How to Crochet a Space-Filling Pancake: the Math, the Art and What Next

Hinke M. Osinga and Bernd Krauskopf
Department of Mathematics, The University of Auckland
Private Bag 92019, Auckland 1142, New Zealand
h.m.osinga@auckland.ac.nz
b.krauskopf@auckland.ac.nz

Abstract

The chaotic behavior of the famous Lorenz system is organized by an amazing surface: the Lorenz manifold. Initial conditions on different sides of this surface will behave differently after some time, which means that the Lorenz manifold encodes how the chaotic dynamics manifests itself in the entrire phase space. To convey it geometric properties, imagine a pancake that grows and grows to fill the entire space without creasing or developing self-intersections. Our mathematical research into the Lorenz manifold naturally led to artistic expression, when we realized that our computational method to find it generates crochet instructions. The crocheted Lorenz manifold not only received media attention, but also gave new insight into the geometry of chaos. Here we survey the developments that followed. They include our involvement with the mathematical art community and our collaboration with artist Benjamin Storch that led to a steel sculpture. In turn, our involvement with art contributed to the development of new ideas for studying and visualizing the Lorenz manifold, for example, by considering its intersection curves on a suitably chosen sphere.

Introduction

The Lorenz system were derived just over 50 years ago by meteorologist Edward Lorenz as a very much simplified model for the dynamics of convection rolls in the atmosphere [7]. Lorenz found that the behavior of the system depends extremely sensitively on the initial condition. Today the Lorenz system is a classic examples of a low-dimensional deterministic chaotic system; see, for example, [2, 15, 17] for more information. It is given by the system of three ordinary differential equations

$$\left\{ \begin{array}{rcl} \dot{x} &=& \sigma(y-x), \\ \dot{y} &=& \rho x - y - xz, \\ \dot{z} &=& xy - \beta z \end{array} \right. \tag{1}$$

for the three variables x, y and z. For the classical values of the parameters given by

$$\sigma = 10, \quad \rho = 28, \quad \beta = 2\frac{2}{3}$$

system (1) is chaotic. More specifically, all trajectories (which can be found by numerical integration) end up at a special object — the famous Lorenz attractor in the three-dimensional (x,y,z)-space. Indeed, it has been shown that the dynamics on the Lorenz attractor is chaotic [16, 17].

The Lorenz attractor is shown in Figure 1, where it is represented by trajectories that spiral around an equilibrium of (1) for a while before moving to spiralling around another equilibrium. Due to its shape, the Lorenz attractor is also referred to as the butterfly attractor. The associated sensitivity to the choice of initial condition is often colloquially represented by a statement such as: the beat of the wing of a butterfly in Brazil may cause a tornado in Texas.

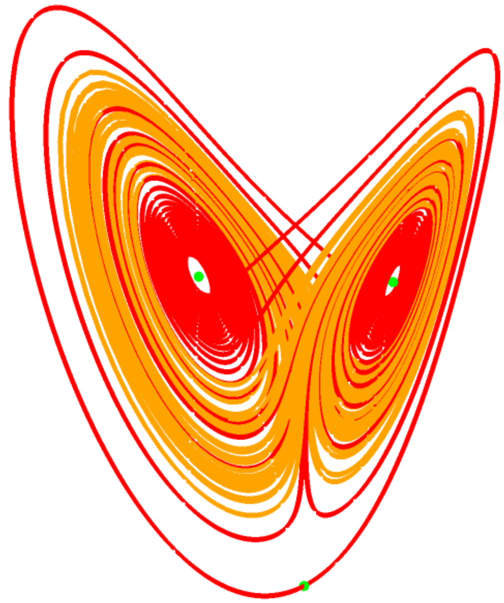

Figure 1: The Lorenz attractor.

An interesting question is how the chaotic dynamics is organized throughout (x,y,z)-space. To understand this, one needs to consider the origin $(x,y,z) = \mathbf{0} := (0,0,0)$, which is an equilibrium of saddle type with two stable directions and one unstable direction. The *Lorenz manifold* $W^s(\mathbf{0})$ is the two-dimensional smooth surface of points that end up under the dynamics of (1) exactly at $\mathbf{0}$. Dynamical systems theory states that $W^s(\mathbf{0})$ is tangent to the space spanned by the two stable eigenvectors of the linearization at $\mathbf{0}$. The origin is actually part of the Lorenz attractor and, locally near $\mathbf{0}$, trajectories of points on one side of the surface $W^s(\mathbf{0})$ go to one wing of the Lorenz attractor, while those of points on the other side go to the other wing. Importantly, a trajectory cannot cross $W^s(\mathbf{0})$, so once it is on one side it must stay on that side of the Lorenz manifold. An important feature of the Lorenz system is its invariance under the transformation $(x,y,z) \mapsto (-x,-y,z)$, which is geometrically a rotation by π around the z-axis in (x,y,z)-space; in particular, this means that the Lorenz attractor and the Lorenz manifold have this rotational symmmetry.

The key question now is what the Lorenz manifold $W^s(\mathbf{0})$ looks like away from the origin $\mathbf{0}$. The problem is that there is no formula for $W^s(\mathbf{0})$ as a surface. Hence, it must be found with dedicated numerical methods that make use of its definition as the set of points in (x,y,z)-space that end up at $\mathbf{0}$ under the dynamics given by (1). We developed such a method: the idea is to start from a small circle in the stable eigenspace of $\mathbf{0}$ and then to build up or grow the surface $W^s(\mathbf{0})$ by adding concentric smooth closed curves of points that have the same geodesic distance from $\mathbf{0}$ (given by the shortest path on the surface); these are also called geodesic level sets. The surface is then represented by a mesh constructed from points on these geodesic level sets; see [3, 4] for details, and the survey [6] for a discussion of alternative approaches. The method is illustrated in Figure 2 where consecutive rings in between successively computed geodesic level sets have alternating color. Panel (a) shows how the Lorenz manifold $W^s(\mathbf{0})$ starts to twist while it squeezes in between trajectories on the Lorenz attractor. Panel (b) is later on in the growth process and it illustrates how $W^s(\mathbf{0})$ rolls up when it goes around one of the wings of the Lorenz attractor.

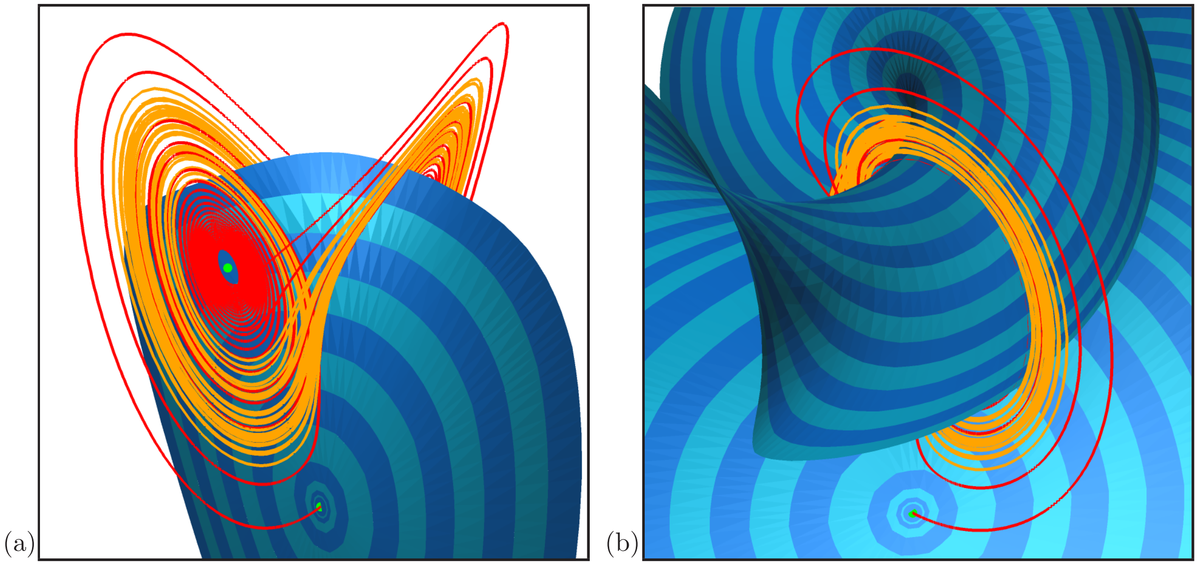

Figure 2: The Lorenz manifold $W^s(\mathbf{0})$ being computed as a collection of geodesic level sets; panels (a) and (b) are two stages of the computation that illustrate how $W^s(\mathbf{0})$ rolls in between trajectories on the Lorenz attractor.

Crocheting the Lorenz manifold and our path to art

After having studied $W^s(\mathbf{0})$ for a number of years by looking at computer-generated images as those in Figure 2, we realized that our method generates a mesh that can be interpreted directly as crochet instructions! In effect, each ring is crocheted by stitches of a certain length, with additions made where prescribed by the output of our numerical method. Of course, we needed to try this out, and the result, together with the full instructions, were published in 2004 in [10]. The process of crocheting the Lorenz manifold is illustrated in Figure 3. Panels (a1)-(a3) shows the very start, when the third ring is started in crochet. The number of stitches needed increases considerably from round to round, and the crocheted object one obtains after following the entire crochet instructions of 25,511 stitches is shown in Figure 3(b). To get this floppy object into shape, it needs to be mounted, for which we use a kiting rod along the z-axis and steel wires of the appropriate length along the boundary and along a curve known as the strong stable manifold; see [10] for details. The result is shown in Figure 3(c), and it corresponds to the Lorenz manifold computed up to geodesic distance 110.75.

We received considerable media attention, including a live interview on a main UK News program, and, to date, we know of 12 further Lorenz manifolds that have been produced throughout the world; see [9]. The artistic aspects of the crocheted Lorenz manifold are able to draw a general audience to a quite advanced topic of mathematics. We found that people are intruiged by the shape and geometry of the object, and would like to understand what lies behind it. Even people with hardly any mathematical background wanted to know more. The combination of crochet and mathematics seems to be the key to drawing people in. We have been using the crocheted Lorenz manifold in numerous public lectures, including Hinke's 2007 London Mathematical Society Popular Lecture [8].

There is also the question: is the crocheted Lorenz manifold art? The answer will depend on one's outlook, and over the years we heard strong arguments in favour and against. In any case, the crocheted Lorenz manifold naturally led to our involvement with the mathematical art community, which really kicked off when we presented it at the 2006 Bridges Conference in London [11]. The most tangible result of our

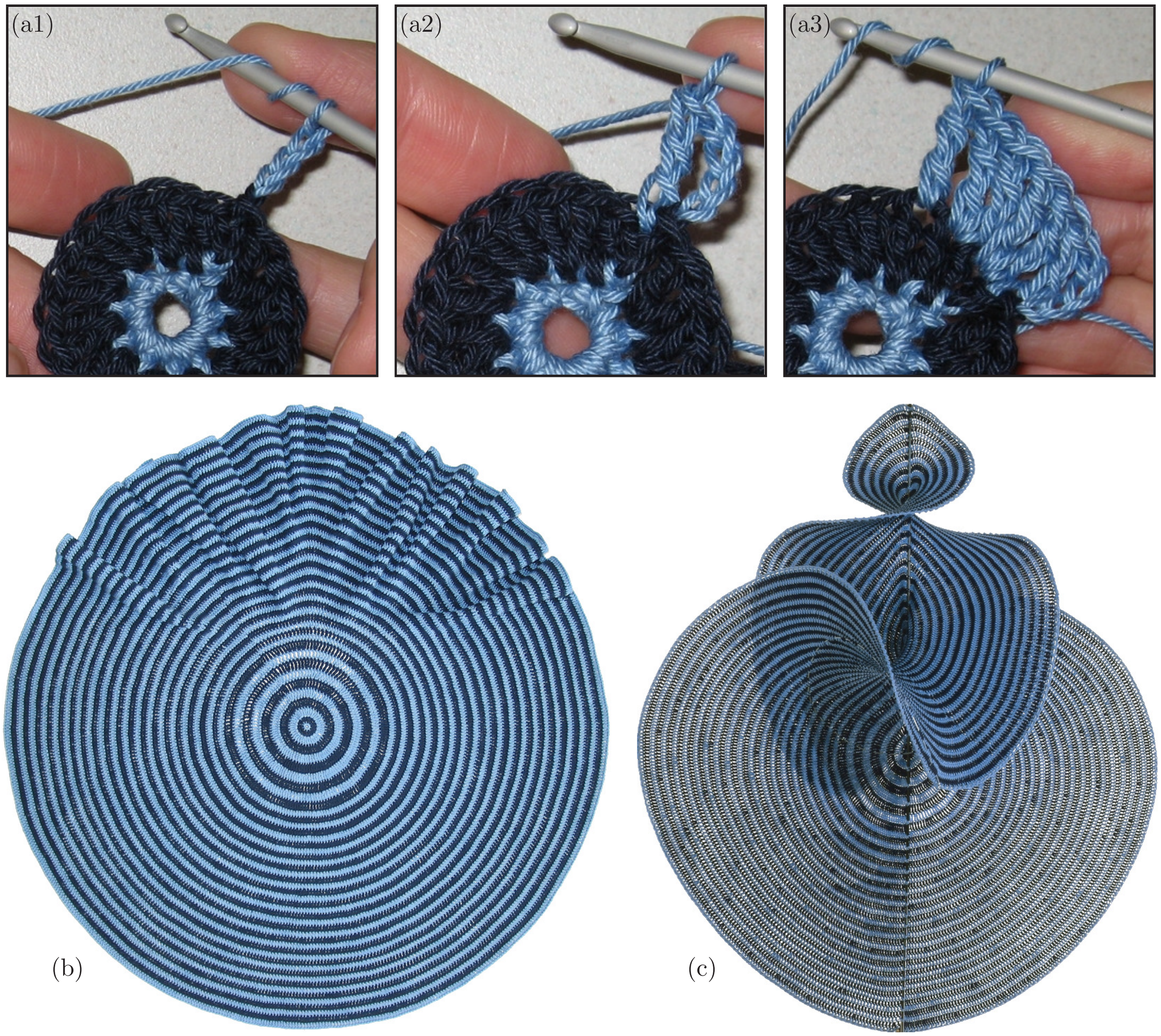

Figure 3: The crocheted Lorenz manifold; panels (a1)–(a3) illustrate the very beginning of the crocheting, panel (b) is the completed result after 25,511 stitches, and panel (c) shows the surface after mounting.

participation is that we met artist Benjamin Storch [13, 14]. We decided there and then to embark on a collaboration with him that led to the creation of the steel sculpture *Manifold*. Benjamin is interested in shapes and forms that express dynamics in some way, and he had already heard of the Lorenz manifold. The idea was to convey the properties of the Lorenz manifold without reproducing it as the crocheted version does. We quickly settled on the idea of producing a band on the manifold, in between two suitably chosen geodesic level sets, in such a way that the whole object supports itself. The chosen band lies in between geodesic distances 120.75 and 140.75, that is, further away from the origin than the boundary of the crocheted piece. Starting from our computed data, Benjamin produced a design and a technical process that split up the chosen band on the Lorenz manifold into 3 pairs of individual pieces. These pieces were then 'flattened out' by Benjamin with the clever use of CAD software to produce the shapes of flat pieces of steel to be cut; see Figure 4(a). The flat pieces were then hammered by Benjamin into their required shapes. This required the

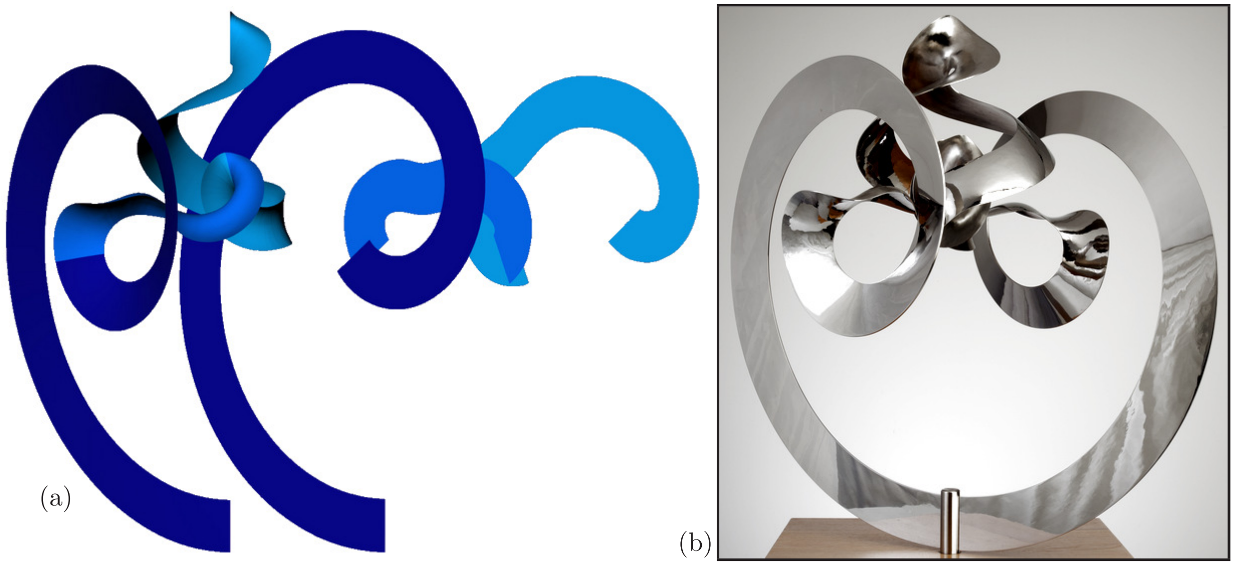

Figure 4: The steel sculpture *Manifold* by Benjamin Storch; panel (a) is a design sketch, and panel (b) shows the finished product.

design of special tools, especially for the anticlastic forming process to create forms of negative curvature. The pieces were then polished, welded together and the entire object surface finished.

The resulting sculpture *Manifold* is shown in Figure 4(b). It has a diameter of about 70 cm and consists of an 8 cm band of steel that is supported by a single pin at the bottom, around which it can swivel on its base. The sculpture was part of the art exhibit of the 2008 Bridges Conference in Leeuwarden [5]. While there is a definite act of artistic decision and considerable craftsmanship behind the sculpture *Manifold*, it is actually an extremely accurate real-world rendering of the mathematical object — the respective band on the Lorenz manifold as computed by our method. And so the question was asked: is the steel sculpture of the Lorenz manifold art? We definitely think so, but nothing beats the experience of the real sculpture! Its swirling shape is very intriguing and mysterious; even experts in dynamical systems would not be able to say immediately that it is an object that one can find in the Lorenz system.

Geometric properties of the Lorenz manifold

Our engagement with the crocheted Lorenz manifold and the steel sculpture *Manifold* has had a definite influence on our subsequent mathematical work on the geometric properties of the Lorenz manifold. First of all, mounting the crocheted object with the use of supporting wires is not entirely straightforward. The wires need to be of the correct length; when they are too long, they introduce extra tension and, hence, additional curvature into the mounted object. The wire along the strong stable manifolds is definitely needed to get the Lorenz manifold into its required shape. The issue is that there are regions on the Lorenz manifold where the local curvature is positive. The tension of the crocheted fabric, on the other hand, results in a surface of negative curvature (like a soap film on a wire frame). This realization led us to consider the overall and local curvature of the Lorenz manifold in [12], which can be derived from our computed triangulation. Figure 5(a) shows a color representation of the local curvature of the Lorenz manifold. The total curvature of the Lorenz manifold is clearly negative, but it has positive curvature in the lighter regions where the vertical helix ‘turns into’ either of the two symmetric scrolls. This knowledge of the local curvature was also crucial for the

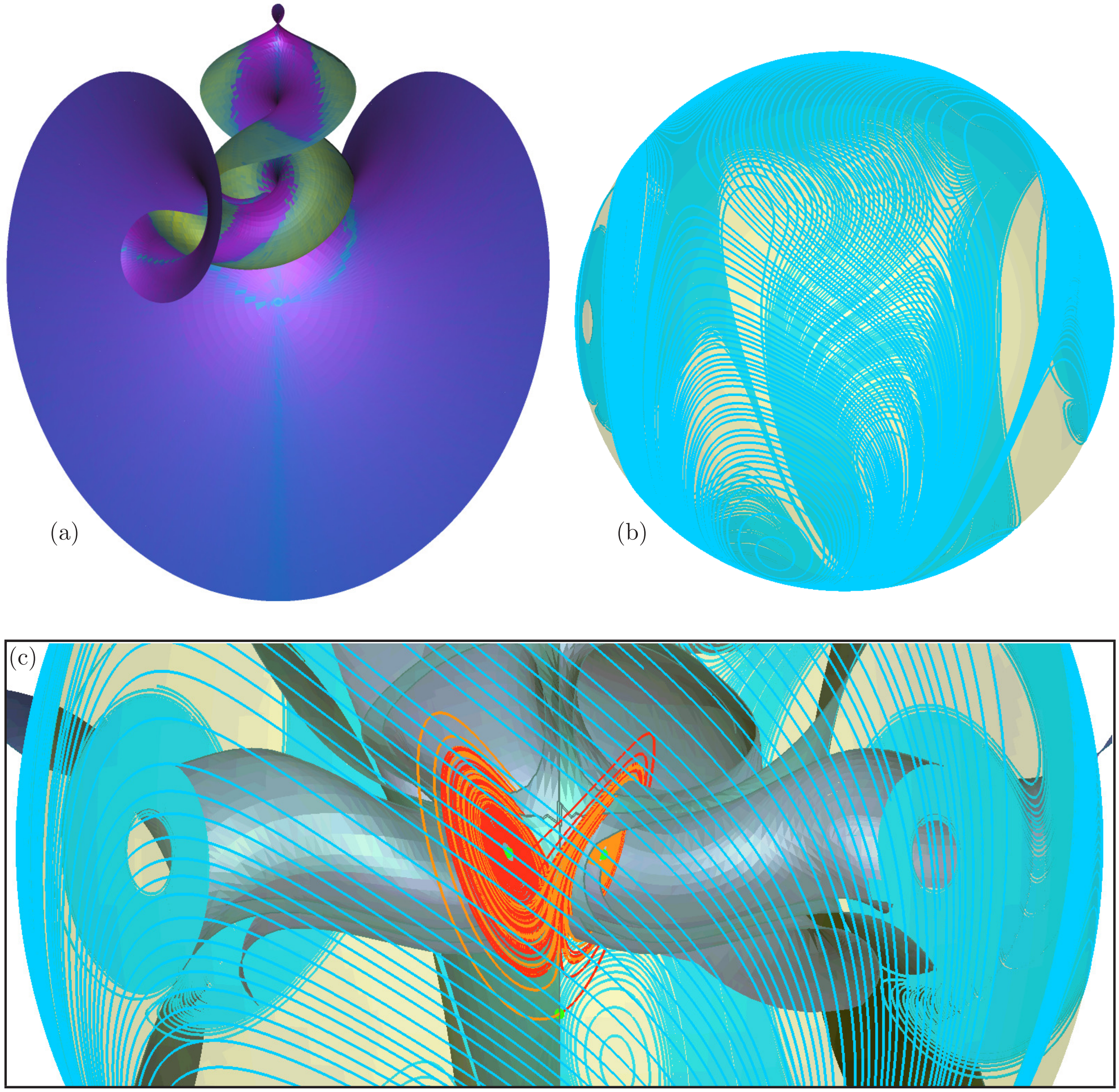

Figure 5: Illustrations of mathematical properties of the Lorenz manifold $W^s(\mathbf{0})$; panel (a) shows the local curvature of $W^s(\mathbf{0})$, panel (b) its many intersection curves with a sphere, and panel (c) is a look inside the sphere with the Lorenz attractor and one half of $W^s(\mathbf{0})$.

production process of the steel sculture.

The crocheted Lorenz manifold in Figure 3(c) represents the first piece up to geodesic distance 110.75. The published crochet instructions do not go further, but the question is: what would the Lorenz manifold look like if one kept crocheting? While crocheting even more may not be practical, the surface can be computed further by our method; in fact, Figure 5(a) actually shows it up to geodesic distance 155.25. When thinking about the properties of the Lorenz manifold, we realized that it has the amazing property of being dense in phase space! To put it colloquially: the Lorenz manifold $W^s(\mathbf{0})$ is a growing smooth, curled-up pancake (a smooth image of the unit disk) of increasing geodesic distance that comes arbitrarily close to any point in three-dimensional space! This property is a direct consequence of the chaotic nature of the Lorenz system (1): the trajectories of any two typical initial points in phase space must eventually move over the Lorenz attractor in a different way; this means that there must be points near $\mathbf{0}$ on these two trajectories that lie on either side of the Lorenz manifold $W^s(\mathbf{0})$. Hence, $W^s(\mathbf{0})$ must already divide any small neighborhood of both initial points, and the result follows.

Our images of $W^s(\mathbf{0})$ as a surface fail to convey its denseness because the manifold would need to be computed up to impractical geodesic distances. The issue is that, in light of the total negative curvature of the Lorenz manifold, the computational effort increases dramatically with geodesic distance. This is why we came up with the idea of studying the properties of $W^s(\mathbf{0})$ by considering its intersection set with a sufficiently large sphere that contains the Lorenz attractor in its interior [1]. The Lorenz manifold $W^s(\mathbf{0})$ intersects the sphere in an infinite number of curves, which fill out the surface of the sphere densely. Importantly, it is possible to compute a great number of these intersection curves directly, via solving a boundary value problem, without the need for computing $W^s(\mathbf{0})$ as a surface. The result of such a computation is shown in Figure 5(b), where the sphere is semi-transparent so that the intersection curves of $W^s(\mathbf{0})$ can be seen on both sides. Figure 5(c) is a close-up view inside the sphere, also showing the Lorenz attractor and the first computed part of the surface $W^s(\mathbf{0})$ that lies in the region of negative y. This gives an impression of how the intersection curves are generated, but also shows how the computed intersection curves corresond to a much larger piece of the surface $W^s(\mathbf{0})$. Our recent and ongoing work focuses on the question of how the Lorenz manifold $W^s(\mathbf{0})$ changes during the transition from simple dynamics for small ρ to chaotic dynamics for $\rho = 28$.

Acknowledgements

We thank Benjamin Storch for an amazing collaboration. The work shown in Figure 5(b)–(c) on the properties of the Lorenz manifold was performed jointly with Eusebius Doedel.

References

[1] E.J. Doedel, B. Krauskopf and H. M. Osinga. Global invariant manifolds in the transition to preturbulence in the Lorenz system. *Indagationes Mathematicae*, **22**(3-4): 223–241, 2011.

[2] J. Gleick. *Chaos, the Making of a New Science*, William Heinemann, London, 1988.

[3] B. Krauskopf and H. M. Osinga. Two-dimensional global manifolds of vector fields. *CHAOS*, **9**(3): 768–774, 1999.

[4] B. Krauskopf and H. M. Osinga. Computing geodesic level sets on global (un)stable manifolds of vector fields. *SIAM Journal on Applied Dynamical Systems*, **2**(4):546–569, 2003.

[5] B. Krauskopf, H. M. Osinga and B. Storch. The sculpture Manifold: a band from a surface, a surface from a band. In R. Sarhangi and J. Sharp (Eds.) Proceedings of 2008 Bridges Leeuwarden, Tarquin Publishing, 2008, pp. 9–14.

[6] B. Krauskopf, H. M. Osinga, E. J. Doedel, M. E. Henderson, J. Guckenheimer, A. Vladimirsky, M. Dellnitz and O. Junge. A survey of methods for computing (un)stable manifolds of vector fields. *International Journal of Bifurcation and Chaos* **15**(3): 763-791, 2005.

[7] E. N. Lorenz. Deterministic nonperiodic flows. *Journal of the Atmospheric Sciences*, **20**: 130–141, 1963.

[8] H. M. Osinga. Chaos and Crochet. 2007 LMS Popular Lecture, produced by mathMedia for The London Mathematical Society; available from the LMS at `http://www.lms.ac.uk/events/popular-lectures-dvds`

[9] H. M. Osinga and B. Krauskopf. Chrocheting the Lorenz manifold. `www.math.auckland.ac.nz/~hinke/crochet/`
[10] H. M. Osinga and B. Krauskopf. Crocheting the Lorenz manifold. *The Mathematical Intelligencer* **26**(4): 25–37, 2004.
[11] H. M. Osinga and B. Krauskopf. The Lorenz manifold: crochet and curvature. In R. Sarhangi and J. Sharp (Eds.) Proceedings of 2006 Bridges London, Tarquin Publishing, 2006, pp. 255–260.
[12] H. M. Osinga and B. Krauskopf. Visualizing curvature on the Lorenz manifold. *Journal of Mathematics and the Arts*, **1**(2): 113–123, 2007.
[13] B. Storch. Anticlastic form — manifestating fields of tension. In R. Sarhangi and J. Barrello (Eds.) Proceedings of 2007 Bridges Donostia, Tarquin Publishing, 2007, pp. 189–194.
[14] B. Storch. Fluid Forms in Metal. `http://www.benjaminstorch.com`
[15] S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Addison Wesley, 1994.
[16] W. Tucker. The Lorenz attractor exists. *Comptes Rendus de l'Académie des Scienes. Série I. Mathématique*, **328**(12):1197–1202, 1999.
[17] M. Viana. What's new on Lorenz strange attractors? *The Mathematical Intelligencer*, **22**(3):6–19, 2000.