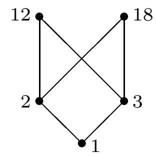
The inverses of some triangular matrices and the Möbius function

Let $(S, \leq)$ denote a partially ordered set and, for $i, j \in S$, let $m_{ij} = 1$, if $j \leq i$, and zero otherwise. Assume that the members of S are ordered in such a way that M is lower triangular with 1 on the diagonals. If S is not finite but has a minimum element, then M is an infinite matrix and represents a linear operator on the set of sequences indexed by the elements of S. Let $\widetilde{m}_{ij}$ denote the (i, j) element of M^{-1}.

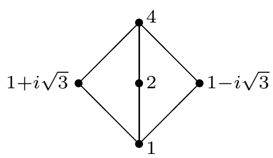
In the following two examples, $\leq$ is defined by divisibility. In the case of Example 2, this is defined, not in the ring $\mathbb{Z}$, but in $\mathbb{Z}[\sqrt{-3}]$ (a ring which does not enjoy the benefits of unique factorization) and the five elements attached to the vertices of the graph are in the order $1, 2, 1 + i\sqrt{3}, 1 - i\sqrt{3}, 4$

Example 1



$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{bmatrix}$$

Example 2

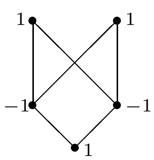


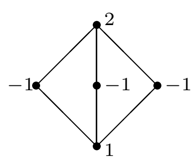
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & -1 & 1 \end{bmatrix}$$

If $j \leq i$ with $j \neq i$ we define the interval $[j, i]$ as the set consisting of each vertex x such that $j \leq x \leq i$. Because row j of M is orthogonal to column i of M^{-1}, it follows that

$$\sum_{j \leq x \leq i} \widetilde{m}_{x,j} = 0.$$

This enables the elements in column i of M^{-1} to be evaluated recursively. In Examples 1 and 2, the calculations of the first columns of M^{-1} are represented on the graphs as shown in the following diagrams





Now consider a countably infinite partially ordered set, where S consists of all sequences of non-negative integers, $(i_1, i_2, \ldots)$, where all but a finite number are zero, and $j \leq i$ means that $j_k \leq i_k$ for all $k = 1, 2, \ldots$. We can order the elements of S so that each element eventually arises, by associating with $i \in S$ a sequence number $n(i)$ equal to

$$n(i) = \prod_{k=1}^{\infty} p_k^{i_k},$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, … is the sequence of primes. For any $i \in S$, we will conventionally write i_k, not only to represent component number k, but also the member of S formed from i by replacing every component, except number k, by zero.

If $\theta \in S$, then the sub-graph associated with the interval $[j + \theta, i + \theta]$ is isomorphic with the subgraph for the interval $[j, i]$. Hence, $\widetilde{m}_{i,j}$ is a function only of $i - j$. In the interpretation provided by the mapping $i \mapsto n(i)$, this means that $\widetilde{m}_{i,j} = \mu\big(n(i)/n(j)\big)$, where μ is said to be the "Möbius function". We will show that $\mu(2^{i_1} 3^{i_2} 5^{i_3} \cdots)$ is zero if any of i_1, i_2, i_3, …, exceeds 1 and otherwise is equal to $(-1)^{i_1 + i_2 + \cdots}$. To calculate the first column of M^{-1}, and hence the value of $\mu(m)$ for m a positive integer, we first consider the case that only one of the i components is non-zero — this corresponds to the evaluation of μ for a prime power. The sub-graph consists of a chain with integers 1 attached to the root, -1 to its neighbour and 0 attached to each other vertex. Now consider an interval $[j, i]$, with $j = (0, 0, \ldots)$ and $i = (i_1, i_2, \ldots, i_N, 0, 0, \ldots)$. For x in $[j, i]$, let $\psi(x)$ denote the value attached to the corresponding vertex in the sub-graph representing $[j, i]$. The fact that $\psi(x) = \prod_{k=1}^{N} \psi(x_k)$, follows by induction because the sums of $\psi(y)$ and $\prod_{k=1}^{N} \psi(y_k)$ over all $y \in [j, x]$ are each zero and because the individual terms are equal if $y \neq x$. Hence $\widetilde{m}_{j,i} = \prod_{k=1}^{N} \widetilde{m}_{j,i_k}$.

If f is a function on the positive integers and F is defined by

$$F(n) = \sum_{d|n} f(d),$$

this can be written as $F = Mf$ and we have the inversion formula

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d),$$

which is another way of writing $f = M^{-1}F$. Applications abound in number theory.

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