Mathematical Apology Number 14: Games with Fibonacci Numbers

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It came as a surprise to me, when I first heard about it, that there is a Journal, "The Fibonacci Quarterly", which devotes itself entirely to the sequence of numbers referred to in its title. How is it possible to go on year by year, publishing four issues per annum, dealing with nothing other than Fibonacci numbers?

Everyone knows that the Fibonacci numbers are formed from the "difference equation"

$$F_n = F_{n-1} + F_{n-2}, \qquad n = 3, 4, 5, \ldots ,$$

with "starting values" $F_1 = 1$, $F_2 = 2$. Other sequences satisfying the same recurrence, but with different starting values, also have names, such as Lucas numbers, but it is the difference equation itself which characterises the long-term behaviour of the sequence. Except for some exceptional values of the starting values, those with a ratio

$$\frac{F_2}{F_1} = -\frac{\sqrt{5}-1}{2},$$

the ratio of F_n/F_{n-1} always has the same limit for large n:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$$

This limiting value is known as "the Golden Ratio" and is given the symbol ϕ.

Many years ago, I heard about a conjecture concerning Fibonacci sequences. I was able to add some extra details and to prove it. I would like to describe the problem in terms of three questions. These are put forward as a challenge to keen readers. At least the first of the three statements in this challenge is relatively easy to verify. My own solution from the seventies is buried in the archives of the Fibonacci Quarterly.

The problem concerns a table made up as follows:

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	$\cdots$
R_1	1	2	3	5	8	13	21	34	$\cdots$
R_2	4	6	10	16	26	42	68	110	$\cdots$
R_3	7	11	18	29	47	76	123	199	$\cdots$
R_4	9	15	24	39	63	102	165	267	$\cdots$
R_5	12	19	31	50	81	131	212	343	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The numbers in the R_1 row are just the Fibonacci numbers. This means that in the S_1 column, we start with 1 and in the S_2 column we put the closest integer to ϕ times 1, which is of course 2. The remaining columns entries in the row are formed from the Fiboncci difference equation, so that the entry in column S_n is the sum of the entries in the previous two columns.

The numbers in R_2 are started by placing in the S_1 position the first positive integer, that is 4, that has not appeared in row R_1. We then place in the S_2 position the closest integer to 4ϕ and carry on to later columns using the Fibonacci formula of simply adding the entries in the previous two columns.

In row R_3, we start by inserting into S_1 the first positive integer that has not appeared in either R_1 or R_2 and we follow this by the closest integer to this number times ϕ. After these entries, 7 and 11, we carry on with the entries 7 + 11 = 18, 11 + 18 = 29, and so on.

Perhaps the pattern will become clear if we point out that the S_1 entry in R_4 is the first positive integer, 9, which has not appeared in any of R_1, R_2 or R_3. Thus we carry on row by row to complete the whole table. Eventually every positive integer n appears, at least once somewhere, because if it had not appeared in any of rows $R_1, R_2, \ldots, R_{m-1}$ but all integers less than n had already appeared, then n will be in the S_1 column of row R_m.

To complete the picture, insert an additional column S_0 defined as the difference between S_2 and S_1. The row names $R_1, R_2, \ldots$, can now be dispensed with and replaced by the additional column S_0.

S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	...
1	1	2	3	5	8	13	21	34	...
2	4	6	10	16	26	42	68	110	...
4	7	11	18	29	47	76	123	199	...
6	9	15	24	39	63	102	165	267	...
7	12	19	31	50	81	131	212	343	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

We will now think of the columns $S_0, S_1, S_2, \ldots$ as sets. We already know that $S_1 \cup S_2 \cup S_3 \cup \ldots$ is the set of all positive integers. A converse to this is built into the first of our three challenge statements.

1. No positive integer occurs in more than one of the sets $S_1, S_2, S_3, \ldots$
2. Every member of S_0 is a member of $S_1 \cup S_2$.
3. $S_0 = S_1 \cup S_2$.

A second Fibonacci-related puzzle is concerned with an unusual way of coding a secret message. The message itself consists of the choice of an integer from the set $\{1, 2, 3, \ldots, m\}$ and the code consists of the sender and receiver of the message in turn taking one or more stones from a pile placed near a tree in a forest. Initially there are exactly n stones in the pile, and both sender and receiver know this. The sender visits the forest when it suits him and removes one or more of the stones. The receiver looks at the pile from time to time and when he sees that it has decreased in size, because the sender has started sending his message, he takes a single stone from the pile himself. This indicates to the sender that he is free to take further stones from the pile when he is ready to continue with the message sending process. The two people keep taking turns in removing stones until there are none left. Because the sequence of stones that the sender can remove, whenever it is his turn, can vary, it is possible to pass on information in this secret manner.

The puzzle now is to find out how large m can be as a function of n. Perhaps an example will help for the case $n = 4$.

The sender can do any of the following:

1. Remove all four stones on his first visit.
2. Remove three stones on his first (and only) visit.
3. Remove two stones on his first (and only) visit.
4. Remove one stone on his first visit and two on the second visit.
5. Remove one stone in each of two visits.

It looks as though $m=5$ is possible when $n=4$. Can you generalise this result, and give a convincing reason for your answer?

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