Mathematical Apology 2

Professor John Butcher, The University of Auckland

In this series of Apologies on issues connected with π, especially the computation of π, we look at a formula that was popular in the seventeenth century. This "Wallis formula" consists of an infinite product with repeated even integers on the top and repeated odd integers on the bottom.

$$\pi = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots .$$

We might well ask where such a formula comes from. Like many formulae involving purely real numbers, the best way to understand results like this is to work in the more esotoric, but in some senses simpler, world of complex numbers. One example which makes complex numbers simpler than real numbers is in the zeros of polynomials. These might not exist in the sense of real numbers and even if they do there might not be enough of them. However, we can always factorise polynomials over the complex numbers and we can always factorise into linear factors. This result is known as "the fundamental theorem of algebra". There are other functions that do not quite achieve this level of simplicity but get quite close to it. One example is the function $\sin(\pi z)/\pi z$ (where it is defined for $z = 0$ to have the limiting value of 1). For this function there are an infinite number of zeros located at $z = \pm 1, \pm 2, \pm 3, \ldots$ and a generalisation of the fundamental theorem enables us to write

$$\frac{\sin(\pi z)}{\pi z} = \left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right)\left(1 - \frac{z^2}{3^2}\right) \cdots . \tag{1}$$

Substitute $z = \frac{1}{2}$ and take the reciprocal of both sides and we arrive at the Wallis product formula for π.

Here is how the first 20 approximations using the formula work out. Also shown (in the third column) is the mean of each two adjacent approximations and this looks as though it is an improvement.

1	4.0000000000	
		3.3333333333
2	2.6666666667	
		3.1111111111
3	3.5555555556	
		3.2000000000
4	2.8444444444	
		3.1288888889
5	3.4133333333	
		3.1695238095
6	2.9257142857	
		3.1346938776
7	3.3436734694	
		3.1579138322
8	2.9721541950	
		3.1372738725
9	3.3023935500	
		3.1522847523
10	3.0021759546	
		3.1386384979
11	3.2751010413	
		3.1491356167
12	3.0231701920	
		3.1394459686
13	3.2557217452	
		3.1471976871
14	3.0386736289	
		3.1399627498
15	3.2412518708	
		3.1459209334
16	3.0505899961	
		3.1403132312
17	3.2300364664	
		3.1450355068
18	3.0600345471	
		3.1405617721
19	3.2210889970	
		3.1443964018
20	3.0677038066	

This is not a very convenient method of computing π to high accuracy because the convergence is so poor. In the next Apology in this series a much better method will be discussed. In the meantime let us consider if any more goddies can be squeezed out of (1).

The series for the sin function is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{2}$$

If you believe that such a series exists then this must be it because the limit of $\sin(x)/x$ (as $x \to 0$) works out right and so does the requirement that the second derivative is given by $\sin''(x) = -\sin(x)$. Another way of realising that this must be the right series is by the relation $\exp(ix) = \cos(x) + i\sin(x)$: all we need to do is to substitute ix into the exponential series and then pick out the purely imaginary terms.

Divide both sides of (2) by x, substitute $x = \pi z$ and we find

$$\frac{\sin(\pi z)}{\pi z} = 1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120} - \frac{\pi^6 z^6}{5040} + \cdots. \tag{3}$$

Even though (1) is written as an infinite product, it is possible to expand it in terms of powers of z. The coefficient of z^2 is easy to find because we need to select the term "1" from all factors on the right hand side of (1) except for just one of these factors. Do this in all possible ways and we find the coefficient of z^2 to be $-1^{-2} - 2^{-2} - 3^{-2} - \cdots$.

Comparing this with (3) we conclude that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{4}$$

Finding the z^4 coefficient in (1) is a little more complicated because we need to choose pairs of factors in of the form $-z^2/m^2$ and $-z^2/n^2$ where $m < n$. Twice the result will give $\sum_{m \neq n} m^{-2} n^{-2}$ and allowing m to equal n in the set of (m, n) pairs in the summation will cause the result to be overstated by $\sum_{n=1}^{\infty} n^{-4}$. Furthermore the sum $\sum m^{-2} n^{-2}$, where there is nor restriction on the positive integers m and n, is exactly the same as $(\sum n^{-2})^2$. Hence, putting this all together and comparing with the z^4 term in (3) leads to

$$2 \times \frac{\pi^4}{120} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 - \sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\frac{\pi^2}{6} \right)^2 - \sum_{n=1}^{\infty} \frac{1}{n^4},$$

implying that

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}. \tag{5}$$

The two sums (4) and (5) are not very practical for evaluating π because even when we have obtained good approximations to the infinite sums, we still have to calculate square (or fourth) roots.

Nevertheless we are much better off than we would have been in the days of Wallis, because we now have computers and it is possible to do the calculations very easily and quickly. The following table shows approximations to π worked out from the partial sums of (4) and (5).

1	2.4494897428	3.0800702882	11	3.0574815067	3.1414341947
2	2.7386127875	3.1271078664	12	3.0642878178	3.1414691945
3	2.8577380332	3.1361523798	13	3.0700753719	3.1414946046
4	2.9226129861	3.1389978894	14	3.0750569156	3.1415134957
5	2.9633877010	3.1401611795	15	3.0793898260	3.1415278307
6	2.9913764947	3.1407217179	16	3.0831930203	3.1415389040
7	3.0117739478	3.1410241579	17	3.0865580258	3.1415475928
8	3.0272978567	3.1412014021	18	3.0895564350	3.1415545057
9	3.0395075896	3.1413120396	19	3.0922450523	3.1415600741
10	3.0493616360	3.1413846225	20	3.0946695241	3.1415646096

Exercises

1. Substitute $z = \frac{1}{6}$ in (1) to obtain a new product expression for π. Is this formula of practical value?

2. Why does a calculation of π using (5) seem to work better than the similar calculation based on (4)?

3. Find a formula for π^6 based on the z^6 coefficient in (3). How would this rate as the basis for a calculation of π?

4. Find similar formulae to (4) and (5) where the summations now run only over the *odd* positive integers.

Correspondence on the exercises, or on any aspect of this column, is welcome. If it seems to be a good idea at the time, some remarks on the exercises will be given in the next issue.

J. C. Butcher
Department of Mathematics
The University of Auckland
Private Bag 92019, Auckland
butcher@math.auckland.ac.nz