Mathematical Apology 3
Approximation of irrational numbers by rational numbers

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In this apology we interrupt our sequence of discussions on the computation of π and consider instead the approximation of irrational numbers, such as π, by rational numbers.

For simplicity we will consider the approxmation of an irrational number x in the interval $[0,1]$. Thus, we could for example deal with $x = \pi - 3$ rather than π itself. Given any denominator d, we can always find a numerator n such that

$$\left| x - \frac{n}{d} \right| < \frac{1}{2d}. \tag{1}$$

All we have to do is choose n as the closest integer to xd.

For some choices of d we can do much better than this. The famous approximation $\pi \approx \frac{22}{7}$ has an error less than $\frac{1}{16\times 7^2}$. We will show that it is possible to choose arbirarily high values of d so that (1) can be replaced by

$$\left| x - \frac{n}{d} \right| < \frac{1}{d^2}. \tag{2}$$

To accomplish this task we introduce what are known as Farey series. Let F_D denote the set of all rational numbers in $[0,1]$ such that the denominator of any member of the set is no greater than D. The first few examples are

$$\begin{aligned}
F_1 &= \{0, 1\} \\
F_2 &= \left\{0, \frac{1}{2}, 1\right\} \\
F_3 &= \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\} \\
F_4 &= \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}
\end{aligned}$$

If n_1/d_1 and n_2/d_2 are two successive members of the Farey series F_D, for $D > 1$, then (i) d_1 and d_2 have no common factor (that is, they are "relatively prime"), (ii) the distance between them is

$$\frac{n_2}{d_2} - \frac{n_1}{d_1} = \frac{1}{d_1 d_2}.$$

and (iii) furthermore $d_1 + d_2 > D$.

To justify these assertions we note first of all that if (iii) were not true, then $(n_1+n_2)/(d_1+d_2)$ would also be in F_D and it is easy to verify that this lies between n_1/d_1 and n_2/d_2, which were supposed to be adjacent members of F_D. We will not worry about (i), because this is an immediate consequence of (ii) which we will prove by induction. Suppose the result has already been proved for F_D and consider $N/(D+1)$ in F_{D+1}. Suppose that this falls between n_1/d_1 and n_2/d_2, two successive members of F_D. We need to prove that

$$\frac{N}{D+1} - \frac{n_1}{d_1} = \frac{K_1}{d_1(D+1)} \qquad \text{and} \qquad \frac{n_2}{d_2} - \frac{N}{D+1} = \frac{K_2}{(D+1)d_2},$$

where the integers K_1 and K_2 are each equal to 1. Add these formulae and we find that

$$\frac{1}{d_1 d_2} = \frac{K_2 d_1 + K_1 d_2}{(D+1) d_1 d_2},$$

implying that $K_2 d_1 + K_1 d_2 = D + 1$ and hence that $(K_2 - 1)d_1 + (K_1 - 1)d_2 = D + 1 - d_1 - d_2$. Since the right-hand side cannot be positive, $K_1 = K_2 = 1$.

We now know enough about Farey series to use them to approximate an irrational number x. Place x between two successive members of F_D, say n_1/d_1 and n_2/d_2 and then compare x with $(n_1 + n_2)/(d_1 + d_2)$. There are two cases: either (a) $n_1/d_1 < x < (n_1 + n_2)/(d_1 + d_2)$ or (b) $(n_1 + n_2)/(d_1 + d_2) < x < n_2/d_2$. In case (a), define the approximation n/d as n_1/d_1 and in case (b) define $n/d = n_2/d_2$. The distance between x and n/d is, in each case, less than the distance between $(n_1 + n_2)/(d_1 + d_2)$ and n/d. Hence, in case (a)

$$\left|x - \frac{n}{d}\right| < \frac{n_1 + n_2}{d_1 + d_2} - \frac{n_1}{d_1} = \frac{(n_1 + n_2)d_1 - (d_1 + d_2)n_1}{d_1(d_1 + d_2)} = \frac{n_2 d_1 - d_2 n_1}{d_1(d_1 + d_2)} = \frac{1}{d_1(d_1 + d_2)},$$

where $n_2 d_1 - d_2 n_1 = 1$ because of the known difference between n_1/d_1 and n_2/d_2. In case (b) a similar calculation gives a bound

$$\left|x - \frac{n}{d}\right| < \frac{1}{d_2(d_1 + d_2)}$$

and in each case we have

$$\left|x - \frac{n}{d}\right| < \frac{1}{d(D+1)} < \frac{1}{d^2}. \tag{3}$$

The last detail to consider is why there should be an infinite number of such choices of d. For an approximation satisfying (3), there exists some $\overline{D}$ such that the error is greater than $1/d(\overline{D}+1)$, and hence a better approximation would have been found if we had used $F_{\overline{D}}$ instead of F_D in which to search for it.

The use of Farey series to show how solutions to (2) can be constructed is not really practical as a method of finding good approximations for particular irrational numbers. Some time in the future a more efficient approach will be discussed. Also on the agenda is at least one more apology concerned wth the evaluation of π.

The author of these Apologies requests some comment on them to make sure that they are not too difficut or to easy for readers of this Magazine.