retirement in 1955, was the first real mathematician I had ever met and I was always in awe of him. He told me that numerical analysis was boring and of no interest to a serious mathematician. I now realise that this was not a well-considered opinion and could only be regarded as a prejudice. Paul Halmos once burst into print under the title "Applied Mathematics is Bad Mathematics". I read this document carefully with an eye open for the irony that I felt must have been present; I did not believe such a great mathematician could, so lightly, dismiss the work of many other mathematicians. But I never recognised even a trace of irony — he really did intend to say what he seemed to be saying. Thus I enjoy a certain measure of perverse satisfaction that a small discovery of mine, which was motivated by nothing more than its application to that small part of Applied Mathematics known as Numerical Analysis, should now have a life of its own. I do not fully understand what a Hopf Algebra is but I have been assured by people who do, that the entity I am describing in this Miniature, is an example of such a structure.

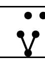
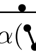
Let T denote the set of all rooted trees and let G denote the set of all mappings from T to $\mathbb{R}$. Multiplication is defined on $G \times G$ to G by the formula

$$(\alpha\beta)(t) = \alpha(t) + \beta(t) + \sum_{u \lhd t} \alpha^*(t \setminus u)\beta(u),$$

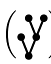
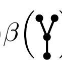
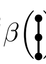
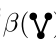
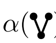
where $u \lhd t$ denotes that u is formed from t by deleting a positive number of vertices, such that u is itself connected and shares the same root with t. In this notation, $t \setminus u$ denotes the graph formed by deleting u from t and α^* denotes the multiplicative extension of α. For example, if

$t =$ , (1)

then there are nine choices for u. These are shown, together with $t \setminus u$, in the following table, which also gives the value of $\alpha^*(t \setminus u)\beta(u)$.

u and $t \setminus u$									
$\alpha^*(t \setminus u)\beta(u)$	$\alpha(\bullet)\beta$()	$\alpha(\bullet)\beta$()	$\alpha(\bullet)\beta$()	$\alpha(\bullet)^2\beta$()	$\alpha(\bullet)^2\beta$()	$\alpha(\bullet)^2\beta$()	$\alpha(\bullet)^3\beta$()	α()β()	$\alpha(\bullet)\alpha$()$\beta(\bullet)$

Combining identical terms, we find

$(\alpha\beta)$() $= \alpha$() $+ \beta$() $+ \alpha(\bullet)\beta$() $+ 2\alpha(\bullet)\beta$() $+ 2\alpha(\bullet)^2\beta$() $+ \alpha(\bullet)^2\beta$() $+ \alpha(\bullet)^3\beta$() $+ \alpha$()β() $+ \alpha(\bullet)\alpha$()$\beta(\bullet)$.

Under this operation, G is a group. Let $r(t)$ denote the "order" (number of vertices) in t. The subset of G denoted by H_p, for p a positive integer, is defined to contain those members which map every $t \in T$ with $r(t) \le p$ to zero. It turns out H_p is a normal subgroup which along with the quotient group G/H_p and G itself are directly related to Runge-Kutta methods. The relation is that every Runge-Kutta method has a corresponding element of G associated with it and the group operation corresponds to composition of Runge-Kutta methods over successive computational steps. The special element $E \in G$ is defined by $E(t) = 1/t!$, where the factorial of a tree is the product over all vertices of the subtree formed by selecting this vertex and its successors. In the example in (1), $t! = 8$. The details of a particular Runge–Kutta method are usually expressed in terms of a matrix A, together with vectors c and b^T. To find the group element corresponding to (A, b^T, c), evaluate for each $t \in T$ an expression formed by replacing every leaf by c and each internal arc by an application of the linear operator A. Replace each internal node by the componentwise product of the quantities branching out from it. Finally, operate on the vector evaluated at the root by the functional b^T. In the case of (1), this gives $\sum_{i,j} b_i c_i a_{ij} c_j^2$. For some well-known Runge-Kutta methods, the corresponding group element is easy to write down. For example, in the case of the Euler method, the single one vertex tree maps to 1 and all other trees map to zero. For the implicit Euler method, $t \mapsto 1$ for all $t \in T$. If α corresponds to a given Runge–Kutta method, then α being in the coset EH_p is equivalent to "the Runge–Kutta method has order p". The concept of "effective order", which allows the computational benefits of high order to be shared more widely, allows α to be *conjugate* to a member of EH_p. For example the implicit midpoint rule for which $t \mapsto 2^{1-r(t)}$ and the implicit trapezoidal rule for which $t \mapsto 2^{-1}$ except for the tree with one vertex, which maps to 1, are conjugates of each other and each has order 2. The easiest way to find out more details is to write to me.

One of my reasons for writing this rather personal note is that I made a mistake in Miniature 14 and this has put me off number theory for a while. Irine Peng pointed out the error and, although further discussions with her led me to see how to repair the error quite nicely, my hope that we would write a correction together was stymied by her departure for graduate studies overseas. Another reason for the choice of subject is that it is an anniversary celebration of Kutta's seminal paper, which appeared in 1901.

John Butcher, `butcher@math.auckland.ac.nz`