Mathematical Apology number 12
A tale of one city: Hamilton and Dublin

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I have never been interested in philately, train spotting or collecting celebrity autographs, but I did try my hand once at mathematical walking. By this I mean going on a ramble with the object of viewing some monument of mathematical significance. I had been attending a massively large conference in Dublin and felt like a break. I asked a friend and colleague, Des Higham, if he felt like walking with me to Brougham Bridge where William Rowan Hamilton, in 1843, had carved a famous piece of graffiti.

Amongst the many objects that mathematicians find useful are the real and the complex numbers. These are both fields in the senses that we are familiar with. We can add, subtract, multiply and divide (except by zero) without being overly fussy about the order in which we do some of these things. Furthermore, the distributive law holds and we can never find two non-zero numbers whose product is zero. Looking at the complex numbers as a vector space over the real numbers, it has dimension 2 with $a + bi$ corresponding to a vector with components a and b. A good question is "are there higher dimensional analogues of the complex numbers?".

Hamilton found the closest thing possible in 4 dimensions but it was not quite a field because multiplication was not commutative. If one goes to even higher dimensions then even such skew-fields are not possible. Legend has it that Hamilton was pondering these questions as he was walking along the Royal Canal in Dublin. What was needed was a set of relations between the i, j and k that he used to generalize the complex number property $i^2 = -1$. As he got to Brougham Bridge he realised that the relations he wanted were

$$i^2 = j^2 = k^2 = ijk = -1.$$

Because the algebra was intended to be associative, it is easily deduced that further relations hold:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ki = j.$$

To pre-multiply a vector $w + xi + yj + zk$ by a second vector $a + bi + cj + dk$

is equivalent to forming the matrix by-vector product

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix},$$

so that the units 1, i, j and k can be represented by the coefficients of a, b, c and d respectively in the matrix appearing in this expression. Denote these four matrices by
$\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ and we have

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{i} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{j} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{k} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We easily check that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1},$$

the same relations satisfied by 1, i, j and k.

If I ever try another mathematical walk, I will make sure I have a map with me and that I have a clear idea of exactly where I am trying to go. On the occasion that Des and I went searching for Brougham Bridge, we neglected these basics and started off in the wrong direction. After a much longer walk than we really wanted on a warm summer day we got there, but only after asking directions several times. I was quite impressed that the people we asked all seemed to know who Hamilton was and were proud of the culture and scientific history of their great city.

When we finally found the bridge, a young man who may have been a bit of a larrikin for all we knew, was sitting on the bridge with his feet dangling over the side so as to obscure the plaque we were looking for. So we waited until he decided to wander off before we inspected the treasure and could photograph each other standing by it. The original marks put there by Hamilton had been removed many years previously, but a brass plate commemorated the discovery of quarternions and stated the relations

$$i^2 = j^2 = k^2 = ijk = -1.$$