

Mathematical Apology 10

The Golden Ratio and the Secant method

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A difference equation of the form

$$u_n = (\alpha + \beta)u_{n-1} - \alpha\beta u_{n-2}$$

has, as its general solution,

$$u_n = A\alpha^n + B\beta^n,$$

except when $\beta = \alpha$, in which case the solution is

$$u_n = (A + nB)\alpha^n.$$

These assertions can be easily checked by substituting the alleged solutions into the difference equation.

A very famous example is the Fibonacci equation for which

$$\alpha + \beta = -\alpha\beta = 1$$

In this case

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\alpha}$$

and the difference equation generates the Fibonacci numbers

$$1, 2, 3, 5, 8, \dots$$

for which the ratio of one member to the previous member tends to the Golden Ratio $\frac{1+\sqrt{5}}{2} \approx 1.61803398874989$.

Another interesting example of this type of difference equation is the choice

$$\alpha + \beta = 4, \quad \alpha\beta = 2$$

so that

$$\alpha = 2 + \sqrt{2}, \quad \beta = 2 - \sqrt{2}.$$

A possible sequence generated by this difference equation is

$$0, 1, 4, 14, 48, \dots$$

and the limiting ratio is, in this case $2 + \sqrt{2}$, from which $\sqrt{2}$ can be computed.

What happens if each member of a sequence is found, not by addition, but perhaps by multiplication? For example, consider the equation

$$v_n = v_{n-1}v_{n-2}.$$

If we start with positive values of v_0 and v_1 , then all members of the sequence will also be positive. However, if neither starting value is zero (which would lead to a trivial and uninteresting sequence), but at least one of them is negative, then we will cycle forever through a sequence of signs like this:

$$+, -, -, +, -, -, +, -, -, \dots$$

To see how the magnitude of v_n behaves, use logarithms and we find that

$$\log(|v_n|) = \log(|v_{n-1}|) + \log(|v_{n-2}|),$$

just as for the Fibonacci equation. If both $\log(|v_0|)$ and $\log(|v_1|)$ have the same sign then all members will also have this sign. For example, if $|v_0| < 1$ and $|v_1| < 1$, then v_n will converge to zero as $n \rightarrow \infty$ and, from our knowledge of Fibonacci sequences, we know that

$$\frac{\log(|v_n|)}{\log(|v_{n-1}|)}$$

will tend to the Golden Ratio. This will mean that, for large n

$$|v_n| \approx |v_{n-1}|^{(1+\sqrt{5})/2}$$

To change direction slightly, we will think about solving an equation

$$f(x) = 0.$$

The famous method of Newton is based on starting with some sort of approximation x_0 and then producing a sequence x_1, x_2, \dots , where

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n = 1, 2, \dots$$

Under fortunate circumstances, the sequence will converge to a solution of the equation. Furthermore, it will converge *quadratically*. That is, if the solution to which convergence is to take place is denoted by ξ and the error in approximation n is denoted by $\epsilon_n = x_n - \xi$, then after many iterations,

$$\epsilon_n = C\epsilon_{n-1}^2,$$

which means that, if the error in approximation number n is represented as $C\epsilon_n$, which is just a rescaling of ϵ_n , then the number of decimal places of accuracy doubles in each iteration.

A standard interpretation of the Newton method is that approximation number n is found as the intercept on the x axis of the tangent to the curve $y = f(x)$ at the point $(x_{n-1}, f(x_{n-1}))$. A similar idea is the ‘‘Secant Method’’, in which x_n is found from the intercept on the x axis of the secant through the pair of points $(x_{n-2}, f(x_{n-2}))$ and $(x_{n-1}, f(x_{n-1}))$. This means that

$$x_n = \frac{x_{n-1}f(x_{n-2}) - x_{n-2}f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}.$$

If the Newton method is second order, in the sense that, asymptotically, the error (scaled in some appropriate way) is squared in each iteration, then what is the order of the Secant Method? We can

try to get an idea of the answer by seeing what happens for the problem defined by $f(x) = x^2 - 2$. For this problem we are, of course, calculating $\sqrt{2}$.

Choose $x_0 = 1$ and $x_1 = 2$. We then find the iterations shown in the table

n	x_n	$x_n - \sqrt{2}$
0	1.0000000000000000	-0.41421356237310
1	2.0000000000000000	0.58578643762690
2	1.3333333333333333	-0.08088022903976
3	1.4000000000000000	-0.01421356237310
4	1.41463414634146	0.00042058396837
5	1.41421143847487	-0.00000212389823
6	1.41421356205732	-0.0000000031577
7	1.41421356237310	0.00000000000000

Obviously we need to work to greater accuracy to really understand what is going on. In the following table, we use information based on calculations carried out to 200 decimal places, to be on the safe side.

n	$\epsilon_n = x_n - \sqrt{2}$	$\frac{\log(\epsilon_n /2\sqrt{2})}{\log(\epsilon_{n-1} /2\sqrt{2})}$
2	$-8.088022903976 \times 10^{-2}$	2.2575165196
3	$-1.421356237310 \times 10^{-2}$	1.4891741599
4	$4.205839683684 \times 10^{-4}$	1.6650523130
5	$-2.123898225031 \times 10^{-6}$	1.6000270777
6	$-3.157745861736 \times 10^{-10}$	1.6249999135
7	$2.371189205858 \times 10^{-16}$	1.6153846154
8	$-2.647270929315 \times 10^{-26}$	1.6190476190
9	$-2.219318361662 \times 10^{-42}$	1.6176470588
10	$2.077174600088 \times 10^{-68}$	1.6181818182
11	$-1.629849922602 \times 10^{-110}$	1.6179775281
12	$-1.196948944367 \times 10^{-178}$	1.6180555556

The factor $1/2\sqrt{2}$ which enters into the heading of the last column was no guess work and it is no coincidence that the entries in this column seem to be getting close to the Golden Ratio, which is claimed to be the order of the secant method. Explanations are invited from readers in anticipation of revelations I intend to make next time.