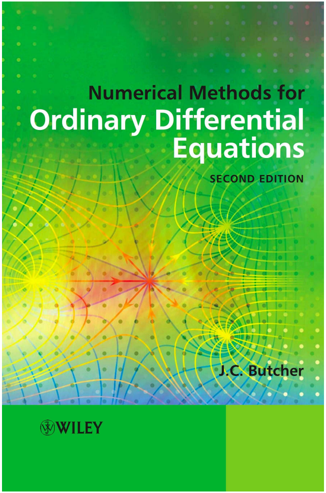
John Butcher's tutorials

Low order Runge–Kutta methods

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$



Review of order conditions

Recall the order-4 conditions for a 4-stage method:

t	$\Phi(t) = \frac{1}{\gamma(t)}$
	$b_1 + b_2 + b_3 + b_4 = 1$
	$b_2c_2 + b_3c_3 + b_4c_4 = \frac{1}{2}$
	$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \frac{1}{3}$
	$b_3a_{32}c_2 + b_4a_{42}c_2 + b_4a_{43}c_3 = \frac{1}{6}$
	$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = \frac{1}{4}$
	$b_3c_3a_{32}c_2 + b_4c_4a_{42}c_2 + b_4c_4a_{43}c_3 = \frac{1}{8}$
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Order 3:

$$\begin{aligned} b_1 + b_2 + b_3 &= 1 \\ b_2 c_2 + b_3 c_3 &= \tfrac{1}{2} \\ b_2 c_2^2 + b_3 c_3^2 &= \tfrac{1}{3} \\ b_3 a_{32} c_2 &= \tfrac{1}{6} \end{aligned}$$

Connection with quadrature

In the special case of a differential equation of the form:

$$\frac{dy}{dx} = \phi(x),$$

integration over a single step using an s-stage Runge–Kutta method is equivalent to the approximation

$$\int_{x_0}^{x_1} \phi(x)dx \approx h \sum_{i=1}^{s} b_i \phi(x_0 + hc_i) \qquad (*)$$

We will examine how well this approximation works for the s special choices of ϕ given by

$$\phi(x) = (x - x_0)^{k-1}, \qquad k = 1, 2, \ldots, s.$$

The error in (*) is equal to

$$-\Big(\sum_{i=1}^{s} b_i c_i^{k-1} - \frac{1}{k}\Big)h^k \qquad (\dagger)$$

To obtain order s, the coefficient of h^k in (†) must be zero for all $k = 1, 2, \ldots, s$.

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From a consideration of special quadrature problems, we have seen that necessary conditions for order s are that

$$\sum_{i=1}^{s} b_i c_i^{k-1} = \tfrac{1}{k}, \qquad k = 1, 2, \ldots, s. \qquad (*)$$

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In the remaining sections of this tutorial, these ideas will be applied to finding methods up to order 4.

Methods with order 2

The two "quadrature" conditions are

$$b_1 + b_2 = 1,$$
$$b_2 c_2 = \tfrac{1}{2},$$

corresponding to the trees  and .

There are no additional trees (or additional order conditions) to satisfy, so all we have to do is choose $c_2 \neq 0$ and immediately we find that $b_2 = 1/2c_2$, $b_1 = 1 - 1/2c_2$.

Here are three methods based on convenient choices of c_2. Note that the first two methods are due to Runge.

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The steps in finding a method are to choose suitable c_i, solve (a), (b) and (c) for the b_i and finally solve (d) for a_{32}.

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Note that in the choice of the c_i and the evaluation of the b_i, the value $b_3 = 0$ must be avoided, otherwise the solution of (d) becomes impossible.

Examples of third order methods:

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}$$

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Recall the conditions for order 4, but ordered differently:

$$b_1 + b_2 + b_3 + b_4 = 1, \tag{1}$$
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Given c_2, c_3, c_4, carry out the three steps:

1. solve for b_1, b_2, b_3, b_4 from (1), (2), (3), (4),
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If c_2, c_3 and c_4 are treated as parameters, and these steps are carried out, it is found that the consistency condition yielded in Step 3 is surprisingly simple. This condition is:

$$c_4 = 1.$$

For specific choices of c_2 and c_3 to be used with $c_4 = 1$, it sometimes happens that some step of the process cannot be carried out, for example because of a vanishing denominator, but fortunately, many cases exist when there is no trouble.

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$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{4} & \frac{1}{4} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 1 & -2 & 2 & \\ \hline & \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{6} \end{array}$$

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ 0 & -1 & 1 & & \\ 1 & -1 & \frac{3}{2} & \frac{1}{2} & \\ \hline & \frac{1}{12} & \frac{2}{3} & \frac{1}{12} & \frac{1}{6} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ 1 & 1 & & & \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} & & \\ 1 & -2 & -1 & 4 & \\ \hline & \frac{1}{6} & \frac{1}{12} & \frac{2}{3} & \frac{1}{12} \end{array}$$

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ \frac{2}{3} & -\frac{1}{3} & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{2}{3} & \frac{2}{3} & & & \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & & \\ 1 & -\frac{5}{4} & \frac{1}{4} & 2 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$