Mathematical Apology 9
Homotopy and $\sqrt{2}$

J. C. Butcher, The University of Auckland

What is the next step in evolution beyond Homo Sapiens? We can get some clue about popularly held opinions on this question through a particular type of cartoon drawing that graces staff notice boards here and there. They show a sequence of more or less human forms changing from a primitive appearance and posture, through to a modern human appearance and then on to some proposed next step. One that I have seen suggests that the wise human has advanced from standing upright to crouching again, this time in front of a computer screen. Others suggest that it is football that distinguishes us, not only from other primates, but also from our more primitive human ancestors. Neither of these claims appeals much to me. I wonder if Mathematics has a stronger claim. While mathematics, as a human behaviour, is as intellectual as almost anything people do, even this does not really constitute such a great step forward as to be regarded as an evolutionary leap. On the other hand, occasionally indulging in pride that we have got somewhere towards understanding a small corner of this vast and difficult subject, acts as an antidote to the party-stopping reaction we usually get when we tell people what we do and what we teach. But why feel either pride or shame in belonging to a club that anyone can join?

Even though the fellowship of mathematics has open membership it has continually changing customs and patterns of behaviour. Mathematics has become more abstract than it used to be and many questions are now looked at from a fresh point of view. Geometry and Analysis, two of the great cornerstones of mathematical understanding a hundred years ago, have been joined by Topology as a profound subject in its own right and also as a unifying force. We will show that even such a mundane question as evaluating the square root of 2 can be looked at from a topological point of view. Thus, continuity and smoothness of change can be just as significant in considering computational questions as the discrete and the digital.

The so-called homotopy approach to computation is to gradually change a problem from one where we *already* know the answer into one for which we *wish* to know the answer. Sometimes we "cannot get there from here" but sometimes we can. I chattered on about Homo Sapiens at the begining of this dissertation in the hope that I could make some sort of pretentious pun out of the words Homo and Topological but it now seems better to use the word "homotopy" purely in its technical sense, as a mapping which takes $t \in [0,1]$ to $\Phi(t)$. We wish to relate $\Phi(1)$ to $\Phi(0)$ using properties of the mapping.

Let $x(t)$ denote the function defined by $x = \sqrt{1+t}$ so that

$$x^2 = 1 + t.$$

In this case $\Phi(t)$ will denote the problem of solving this equation. When $t = 0$ we have a solution equal to $x = 1$ and we want to move towards $t = 1$ for which $x = \sqrt{2}$. How then are x and t related? They could for example be connected by the initial value differential equation problem

$$\dot{x} = \frac{1}{2x}, \qquad x(0) = 1, \tag{1}$$

or perhaps by the problem

$$\dot{x} = \frac{x}{2(1+t)}, \qquad x(0) = 1. \tag{2}$$

The twentieth century was a great time for discoveries about numerical methods for solving this type of problem. The year 2001, for example, celebrates the centenary of the paper by Kutta in which the Runge-Kutta method became firmly established as an accurate and reliable method for solving initial value problems. We will select the most famous member of this family of numerical schemes. To advance an approximation to the solution of the problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

to a nearby point $t_1 = t_0 + h$, write the result $y_1 \approx y(t_0 + h)$ in the form

$$y_1 = y_0 + \frac{h}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right),$$

where

$$\begin{aligned} F_1 &= f(t_0, y_0), & F_2 &= f(t_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}hF_1), \\ F_3 &= f(t_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}hF_2), & F_4 &= f(t_0 + h, y_0 + hF_3). \end{aligned}$$

The cunning way that the benefits of Simpson's rule are brought to the aid of differential equations is as remarkable as it is simple. The scheme usually works much better, for example, than the method of Euler which uses only the F_1 , instead of a linear combination of F_1, F_2, F_3 and F_4, to approximate the average derivative of y over the step. Thus the Euler method gives the result

$$y_1 = y_0 + hf(t_0, y_0).$$

It costs only a quarter as much in terms of f evaluations but is, nevertheless, usually not as efficient.

When we carry out a single step of the Euler method, using either problem (1) or problem (2), we obtain the same answer as we would find by a single step of the Newton-Raphson method for finding square roots. However, if we take n steps, with $h = 1/n$, we get a sequence of increasing good approximations to $\sqrt{2}$ for each of the two problems

n	Problem (1)	Problem (2)
1	1.50000000	1.50000000
2	1.45000000	1.45833333
3	1.43679654	1.44375000
4	1.43071119	1.43638393

For the Runge-Kutta method there is an enormous improvement

n	Problem (1)	Problem (2)
1	1.41437908	1.41493056
2	1.41422370	1.41427766
3	1.41421549	1.41422757
4	1.41421415	1.41421819

Notice that, even for a single step of the Runge-Kutta method, the result is much closer to the correct answer (1.41421356) than for four small steps of the Euler method.

Having obtained an approximation after one application of one of these homotopy schemes, we do not need to rest there. We can then start a new homotopy for the problem

$$x^2 = X^2(1 - t) + 2t,$$

where X is the approximation already found. For $t = 0$ we have $x = X$ and for $t = 1$, we hope to get close to $x = \sqrt{2}$.