A global bifurcation analysis of the subcritical Hopf normal form subject to Pyragas time-delayed feedback control

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Abstract

Unstable periodic orbits occur naturally in many nonlinear dynamical systems. They can generally not be observed directly, but a number of control schemes have been suggested to stabilize them. One such scheme is that by Pyragas [35, 36, 40], which uses time-delayed feedback to target a specific unstable periodic orbit of a given period and stabilize it. This paper considers the global effect of applying Pyragas control to a nonlinear dynamical system. Specifically, we consider the standard example of the subcritical Hopf normal form subject to Pyragas control, which is a delay differential equation (DDE) that models how a generic unstable periodic orbit is stabilized. Our aim is to study how this DDE model depends on its different parameters, including the phase of the feedback and the imaginary part of the cubic coefficient, over their entire ranges. We show that the delayed feedback control induces infinitely many curves of Hopf bifurcations, from which emanate infinitely many periodic orbits that, in turn, have further bifurcations. Moreover, we show that, in addition to the stabilized target periodic orbit, there are possibly infinitely many stable periodic orbits. We compactify the parameter plane to show how these Hopf bifurcation curves change when the $2\pi$-periodic phase of the feedback is varied. In particular, the domain of stability of the target periodic orbit changes in this process and, at certain parameter values, it disappears completely. Overall, we present a comprehensive global picture of the dynamics induced by Pyragas control.

1 Introduction

The control of unstable dynamics is an area of significant interest in engineering [1], biotechnology [12, 29, 42], chemistry [30] and other scientific disciplines. The motivation for controlling unstable behavior stems from applications where stable periodic motions or equilibrium solutions are favored over the unpredictability associated with unstable or chaotic dynamics. For instance, in [21] the unstable motion of a cutting tool, known as machine chatter, is undesirable as it results in an imperfect cut. The majority of available control methods, such as gain scheduling [25] and feedback linearization [17], focus on controlling an equilibrium solution of a nonlinear system. In this paper we are interested solely in the control of unstable periodic orbits, these are also called UPOs [3, 9, 32]. An unstable periodic orbit is a periodic solution of a dynamical system that has at least one Floquet multiplier outside of the unit circle in the complex plane. The motion of the unstable periodic orbit may be desirable from an experimental point of view, but to make this motion observable it must be stabilized or controlled. For example, in [12] a stable periodic motion of the cardiac system is required, but the addition of a drug into the system makes the desired periodic orbit unstable; these authors apply a control scheme to stabilize the unstable periodic orbit.

Our starting point is a system of (autonomous) ordinary differential equations (ODEs)

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implement in experiments, especially where detailed analytical information on the system is unavailable. Directions of the target unstable periodic orbit. For this reason, this method has sometimes proved difficult to implement in experiments. The method requires the user to have a priori knowledge about the linear stable and unstable directions. It has been implemented successfully in a number of applications, most notably in laser [41] and chemical [30,31] systems. Furthermore, Pyragas feedback control scheme has been successfully implemented in a number of applications, including experiments on laser [2], electronic [13,40], engineering [11,49] and chemical [22,45] systems. Furthermore, Pyragas controlled systems can be embedded into a continuation scheme [48,49], which automatically determines the period \( T \) of the target periodic orbit \( \Gamma \) as a system parameter is changed, and sets the delay appropriately.

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The OGY method was one of the first control schemes suggested to stabilize an unstable periodic orbit, and it is mainly used for the control of periodic orbits that lie within a chaotic attractor. The OGY method has been implemented successfully in a number of applications, most notably in laser [41] and chemical [30,31] experiments. The method requires the user to have a priori knowledge about the linear stable and unstable directions of the target unstable periodic orbit. For this reason, this method has sometimes proved difficult to implement in experiments, especially where detailed analytical information on the system is unavailable.

The method introduced by Pyragas [36] applies a continuous time-delayed feedback term that forces the system towards the desired periodic dynamics. Suppose system (1) has an unstable periodic orbit \( \Gamma \) with period \( T \). The Pyragas control scheme is then

\[
\dot{x}(t) = f(x(t), \mu),
\]

where \( x \in \mathbb{R}^n \), \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is a smooth function and \( \mu \in \mathbb{R}^m \) is a vector of scalar parameters. Here \( \mathbb{R}^n \) is the phase space of the ODE, meaning that an initial condition \( x_0 \in \mathbb{R}^n \) uniquely defines the solution of the initial value problem (IVP). Suppose that system (1) has an unstable periodic orbit \( \Gamma \), which we would like to stabilize. Unstable periodic orbits such as \( \Gamma \) can arise in system (1) in various ways. In particular, unstable periodic orbits (of saddle type) can be found as part of chaotic attractors, where they lie dense. In both cases it may be desirable to find and stabilize an unstable periodic orbit. Two of the most successful methods designed to control and stabilize an unstable periodic orbit \( \Gamma \) are that by Ott, Grebogi and Yorke (OGY) [28] and that by Pyragas [36]. Both of these methods are non-invasive, that is, when the target periodic orbit \( \Gamma \) has been stabilized the control force vanishes.

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\[
\dot{x}(t) = f(x(t), \mu) + K[x(t - \tau) - x(t)], \quad \text{with} \quad \tau = T.
\]

Here \( K \) is a \( n \times n \) feedback gain matrix [3,32,33], coupling the feedback term to the original system of ODEs (1). Pyragas control targets the unstable periodic orbit \( \Gamma \) by setting the delay equal to its period \( T \). The control force depends on the difference between the signals \( x(t - T) \) and \( x(t) \) in (2); when the control is successful (that is, \( K \) is chosen suitably) system (2) converges toward the target state, which is the periodic orbit \( \Gamma \). As \( \Gamma \) is approached the difference in the control term becomes smaller. When the target periodic orbit \( \Gamma \) has been stabilized, the system follows the desired periodic motion effectively with zero control force. Hence, the Pyragas control method is indeed non-invasive.

It is often stated that the Pyragas control scheme is easy to implement [2, 10, 46], as the user requires only knowledge about the period \( T \) of the target unstable periodic orbit \( \Gamma \), and, furthermore, setting the system delay \( \tau \) equal to \( T \) provides an in-built targeting mechanism. Pyragas control is a particularly simple and useful method for stabilizing unstable periodic orbits embedded in chaotic attractors. By picking some value of the period \( \tau = T \), it is likely that (2) will converge to one of the many saddle periodic orbits inside the chaotic attractor. However, the situation is different when a specific periodic orbit is targeted. Even when its exact period is known, care must be taken when choosing the parameter values to ensure the system follows the target periodic orbit.

The Pyragas feedback control scheme has been successfully implemented in a number of applications, including experiments on laser [2], electronic [13,40], engineering [11,49] and chemical [22,45] systems. Furthermore, Pyragas controlled systems can be embedded into a continuation scheme [48,49], which automatically determines the period \( T \) of the target periodic orbit \( \Gamma \) as a system parameter is changed, and sets the delay appropriately.

A result of implementing the Pyragas scheme is that the controlled system (2) is a delay differential equation (DDE) with a single fixed delay \( \tau \). One of the main differences between the analysis of a DDE and an ODE is the nature of the phase space. For a DDE with a fixed delay, an initial condition takes the form of a continuous function \( \phi(t) \) on the interval \([-\tau, 0]\) [6]. In other words, to determine how (2) will evolve, the user must define a history of length \( \tau \). This means that the phase space of the DDE is the infinite-dimensional space \( C([-\tau, 0]; \mathbb{R}^n) \) of continuous functions from \([-\tau, 0]\) to \( \mathbb{R}^n \); here \( \mathbb{R}^n \) is referred to as the physical space.
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As with ODEs the stability of an equilibrium point of a DDE is determined by the roots of the characteristic equation of the system. However, the characteristic equation of a DDE is quasi-polynomial, that is to say, it contains terms of the form $e^{\lambda t}$ (for eigenvalues $\lambda$). Hence, it has infinitely many roots in the complex plane. For an equilibrium solution of a DDE to be stable, all such roots must have a negative real part. It has been shown that there are only a finite number of roots with positive real part [6]. Therefore, one need only consider the root (or complex conjugate pair of roots) with the largest real part to determine stability. This means that standard bifurcation theory (i.e. that for ODEs) is still valid for a a system such as (2); for example, a Hopf bifurcation occurs when a pair of complex conjugate eigenvalues have zero real part [6].

The periodic orbit $\Gamma$ is a solution of (2) such that after time $T$, where $T$ is the period of $\Gamma$, a function section of length $\tau$ is repeated. The stability of $\Gamma$ is determined by its Floquet multipliers. A periodic orbit $\Gamma$ has a countable number of Floquet multipliers with only a finite number outside of the unit circle in the complex plane [15]. When Floquet multipliers move out of the unit circle, the periodic orbit has undergone a bifurcation; thus the analysis of bifurcations of periodic orbits in systems such as (2) is the same as in systems without delay.

Although DDEs are generally more difficult to study analytically than ODEs, there are several computational tools available for the study of DDEs. The routine dde23 implemented in Matlab [26] is a numerical integrator that solves DDEs with constant time-delays. DDE-Biftool [7] is a software package, implemented in Matlab [26], that is capable of continuing equilibria, periodic orbits and bifurcations of equilibria. We use DDE-Biftool in its extended form (version 3.0) developed by Sieber [47] to compute the bifurcation curves presented in this paper, including those of periodic orbits.

The aim of this paper is to understand the global or overall dynamics that may arise when Pyragas time-delayed feedback control is implemented. Here, the term global is used with respect to the parameter space. To understand these dynamics, we aim to answer the following questions. Can stabilization of the target periodic orbit be achieved for all values of system parameters? Does the addition of feedback induce further stable periodic orbits that are not the target? As we will see, even in a simple setting, successful implementation of Pyragas control is actually quite challenging. Firstly, the control scheme is simple to setup, but determining the exact period can be difficult. There are several methods that can be used to automatically determine the period, such as the iterative procedure of Schikora et al. [43, 44], the continuation procedure of Sieber and Krauskopf [48] and the adaptive algorithm developed by Pyragas [40]. However, all of these methods add complexity and computational effort. In our paper [34], we show that at least a linear approximation of the parameter-dependent target period $T$ is required for stabilization to be successful. Here, we take a global view of Pyragas control and this reveals an additional issue: even when the exact parameter-dependent function of the period is known and used, there may actually be stable periodic orbits other than the target periodic orbit. Thus, if the user does not choose parameter values carefully, the system could quite easily follow one of the other stable periodic orbits instead of the targeted one.

To reach this conclusion, we analyze the generic subcritical Hopf normal form subjected to Pyragas feedback control, as first proposed by Fiedler et al [9], and given by

$$\dot{z} = (\lambda + i)z(t) + (1 + i\gamma)|z(t)|^2z(t) + b_0e^{i\beta}[z(t - \tau) - z(t)].$$

(3)

Here $z \in \mathbb{C}$ and $\lambda, \gamma \in \mathbb{R}$; the complex number $b_0e^{i\beta}$ is the feedback gain $K$; $b_0 \in \mathbb{R}$ is the control amplitude; and $\beta$ is the phase of the feedback. We use the convention that $b_0 \geq 0$ and $\beta \in [0, 2\pi]$. As the feedback gain of the Pyragas control is dependent on both $b_0$ and $\beta$, and as $b_0e^{i\beta} = -b_0e^{i(\beta \pm \pi)}$, a reflection of the same dynamics as those presented in this paper can be achieved for a negative $b_0$ through a shift of $\pi$ in the $2\pi$-periodic feedback phase $\beta$.

We consider system (3) because the period of the target periodic orbit and the local mechanism of stabilization can be understood by analytical work, which is rare for non-linear time-delay systems. System (3) has been well studied in [3, 9, 19] near the Hopf bifurcation point; a summary of this work is given in the next section. On the other hand, the global dynamics resulting from the addition of feedback have not been fully considered yet.

The generic subcritical Hopf normal form is given by the ODE part of (3), which is

$$\dot{z} = (\lambda + i)z(t) + (1 + i\gamma)|z(t)|^2z(t).$$

(4)
As in (3), system (4) is written in the complex variable \( z \in \mathbb{C} \). There is a subcritical Hopf bifurcation at \( \lambda = 0 \), which we denote \( H_P \). From \( H_P \) emanates an unstable periodic orbit \( \Gamma_P \), which has one unstable Floquet multiplier. In the literature \( \Gamma_P \) is referred to as the Pyragas periodic orbit, and \( \Gamma_P \) is the control target [3,9,19].

In this paper we take a global view of the controlled system (3) and perform a detailed bifurcation analysis in the \((\lambda, b_0)\)-plane. We analyze how the targeted unstable periodic orbit \( \Gamma_P \) is stabilized and also show its domain of stability locally as well as globally. Moreover, we show that the system (3) has infinitely many Hopf bifurcations in addition to \( H_P \), which are induced by the delay term. We analyze bifurcations of the target periodic orbit and those periodic orbits that emanate from delay-induced Hopf bifurcations. In particular, we find that there are other stable periodic orbits in addition to the target periodic orbit \( \Gamma_P \). To present the global dynamics of the system we show bifurcation sets in a compactified \((\lambda, b_0)\)-plane, thus allowing us to present the limiting behavior at infinity. Using this compactification we are able to see how the structure and geometry of system (3) changes globally as the feedback phase \( \beta \) is varied through a period of \( 2\pi \). In particular, we find a cyclic-type transition of the infinitely many delay-induced Hopf bifurcation curves. We also show how the domain of stability of \( \Gamma_P \) in the \((\lambda, b_0)\)-plane changes as \( \beta \) is varied. Finally, we also consider the effect that changing the normal form parameter \( \gamma \) has on the dynamics of system (3); specifically, we determine for which values of \( \gamma \) the Pyragas control scheme fails.

This paper is organized as follows. Section 2 gives some background on (3) and a synopsis of the literature on its study. Section 3 presents the dynamics of system (3) close to the Hopf bifurcation \( H_P \) in the \((\lambda, b_0)\)-plane. Section 4 considers a more global view of the \((\lambda, b_0)\)-plane and shows further dynamics induced by the feedback. Section 5 studies the effect that changing the parameter \( \beta \) has on the domain of stability of the targeted periodic orbit \( \Gamma_P \) and the delay-induced Hopf bifurcation curves. Section 6 considers the effect of the parameter \( \gamma \) on the stabilization of \( \Gamma_P \). Conclusions and a discussion can be found in section 7.

2 Notation & Background

In system (3) Pyragas control is applied to the subcritical Hopf normal form. A normal form is a reduced analytic expression of a general problem on the centre manifold, achieved through successive coordinate transformations. Analysis of the normal form is often easier than that of the unsimplified system, yet it yields a precise qualitative overview of the dynamics of the system. Brown et al. [3] showed that near a Hopf bifurcation, the same dynamics are induced, whether Pyragas control is added to the normal form or to the original equations.

System (4) displays the symmetry associated with the Hopf normal form, namely it is invariant with respect to any rotation in the complex plane about the origin \((S_1 \text{ group operation})\). In fact, the normal form highlights this symmetry property near the Hopf bifurcation, which would otherwise not be discernible in the full system [14]; this symmetry can aid in the analysis of the dynamics. When reducing a system near a Hopf bifurcation to its normal form all quadratic terms can be removed under a non-linear change of coordinates [14]. However, cubic terms of the form \(|z(t)|^2z(t)\) cannot be removed as they are resonant, that is, invariant under the aforementioned symmetry. Therefore, Hopf bifurcations of the zero equilibrium solution of (4) with a pair of purely imaginary eigenvalues \( \eta = \pm i \) produce rotating wave solutions of the form

\[
z(t) = r_P e^{i \frac{2\pi}{T_P} t},
\]

where \( T_P \) denotes the period of the bifurcating periodic solution \( \Gamma_P \). System (4) is transformed to polar coordinates by the ansatz (5), yielding,

\[
\begin{cases}
0 = (\lambda + r_P^2) r_P \\
\frac{2\pi}{T_P} = (1 + \gamma r_P^2)
\end{cases}
\]

From the equations (6) we find that \( \Gamma_P \) has amplitude \( r_P = \sqrt{-\lambda} \), exists for \( \lambda < 0 \) and has period \( T_P = \frac{2\pi}{1 + \gamma \lambda} \). System (3) maintains the symmetry properties of the Hopf normal form as it is a DDE with a fixed delay, where the delay term enters linearly; that is, the delay term is also invariant under any complex rotation about the origin. The basic periodic orbits of (3) are circular and, hence, can be represented by their radius. Since the unstable periodic orbit \( \Gamma_P \) is the target state of the Pyragas control, in (3) we set
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Throughout our analysis of (3) in this paper we fix \( \tau \) to the expression (7). The Hopf bifurcation \( H_P \) is also always at \( \lambda = 0 \) in the DDE (3), but its criticality may differ from that of \( H_P \) in the normal form (4), where it is always subcritical.

Fiedler et al. [9] first introduced and analyzed system (3). The main motivation of these authors was to use (3) as a counterexample to the ‘odd number limitation’, which states that only unstable periodic orbits with an even number of real unstable Floquet multipliers can be stabilized with Pyragas control [27]. The principal focus of [9] is the successful local stabilization of the target unstable periodic orbit \( \Gamma_P \), which emanates from the subcritical Hopf bifurcation \( H_P \). The periodic orbit \( \Gamma_P \) has one real Floquet multiplier greater than unity, hence, its stabilization provided a counterexample to the odd-number limitation. Fundamentally, this successful stabilization is possible due to the choice of the feedback gain as \( b_0 e^{i\beta} \) in (3). The authors of [9] do not fix the delay \( \tau \); they discuss the mechanism for local stabilization and present the results in the two-parameter \((\lambda, \tau)\)-plane. Fiedler et al. show that the addition of feedback induces a secondary Hopf bifurcation. From this Hopf bifurcation emanates a stable periodic orbit. This stable periodic orbit then undergoes a transcritical bifurcation with the target state \( \Gamma_P \). In this transcritical bifurcation the two periodic orbits exchange stability, resulting in \( \Gamma_P \) becoming stable. Beyond discussing how stabilization works, Fiedler et al. [9] establish a domain within the \((\lambda, \tau)\)-plane in which the target orbit \( \Gamma_P \) is stable. In particular, they develop analytical expressions for the upper and lower limits of the control amplitude \( b_0 \) for this domain when \( \lambda \) is small. This analysis also yields the result that stabilization fails when the feedback gain \( b_0 e^{i\beta} \) is real, that is to say, when \( \beta = 0 \) or \( \beta = \pi \).

Just et al. [19] conducted a detailed local bifurcation analysis of system (3), presented in the \((\lambda, \tau)\)-plane and in the \((b_0, \tau)\)-plane. These authors derive the critical level of feedback amplitude given by the \( b_0 \) value

\[
b_0^c = \left\{ (\lambda, b_0) = \left(0, \frac{-1}{2\pi(\cos \beta + \gamma \sin \beta)} \right) \right\},
\]

immediately above which the target state \( \Gamma_P \) bifurcates stably from \( H_P \) at \( \lambda = 0 \) (for larger values of \( b_0 \) other instabilities may occur; these are discussed in section 4). Moreover, [19] also gives the condition

\[
b_0 \tau = \frac{-1}{(\cos \beta + \gamma \sin \beta)},
\]

at which a transcritical bifurcation occurs.

Brown et al. [3] analyze how Pyragas control stabilizes an unstable periodic orbit that emanates from a generic subcritical Hopf bifurcation in a \( n \)-dimensional dynamical system. Critical to this successful stabilization is the correct choice of feedback gain \( b_0 e^{i\beta} \), and [3] presents explicit formulae for choosing both \( b_0 \) and \( \beta \). Furthermore, these authors perform a linear stability analysis and a center manifold reduction to show that there exists a degenerate Hopf bifurcation at the point \( b_0^c \). In fact, [3] shows that the point \( b_0^c \) in the \((\lambda, b_0)\)-plane is a Hopf bifurcation point with a further degeneracy; namely for a solution \( \eta \) of the characteristic equation of the linearized system, one has that \( \text{Re}[\eta(0)] = 0 \) when \( b_0 \) has the value in (8). Moreover, at \( b_0^c \), the cubic coefficient of the normal form is purely imaginary. Brown et al. [3] also present a schematic of the local bifurcation set in the \((\lambda, b_0)\)-plane near the point \( b_0^c \). In particular, they detail where the target unstable periodic orbit \( \Gamma_P \) is stabilized. The authors of [3] also show that stabilization of \( \Gamma_P \) is impossible when the real part of the imaginary cubic coefficient of system (3) is zero, that is, when \( \gamma = 0 \).

Erzgräber et al. [8] discuss the effect of the feedback phase \( \beta \) on the dynamics of system (3). In particular, they focus on the changes to the domain of stabilized periodic orbits, as \( \beta \) is varied over a period of \( 2\pi \). These authors present a detailed bifurcation analysis of the equilibrium solution and the target unstable periodic orbit. Furthermore, they also conducted a bifurcation analysis of the stable periodic orbits induced by the addition of the feedback term [9, 19]. These results are presented in the \((b_0, \beta)\)-plane. The analysis of [8] reveals that the target unstable periodic orbit \( \Gamma_P \) and the delay-induced periodic orbits undergo not only the transcritical bifurcation discussed in [3] and [19], but also torus bifurcations.

\[
\tau = T_P = \frac{2\pi}{1 - \frac{\gamma}{\lambda}}.
\]
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Figure 1: One-parameter bifurcation diagrams in $\lambda$ of (3). Panel (a) is for $b_0 = 0$ and shows the Hopf bifurcation point $H_P$ (black dot) of the equilibrium and the bifurcating unstable periodic orbit $\Gamma_P$ (green dashed). Panel (b) is for $b_0 = 0.022$ and shows further delay-induced Hopf bifurcations (black dots). From the bifurcation $H_L$ emanates a stable periodic orbit $\Gamma_L$ (red), which exchanges stability with $\Gamma_P$ at a transcritical bifurcation TC (black square). Solid (dashed) curves indicate stable (unstable) periodic orbits, the solid black (grey) lines indicate where the equilibrium is stable (unstable).

3 The ($\lambda, b_0$)-plane near the point $b_0^c$

In this section we discuss the dynamics of system (3) near the Hopf bifurcation $H_P$ that occurs for $\lambda = 0$. Throughout this analysis near $b_0^c$ we fix $\gamma = -10$ and $\beta = \frac{\pi}{4}$ as was done in [9, 19]. In particular, we show the mechanism by which Pyragas control stabilizes the target unstable periodic orbit $\Gamma_P$. The analysis is presented in both one-parameter bifurcation diagrams and two-parameter bifurcation sets.

Figure 1(a) shows the one-parameter bifurcation diagram in $\lambda$ of (3) for $b_0 = 0$, that is, for system (4) which has no control term; the unstable periodic orbit $\Gamma_P$ is represented by the (green) dashed curve. The equilibrium solution is stable (black) before it undergoes a Hopf bifurcation at the point $H_P$, from which $\Gamma_P$ bifurcates. Figure 1(b) is the one-parameter bifurcation diagram of (3) for $b_0 = 0.022 < b_0^c$. The point $H_P$ is unaffected by the presence of delayed feedback and remains at $\lambda = 0$. However, further Hopf bifurcations are now present and they are shown as the additional black dots on the bottom axis of Fig. 1(b). From one of these delay-induced Hopf bifurcations, which we label $H_L$, emanates a branch of stable periodic orbits. The target periodic orbit $\Gamma_P$ and the delay-induced stable periodic orbit $\Gamma_L$ exchange stability at the transcritical bifurcation TC. The target unstable periodic orbit $\Gamma_P$ is thereby stabilized.

Figure 2 shows the two-parameter bifurcation set in the ($\lambda, b_0$)-plane of (3), near the critical level of feedback amplitude $b_0^c$ given by the point $b_0^c$ from (8). This figure is a computed version of the schematic presented in [3]. The curve $H_P$ is the vertical (green) curve at $\lambda = 0$. The curve of delay-induced Hopf bifurcations $H_L$ is represented by the (red) curve that intersects $H_P$ at the point $b_0^c$. From the point $(0, b_0^c)$ emerges the transcritical bifurcation curve TC (purple), the curve extends into the region where $\lambda < 0$. An analytical expression for the curve TC can be found, by combining equations (7) and (9), as

$$b_0 = TC(\lambda) \equiv -1 + \frac{\gamma \lambda}{2\pi (\cos \beta + \gamma \sin \beta)}.$$  

The (blue) curve labeled $S_L$ is a saddle-node of limit cycles (SNLC) bifurcation (or fold bifurcation); where a stable delay-induced periodic orbit bifurcates. The curve $S_L$ emanates from the degenerate Hopf bifurcation point $DH_L$ on $H_L$ and extends left in the ($\lambda, b_0$)-plane for decreasing values of $\lambda$ and $b_0$. The shaded region in Fig. 2 is the domain of stabilization for the targeted unstable periodic orbit $\Gamma_P$ near the point $b_0^c$. The curve TC forms its lower boundary [3] and the domain is bounded on the right by the Hopf bifurcation curve $H_P$.

Below $b_0^c$, the Hopf bifurcation $H_P$ is subcritical and the equilibrium is stable for $\lambda < 0$. In the region to the left of the curve $H_P$ and to the right of the curve $H_L$ the equilibrium is unstable. At $b_0^c$ the Hopf bifurcation
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Figure 2: Bifurcation set in the $(\lambda,b_0)$-plane of (3) near the point $b_0^c$, where the curves $H_P$ (green) and $H_L$ (red) intersect. The transcritical bifurcation curve $TC$ (purple) begins at the point $b_0^c$. The SNLC bifurcation curve $S_L$ (blue) emerges from the degenerate Hopf bifurcation point $DH_L$ ($\ast$). The domain of stability of $\Gamma_P$ is shaded.

$H_P$ changes criticality, hence, immediately above $b_0^c$, $H_P$ is supercritical. Also, immediately above $b_0^c$, the target periodic orbit $\Gamma_P$ bifurcates stably from $H_P$.

The curve $H_L$ also has sections of differing criticality. Between the point $b_0^c$ and the degenerate Hopf bifurcation point $DH_L$, the curve $H_L$ is supercritical; from this section bifurcates a stable periodic orbit; see Fig. 1(b). If $b_0$ is decreased this periodic orbit then undergoes a saddle-node of limit cycles bifurcation along the curve $S_L$. If $b_0$ is increased it exchanges stability with $\Gamma_P$ at the transcritical bifurcation $TC$; see Fig. 2. At the point $DH_L$ the critical complex conjugate pair of eigenvalues do not satisfy the eigenvalue crossing condition, that is the eigenvalues do not actually cross the imaginary axis.

To the left of $b_0^c$ the curve $H_L$ is subcritical; from this section bifurcates an unstable periodic orbit, which exists for decreasing $\lambda$ (to the left of $H_L$). To the right of $DH_L$ the curve $H_L$ is also subcritical; from this section of the curve bifurcates an unstable periodic orbit, which also exists for decreasing $\lambda$ (to the left of $H_L$).

4 A more global view of the $(\lambda,b_0)$-plane

We now consider the $(\lambda,b_0)$-plane more globally, again for $\gamma = -10$ and $\beta = \frac{\pi}{4}$. By exploring more of the $(\lambda,b_0)$-plane we are able to show that the Pyragas time-delayed feedback induces further Hopf bifurcations. From these bifurcations emanate delay-induced periodic orbits; we analyze the stability and bifurcations of these periodic orbits and show that there are, in fact, infinitely many stable delay-induced periodic orbits of (3). We present the overall domain of stabilization for the target periodic orbit $\Gamma_P$ and also present a selection of the domains of stability for the stable delay-induced periodic orbits.

4.1 Families of Hopf bifurcations and bifurcating periodic orbits

Figure 3 shows Hopf bifurcation curves of system (3) in the $(\lambda,b_0)$-plane; for reference, note that Fig. 2 is an enlargement of Fig. 3 near the point $b_0^c$. The $(\lambda,b_0)$-plane shown in Fig. 3 is bounded below at $b_0 = 0$ as is the convention [3,9,19]. It also has a left-hand boundary at $\lambda = 0$ (in this parameter regime where $\gamma = -10$ this is at $\lambda = -0.1$); at this boundary the delay $\tau$ becomes undefined. For $\lambda < \frac{1}{2}$ the delay $\tau$ defined in (7) is negative and (3) is an advanced equation rather than a delay equation, analysis of which is beyond the scope of this paper. In particular, for $\lambda < \frac{1}{2}$ system (3) no longer describes Pyragas control.

The curve $H_P$ is again the vertical (green) curve at $\lambda = 0$. By considering a larger view of the $(\lambda,b_0)$-plane,
the Hopf bifurcation curve $H_L$ can now be seen to form a closed loop. Namely, both of its end points meet at the point $b_0^*$ on the left-hand boundary ($\lambda = \frac{1}{7}$) of Fig. 3. In addition to crossing $H_P$ at the point $b_0^*$, the curve has a second crossing of $H_P$ at the point $HH_0$. The point $HH_0$ is a non-degenerate double Hopf bifurcation point, at which two pairs of purely imaginary eigenvalues exist; it is given by

$$HH_0 = \left\{ (\lambda, b_0) = \left( 0, \frac{\pi - \beta}{2\pi(\sin \beta)} \right) \right\}. \quad (11)$$

At $HH_0$ the criticality of $H_P$ changes, it is again subcritical and remains subcritical above $HH_0$. Above $HH_0$ a periodic orbit bifurcates unstably from $H_P$ with a complex conjugate pair of unstable Floquet multipliers. Thus, the target orbit $\Gamma_P$ is only stable when it bifurcates from $H_P$ between the points $b_0^*$ and $HH_0$.

In addition to the curve $H_L$, two further delay-induced Hopf bifurcation curves $H^j_1$ and $H^j_R$ are shown in Fig. 3. Both of these curves also emerge from the point $b_0^*$.

The curve $H^j_1$ stretches downwards in the plane from $b_0^*$ before turning upwards and extending to infinity in the direction of $b_0$. The curve $H^j_1$ can be split into sections of differing criticality. Figure 3 shows a degenerate Hopf bifurcation point $DH_1$ near the minimum of the curve $H^j_1$. At this point the critical complex conjugate pair of eigenvalues reach, but do not actually cross the imaginary axis. To the left of $DH_1$, the curve $H^j_1$ is subcritical. From this section bifurcates an unstable periodic orbit, which exists for decreasing values of $b_0$. Between the point $DH_1$ and the point $HH_0$, the curve $H^j_1$ is supercritical and from this section emanates a stable periodic orbit that exists for decreasing values of $\lambda$ and, which we call $\Gamma^j$. Above $HH_0^j$, a periodic orbit that emanates from $H^j_1$ is unstable and exists for decreasing $\lambda$.

The curve $H^j_R$ starts at the point $b_0^*$ before crossing the curve $H_P$ at the double Hopf bifurcation point $HH_1$ and then extending to infinity in both $\lambda$ and $b_0$. All periodic orbits that emanate from $H^j_R$ are unstable and bifurcate from $H^j_1$ for decreasing values of $b_0$. Those periodic orbits that bifurcate from $H^j_R$ to the right of the point $HH_1$ have three unstable Floquet multipliers. The unstable periodic orbits that emanate immediately to the left of $HH_1$ have five unstable Floquet multipliers.

An equation for these delay-induced Hopf bifurcation curves can be found by using the ansatz $z(t) = r e^{i\omega t}$ in system (3): setting $r = 0$ yields the relationship

$$b_0 = H(\lambda) \equiv \frac{-\lambda^2 - (\omega - 1)^2}{2((\omega - 1) \sin \beta - \lambda \cos \beta)}. \quad (12)$$

where $\omega$ denotes the frequency of the Hopf bifurcation (which is derived in Appendix A). At the point $b_0^*$, numerical calculations show that the delay-induced Hopf bifurcations have frequency $\omega = 0$. Substituting $\omega = 0$ and $\lambda = \frac{1}{7}$ into equation (12) gives the expression

$$b_0^* = \left\{ (\lambda, b_0) = \left( \frac{1}{\gamma}, \frac{1^2}{2\gamma^2 + 3 \cos \beta + \sin \beta} \right) \right\}. \quad (13)$$

Note that $b_0^* \approx (-0.1, 0.7935)$ for $\gamma = -10$ and $\beta = \frac{\pi}{7}$ as shown in Fig. 3.

Figure 4(a) shows the same part of the $(\lambda, b_0)$-plane as Fig. 3, but with additional Hopf bifurcation curves. More specifically, the curves $H^j_1$ and $H^j_R$ are, in fact, part of the respective families of curves $H^k_1$ and $H^k_R$, with elements $H^j_1$ and $H^j_R$ respectively ($k = 1, 2, 3, \ldots$). A selection of curves for each of these families is shown in Fig. 4(a). In addition, two of the double Hopf bifurcation points, where delay-induced Hopf bifurcation curves cross each other, are marked by violet dots and labelled $HH_1^j$ and $HH_2^j$. These two points form part of a set of double Hopf bifurcation points $HH_D$, where delay-induced Hopf bifurcation curves cross each other.

The curves in the family $H^j_1$ slope downwards in the plane, before reaching a minimum and then extending to infinity in the $b_0$-direction. For increasing $k$, each curve in the family has a minimum with a higher value of $b_0$ and lower value of $\lambda$ than the preceding curve. Each curve in the family has a degenerate Hopf bifurcation point $DH_k$ to the right of the minimum of the curve. The curves $H^j_1$ converge in this fashion on the left-hand boundary of the plane at $\lambda = \frac{1}{7}$ as $k \to \infty$.

The curve $H^j_1$ intersects other delay-induced Hopf bifurcation curves at double Hopf bifurcation points of the set $HH_D$. The periodic orbit that emanates from $H^j_1$ above a point in this set has an additional complex pair of Floquet multipliers to one that bifurcates from below the point.
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Figure 3: Bifurcation set in the \((\lambda,b_0)\)-plane of (3), showing the Hopf bifurcation curve \(H_P\) (green) and the delay-induced Hopf curves \(H_L, H^1_J\) and \(H^1_R\) (red). Also shown are the double Hopf bifurcation points \(HH_0\) (green), \(HH_1\) (green), \(HH^1_D\) (violet) and \(HH^1_H\) (violet), as well as the points \(b_0^c\) and \(b_0^e\) (black). Degenerate Hopf bifurcation points on \(H_L\) and \(H^1_J\) are marked with an asterisk.

The criticality of the subsequent curves of the family \(H^k_J\) is similar to that of \(H^1_J\). More specifically, the section of \(H^k_J\) between the degenerate Hopf bifurcation point \(DH_k\) and the first crossing with another delay-induced Hopf bifurcation curve is supercritical. From this section of \(H^k_J\), bifurcates a stable periodic orbit \(\Gamma^k_J\). These stable delay-induced periodic orbits are discussed in more detail in section 4.3.

Each curve of the family \(H^k_R\) emanates from the point \(b_0^e\) and extends to the right of the plane with positive gradient, crossing the curves \(H^1_J\) and \(H_P\) at double Hopf bifurcation points before extending to infinity in both \(\lambda\) and \(b_0\). At the left-hand boundary where \(\lambda = \frac{1}{2}\), as \(k \to \infty\), each curve \(H^k_R\) has a steeper gradient than the preceding curve in the family and the curves are closer together.

As \(\lambda\) is decreased the curve \(H^1_R\) intersects other delay-induced Hopf bifurcation curves at double Hopf bifurcation points that are part of the set \(HH_D\). The unstable periodic orbit that bifurcates from \(H^1_R\) to the left of one of these points has an additional complex pair of unstable Floquet multipliers. The criticality of the subsequent curves in the family are as that of the curve \(H^1_R\).

Figure 4(b) is an enlargement of the lower left part of Fig. 4(a). It also shows the transcritical bifurcation curve \(TC\) (10) (purple) from Fig. 2 over a much larger range of \(\lambda\). Figure 4(b) shows that \(TC\) ends at the left-hand boundary of the plane at the point \((\lambda, b_0) = (\frac{1}{2}, 0)\). Figure 4(b) also shows a selection of SNLC bifurcation curves \(S^k_J\) and \(S^k_R\) of the delay-induced periodic orbits that bifurcate from the \(H^k_J\) and \(H^k_R\) families of Hopf bifurcation curves respectively.

The curves labelled \(S^k_J\) start at degenerate Hopf bifurcation points \((DH_k)\) on \(H^k_J\) and end at the point \((\lambda, b_0) = (\frac{1}{2}, 0)\) (on the left-hand boundary). These are SNLC bifurcation curves of periodic orbits that bifurcate from the respective curves in the family \(H^k_J\). For a curve \(H^1_J\) the unstable periodic orbit that bifurcates from the section between \(b_0^e\) and \(DH_k\) undergoes a SNLC bifurcation along the curve \(S^k_J\).

The curves labelled \(S^k_R\) are SNLC bifurcation curves of periodic orbits that emanate from the respective curves of the family \(H^k_R\). The curves start at the point \((\lambda, b_0) = (\frac{1}{2}, 0)\) (on the left-hand boundary) and extend to infinity in both \(\lambda\) and \(b_0\). The unstable periodic orbit that bifurcates from a curve \(H^k_R\) undergoes a SNLC
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Figure 4: Panel (a) shows Hopf bifurcation curves of (3) in the \((\lambda, b_0)\)-plane; shown are the curves \(H_P\) (green), \(H_L\) (red) and a selection of curves from the families \(H^K_1\) (red) and \(H^K_2\) (red). The enlargement in panel (b) also shows the transcritical bifurcation curve TC (purple) and a selection of curves from the two families of SNLC bifurcations \(S^K_0\) (blue) and \(S^K_R\) (blue).

4.2 Torus bifurcations and the domain of stability of \(\Gamma_P\)

The double Hopf bifurcation points, where the curve \(H^K_R\) crosses \(H_P\) at the point \(HH_K\) form a set \(HH_K\). The points in the set \(HH_K\) are given by the expression (derived in Appendix B)

\[
HH_k = \left\{ (\lambda, b_0) = \left( 0, \frac{\pi(1 + k) - \beta}{2\pi(\sin \beta)} \right) \right\}.
\]  

(14)
At each point in the set \( HH_k \), the frequency \( \omega \) of the delay-induced Hopf bifurcation is either \( \omega = 1 \) or \( \omega = k + \frac{\beta}{\pi} \). Therefore, when \( \beta = \frac{\pi}{4} \) the double Hopf bifurcation points in the set \( HH_k \) are resonant. In this parameter regime \( HH_0 \) is a double Hopf bifurcation point of 1:4 resonance; see also [5, 24].

As we have already discussed, above \( HH_0 \), all periodic orbits that bifurcate from \( H_P \) are unstable. In fact, above each double Hopf bifurcation point of the set \( HH_k \) there are further instabilities in the form of an extra complex conjugate pair of unstable Floquet multipliers. That is to say, those periodic orbits bifurcating from \( H_P \) between \( b_c^0 \) and \( HH_0 \) have no unstable Floquet multipliers, those that emanate between \( HH_0 \) and \( HH_1 \) have two unstable Floquet multipliers, those that bifurcate from between \( HH_1 \) and \( HH_2 \) have four unstable Floquet multipliers and so on. In other words, the point \( b_c^0 \) forms the lower right corner of the overall domain of stability of \( \Gamma_P \) and \( HH_0 \) forms its upper right corner.

Figure 5 is an enlargement of the lower left part of Fig. 4(a). It shows the overall domain of stability (shaded) of the target periodic orbit \( \Gamma_P \) and a selection of torus bifurcation curves (grey and black). It is common for double Hopf bifurcation points to be the starting point of torus bifurcation curves, and here we show a single torus bifurcation curve emerging from a few of the double Hopf bifurcation points [14, 23].

The curves \( T^0_P \), \( T^1_P \) and \( T^2_P \) are torus bifurcations of the target periodic orbit \( \Gamma_P \) that emanates from the curve \( H_P \). They start at the points \( HH_0 \), \( HH_1 \) and \( HH_2 \) respectively and the curves \( T^1_P \) and \( T^2_P \) end at the point \( (\lambda, b_0) = (\frac{1}{4}, 0) \) on the left-hand boundary. The curve \( T^0_P \) ends on the transcritical bifurcation curve TC at the 1:1 resonance point \( R_1 \). That is, the periodic orbits undergoing the transcritical bifurcation have the same frequency as the torus bifurcation.

The overall domain of stability of \( \Gamma_P \) is, therefore, bounded below by the curve TC between the points \( b_c^0 \) and \( R_1 \), to the left by the curve \( T^0_P \) and to the right by the curve \( H_P \). This shows that the stabilized periodic orbit \( \Gamma_P \) does not remain stable throughout the plane as suggested by the local unbounded domain that was developed by [3] and shown in Fig. 2.
Each subsequent torus bifurcation \((T^1_J, T^2_J, \ldots)\) results in an additional unstable complex pair of Floquet multipliers. Thus, as \(\lambda\) is decreased, a periodic orbit \(\Gamma_P\) bifurcating from \(H_P\) (below \(HH_D\)) is first stabilized by Pyragas control before becoming unstable in the torus bifurcation \(T^0_P\). It then becomes increasingly unstable as it undergoes further torus bifurcations (in the set \(T^K_P\)) as it approaches the left-hand boundary of dynamics in the plane at \(\lambda = \frac{3}{2}\). At this boundary, the period of \(\Gamma_P\) (given by \(T = \frac{2\pi}{\omega}\)) goes to infinity.

The periodic orbits that bifurcate from the curve \(H_L\) are also destabilized in torus bifurcations. These torus bifurcation curves emerge from the double Hopf bifurcation points of the set \(HH_D\) where the curve \(H_L\) intersects curves in the family \(H^K_J\).

The curves labelled \(T^1_J, T^2_J\) and \(T^3_J\) are torus bifurcations of the delay-induced periodic orbits that emanate from the family of curves \(H^K_J\). The curves start at the double Hopf bifurcation points in the set \(HH_D\) marked in Fig. 5 by violet dots. These torus bifurcation curves also terminate at the point \((\lambda, b_0) = (\frac{3}{2}, 0)\) (on the left-hand boundary). The previously mentioned stable periodic orbits that bifurcate from the family of curves \(H^K_J\) are destabilized in these torus bifurcations. As they approach the left-hand boundary of the plane the periodic orbits are further destabilized in additional torus bifurcations that emerge from other points in the set \(HH_D\).

The unstable periodic orbits that bifurcate from the family of SNLC bifurcation curves \(S^K_J\) almost immediately undergo a torus bifurcation, in which they are further destabilized. Further to the left of the \((\lambda, b_0)\)-plane, these unstable periodic orbits undergo further torus bifurcations that emerge from double Hopf bifurcation points in the set \(HH_D\) in which they are further destabilized.

We notice two things from numerical evidence. Firstly, at the left-hand boundary of the \((\lambda, b_0)\)-plane the frequency \(\omega\) of all the delay-induced Hopf bifurcations is zero, that is \(\omega = 0\). Secondly, at the left-hand boundary, the period of all delay-induced periodic orbits goes to infinity.

### 4.3 Other regions of stable periodic orbits

We now consider in more detail the stable periodic orbits \(\Gamma^K_J\) that bifurcate from the family of Hopf bifurcation curves \(H^K_J\). First, we take a slice at \(b_0 = 0.295\) of the \((\lambda, b_0)\)-plane shown in Fig. 5. Figure 6(a) shows the resulting one-parameter bifurcation diagram of (3) in \(\lambda\), where the bottom axis shows the stability of the equilibrium solution. The green curve is the stable periodic orbit \(\Gamma_P\), which bifurcates from the Hopf bifurcation \(H_P\). Figure 6(a) also shows the stable delay-induced periodic orbits \(\Gamma^1_J, \Gamma^2_J\) and \(\Gamma^3_J\) (solid red) that bifurcate supercritically from the Hopf bifurcations \(H^1_J, H^2_J\) and \(H^3_J\). At each of these Hopf bifurcations the equilibrium changes from being stable (black) to unstable (grey) as \(\lambda\) is reduced. The stable periodic orbits \(\Gamma^1_J, \Gamma^2_J\) and \(\Gamma^3_J\) become unstable in the torus bifurcations \(T^1_J, T^2_J\) and \(T^3_J\), respectively. The grey curves represent already unstable delay-induced periodic orbits that bifurcate from \(H_L, H^1_J, H^2_J\) and \(H^3_J\).

Figure 6(b) shows the corresponding domains of stability (shaded grey) in the \((\lambda, b_0)\)-plane, namely those of \(\Gamma_P\) and of the stable delay-induced periodic orbits \(\Gamma^1_J, \Gamma^2_J\) and \(\Gamma^3_J\). The regions where the equilibrium is stable are shaded blue. Darker blue shading indicates regions of bistability, where the equilibrium and a periodic orbit are both stable.

As previously discussed, the overall domain of stability of \(\Gamma_P\) is bounded by the curves \(H_P, TC\) and \(T^0_P\). In the part of the domain of stability enclosed by the curves \(H_L, T^0_P\) and \(TC\) the equilibrium is also stable. This bistability adds to the complexity of implementing the Pyragas control scheme, because the system may not reach the target periodic orbit even though it is stable.

Similarly to \(\Gamma_P\), the domain of the stable delay-induced periodic orbit \(\Gamma^K_J\) is bounded on the right by the Hopf bifurcation curve \(H^K_J\) between the degenerate Hopf bifurcation point \(DH_K\) and the double-Hopf bifurcation point \(HH_D\). Its left-hand boundary is the torus bifurcation curve \(T^K_J\) and its lower boundary the SNLC bifurcation curve \(S^K_J\). There are also regions where the equilibrium and the delay-induced periodic orbit \(\Gamma^K_J\) are both stable. This region is below the curve \(H^K_J\) and between the curves \(T^K_J\) and \(S^K_J\).

Our calculations clearly indicate that there are infinitely many curves in the family \(H^K_J\), which accumulate on the left-hand boundary \(\lambda = \frac{3}{2}\) of the \((\lambda, b_0)\)-plane. Indeed when considering the control problem (3) it is common to take \(\lambda\) as the main bifurcation parameter. However, to make the connection to the stability problem of a general DDE, we now take the delay \(\tau\) as the main bifurcation parameter. Hence, we consider the bifurcation set of (3) in the \((\tau, b_0)\)-plane as shown in Fig. 7(a). The bifurcation curves in Fig. 6(b) translate...
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Figure 6: Panel (a) is the one-parameter bifurcation diagram of (3) in \( \lambda \) for \( b_0 = 0.295 \). Shown are the stable periodic orbit \( \Gamma_P \) bifurcating from \( H_P \) and further delay-induced stable periodic orbits \( \Gamma_1^j \), \( \Gamma_2^j \) and \( \Gamma_3^j \) bifurcating from \( H_1^j \), \( H_2^j \) and \( H_3^j \). These stable periodic orbits become unstable at the torus bifurcations \( T_1^j \), \( T_2^j \) and \( T_3^j \) (black dots). Solid (dashed and grey) curves indicate stable (unstable) periodic orbits, the solid black (grey) lines indicate where the equilibrium is stable (unstable). Panel (b) shows the domains of stability (shaded grey) of the target periodic orbit \( \Gamma_P \) and the stable delay-induced periodic orbits \( \Gamma_1^j \), \( \Gamma_2^j \) and \( \Gamma_3^j \). The regions where the equilibrium solution is stable are shaded blue; above the horizontal boundary at \( b_0 = 0.28 \), the stability region is not defined, as further curves of the family \( H^K_j \) are not shown.

Figure 7(a) shows that the curves of the family \( H^K_j \) form a lobe structure that is common in systems with delay [4, 16, 21, 50–52]. The right end of the curve \( H_1^j \) has a horizontal asymptote at \( b_0 \approx 0.7935 \) (the \( b_0 \) value directly to those shown in Fig. 7(a) via the parameter transformation \( \tau = \frac{2\pi}{1-\gamma \lambda} \) from (7). The delay \( \tau \) goes to infinity as \( \lambda \) approaches the left-hand boundary \( \lambda = \frac{1}{\gamma} \) of the \((\lambda, b_0)\)-plane. Fig. 7(a) shows the Hopf bifurcation curve \( H_P \) (green) at \( \tau = 2\pi \) and the delay-induced Hopf bifurcation curves \( H_1^j \), \( H_2^j \), \( H_3^j \) and \( H_4^j \) (red). The domains of stability of the target periodic orbit \( \Gamma_P \) and the stable delay-induced periodic orbits \( \Gamma_1^j \), \( \Gamma_2^j \), \( \Gamma_3^j \) and \( \Gamma_4^j \) are shaded in grey. The equilibrium solution is stable in the regions shaded blue and the darker blue areas are the regions of bistability discussed earlier in this section. These domains are bounded by the same bifurcation curves as those in the \((\lambda, b_0)\)-plane discussed previously.

Figure 7(a) shows that the curves of the family \( H^K_j \) form a lobe structure that is common in systems with delay [4, 16, 21, 50–52]. The right end of the curve \( H_1^j \) has a horizontal asymptote at \( b_0 \approx 0.7935 \) (the \( b_0 \) value
of the point $b_0^*$) as $\tau \to \infty$, but its left end has a vertical asymptote at a finite value of $\tau$ as $b_0 \to \infty$. In fact, for $b_0 \to \infty$ the curves $H_k^1$ are spaced at $2\pi$ intervals from each other. This can be shown analytically by considering purely imaginary eigenvalues $\eta = i\omega$ of the characteristic equation of (3). Solving this equation for the frequency $\omega$ (simplifying equations (20) in Appendix A) yields $\omega$ as a function of $\lambda$ as

$$\omega(\lambda) = 1 - b_0 \left[ \sqrt{1 - \left( \frac{\lambda}{b_0} + \cos \beta \right)^2} - \sin \beta \right],$$  \hspace{1cm} (15)$$

and as a function of $\tau$ as

$$\omega(\tau) = 1 - b_0 \left[ \sqrt{1 - \left( \frac{1 - \frac{2\pi}{\gamma b_0}}{\tau} + \cos \beta \right)^2} - \sin \beta \right].$$  \hspace{1cm} (16)$$

When $b_0 \to \infty$ the part of equations (15) and (16) inside the square root becomes $1 - \cos^2(\beta) = \sin^2(\beta)$. Thus, the expressions in the square brackets become 0 and, therefore, the frequency $\omega$ tends to 1. Hence, the period $T = \frac{2\pi}{\omega}$ tends to $2\pi$ as $b_0 \to \infty$. In [53], Yanchuk and Perlikowski derive the relationship $\tau_k = \tau_0 + kT_0$ (for

Figure 7: Panel (a) is the bifurcation set of (3) from Fig. 6(b) but now shown in the $(\tau,b_0)$-plane. Panel (b) is the one-parameter bifurcation diagram for $b_0 = 0.295$ from Fig. 6(a) but now shown in terms of the parameter $\tau$. 
\( k = 0, 1, 2, 3, \ldots \) between delay and periodicity, which states that periodic orbits reappear infinitely many times for specific values of \( \tau \). Here, \( \tau_1 = T_0 = 2\pi \), which implies that \( \tau_k = 2k\pi \) is the vertical asymptote of the curve \( H_k^\ast \). In particular, it follows that there are infinitely many Hopf bifurcation curves in the family \( H_k^\ast \).

Owing to the different slopes of the curves in the family \( H_k^\ast \), the spacing of associated points on the curves is no longer \( 2\pi \) for finite \( b_0 \). Then the spacing of the curves in the family \( H_k^\ast \) can still be described by the expression linking delay and periodicity given in \([53]\) but with a \( b_0 \)-dependent stretching factor. When considering the bifurcation set of (3) in the \((\lambda, b_0)\)-plane, this means that there are infinitely many Hopf bifurcation curves that accumulate on the left-hand boundary; see Fig. 4(a).

The theory of \([53]\) also suggests that there may be infinitely many stable delay-induced periodic orbits. Figures 6(b) and 7(a) show that, for large \( k \) the domain of stability of the stable delay-induced periodic orbit \( \Gamma_k^\ast \) becomes impractically small. On the other hand they also show that the domains of stability of \( \Gamma_1^\ast \) and \( \Gamma_2^\ast \), are large enough to cause concern when implementing Pyragas control. As we have already discussed, if an initial condition is not carefully chosen, the system could reach one of these stable periodic orbits rather than \( \Gamma_P \).

Figure 7(b) is the one-parameter bifurcation diagram in the delay \( \tau \) of (3) for \( b_0 = 0.295 \); compare with Fig. 6(a). It shows the periodic orbit \( \Gamma_P \) (green) and the stable delay-induced periodic orbits \( \Gamma_1^\ast \), \( \Gamma_2^\ast \), \( \Gamma_3^\ast \) and \( \Gamma_4^\ast \) (red); the already unstable periodic orbits that bifurcate from delay-induced Hopf bifurcation curves are again shown in grey. The stable periodic orbits bifurcate from the Hopf bifurcation points \( H_1^\ast \), which are equally spaced in \( \tau \); we calculate the spacing for \( b_0 = 0.295 \) to be approximately \( 1.1 \times 2\pi \). Notice from Fig. 7(b) that for larger values of \( k \) the periodic orbit \( \Gamma_k^\ast \) is stable for a smaller range of \( \tau \), which corresponds to the shrinking domains of stability shown in Fig. 7(a).

5 The effect of the \( 2\pi \)-periodic feedback phase \( \beta \)

Most previous work on (3) fixes the \( 2\pi \)-periodic feedback phase at \( \beta = \frac{\pi}{4} \). We now vary \( \beta \), firstly increasing it and then decreasing it from \( \frac{\pi}{4} \). In particular, we evaluate the effect that \( \beta \) has on the domain of stability, the geometry of delay-induced Hopf bifurcation curves and the positions of the points \( b_0^6 \), \( HH_0 \) and \( b_0^5 \).

5.1 The effect on the domain of stability

Figure 8 shows how the domain of stability of \( \Gamma_P \) (shaded area) in the \((\lambda, b_0)\)-plane changes as the parameter \( \beta \) is increased. Panel (a) shows the domain when \( \beta = \frac{\pi}{4} \) as in Fig. 5. Recall that for \( \beta = \frac{\pi}{4} \) the lower boundary of the stability region in the \((\lambda, b_0)\)-plane is the transcritical bifurcation curve TC. The upper boundary is the torus bifurcation curve \( T_0^\ast \) and the right-hand boundary is the Hopf bifurcation curve \( H_P \) between the points \( b_0^5 \) and \( HH_0 \).

As \( \beta \) is increased, as in Fig. 8(b) and (c) for \( \beta = \frac{3\pi}{4} \) and \( \beta = 3 \), respectively, the domain of stability of \( \Gamma_P \) becomes smaller in area, but is still bounded by the same curves. Also, the range of \( b_0 \) (the difference in \( b_0 \) between the points \( HH_0 \) and \( b_0^5 \)) for which \( \Gamma_P \) bifurcates stably from \( H_P \), decreases. More specifically, the curve \( H_0 \) shifts left in the plane, the two points \( b_0^5 \) and \( HH_0 \) move closer together, and the end of the curve \( T_0^\ast \) moves right along the curve TC. As \( \beta \) approaches \( \pi \), the area of the domain of stability shrinks to zero; see panels (d) and (e) for \( \beta = 3.12 \) and \( \beta = \pi \) respectively.

Figure 8(e) shows that as \( \beta = \pi \), the domain of stabilization has disappeared. Here, the curve \( H_4 \) is tangent to the curve \( H_P \) at \((\lambda, b_0) = \left(0, \frac{1}{2\pi}\right)\). The points \( b_0^5 \) and \( HH_0 \) are both at the point \((\lambda, b_0) = \left(0, \frac{1}{2\pi}\right)\). When \( \beta = \pi \), the periodic orbits bifurcating from \( H_4 \) are no longer stable. The targeted unstable periodic orbit that emanates from \( H_P \) cannot be stabilized through a transcritical bifurcation (see section 3). A periodic orbit that bifurcates from \( H_P \) below the point \((\lambda, b_0) = \left(0, \frac{1}{2\pi}\right)\) has one unstable Floquet multiplier, above \((\lambda, b_0) = \left(0, \frac{1}{2\pi}\right)\) a periodic orbit that bifurcates from \( H_P \) has a complex conjugate pair of unstable Floquet multipliers.

As \( \beta \) is further increased the curve \( H_4 \) has shifted right in the plane. Furthermore, the points \( b_0^5 \) and \( HH_0 \) have moved through each other; this is shown in panel (f) of Fig. 8. Below the point \( HH_0 \), an unstable periodic orbit that bifurcates from the curve \( H_P \) has one unstable Floquet multiplier. An unstable periodic orbit that bifurcates from \( H_P \) between \( HH_0 \) and \( b_0^5 \) has three unstable Floquet multipliers. Above the point \( b_0^5 \), an
unstable periodic orbit that bifurcates from $H_P$ has two unstable Floquet multipliers. There exist double Hopf bifurcation points of the set $HH_K$ further up the curve $H_P$. Above each of these points the unstable periodic orbits that bifurcate from $H_P$ have an additional complex pair of unstable Floquet multipliers to those that bifurcate from $H_P$ below the point. As $\beta$ is further increased, the point $b_0^c \to \infty$ as $\beta \to [\arctan(-\frac{1}{\gamma}) + \pi]$, then for $\beta > [\arctan(-\frac{1}{\gamma}) + \pi]$ the point $b_0^c$ is negative.

The effect on the domain of stability of $\Gamma_P$ when the feedback phase $\beta$ is decreased is considered in Fig. 9. Panel (a) again shows the domain (shaded) for $\beta = \frac{\pi}{4}$ as in Fig. 5. As $\beta$ is decreased both the points $b_0^c$ and $HH_0$ have moved upwards and the point $R_1$ has moved left along the curve $TC$; this is shown in panels (b) and (c) for $\beta = 0.5$ and $\beta = 0.2$, respectively. In both of these panels the area of the domain of stability and the

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Figure 8: The overall domain of stability (shaded) in the $(\lambda, b_0)$-plane for different values of increasing $\beta$. Panel (a) is for $\beta = \frac{\pi}{4}$ as it is in Fig. 5, and panels (b)–(f) are for the given value of $\beta$. Each panel shows the curves $H_P$ (green) and $H_L$ (red) as well as the points $b_0^c$ (black dot) and $HH_0$ (green dot). The transcritical bifurcation curve $TC$ (purple) and the torus bifurcation curve $T_{P}^{\omega}$ (grey) meet at the point $R_1$ (light green dot).
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Figure 9: The overall domain of stability (shaded) in the \((\lambda, b_0)\)-plane for different values of decreasing \(\beta\). Panel (a) is for \(\beta = \frac{\pi}{2}\) as it is in Fig. 5, and panels (b)–(f) are for the given value of \(\beta\). Each panel shows the curves \(H_P\) (green) and \(H_L\) (red) as well as the points \(b_0^c\) (black dot) and \(HH_0\) (green dot). The transcritical bifurcation curve TC (purple) and the torus bifurcation curve \(T_P^0\) (grey) meet at the point \(R_1\) (light green dot).

range of stability in \(b_0\) increase. As \(\beta\) is decreased below 0.2, the point \(b_0^c\) moves upwards at a much faster rate than the point \(HH_0\) and the point \(R_1\) moves right along the curve TC. Thus, the area of the domain of stability of \(\Gamma_P\) becomes smaller; an example of this is shown in Fig. 9(d) for \(\beta = 0.11\). At approximately \(\beta = 0.103\), as shown in panel (e), the points \(b_0^c\) and \(HH_0\) are equal where the curve \(H_L\) has a point of self-intersection on the curve \(H_P\) and the domain of stability of \(\Gamma_P\) has disappeared. Below this point on \(H_P\), a periodic orbit bifurcates unstably from \(H_P\) with one unstable Floquet multiplier and above it a periodic orbit bifurcates unstably with a complex conjugate pair of unstable Floquet multipliers. The maximum range of stability in \(b_0\) is reached at \(\beta \approx 0.12\). As \(\beta\) is further decreased, the point \(b_0^c\) rises rapidly, going to infinity as \(\beta \to \arctan(-\frac{1}{\gamma})\). Below this value of \(\beta\) the point \(b_0^c\) is negative.

Overall, Fig. 8 and Fig. 9 show that stabilization of \(\Gamma_P\) is possible only when the point \(b_0^c\) is below the point...
5.2 The effect of $\beta$ on the delay-induced Hopf bifurcation curves

Figures 8 and 9 show that a change in $\beta$ causes the Hopf bifurcation curve $H_L$ to deform and move in the plane. This movement of $H_L$ plays a crucial part in changes to the domain of stability for $\Gamma_P$. Therefore, we now consider in more detail the effect that changing the parameter $\beta$ has on the three families of Hopf bifurcation curves (shown in Fig. 4(a)).

To gain a truly global overview of how the delay-induced Hopf bifurcation curves move in the $(\lambda, b_0)$-plane we must consider the curves beyond the region shown in Fig. 3. Hence, we now compactify the $(\lambda, b_0)$-plane so that we can consider how the bifurcation curves behave near infinity. There are several ways to compactify a plane. One common approach is to transform the plane to the Poincaré disk. Here, we choose alternatively to compactify the parameter space as the unit square via individual stereographic transformations for $\lambda$ and $b_0$ given by,

\begin{align}
\hat{b}_0 &= \frac{b_0}{1 + b_0}, \\
\hat{\lambda} &= \frac{\lambda - \frac{1}{2}}{0.25 + (\lambda - \frac{1}{2})}. \quad (17a, 17b)
\end{align}

Figure 10: The first curves of Hopf bifurcations from each family in the compactified $(\hat{\lambda}, \hat{b}_0)$-plane. Shown are the curve $H_P$ (green), the delay-induced curves $H_L$, $H^1_J$ and $H^1_R$ (red), the points $b_c^0$ and $b_0^*$ (black dots) and the double Hopf bifurcation points $HH_0$ and $HH_1$ (green dots).

$HH_0$. For a positive value of $b_0$, this only occurs when $\beta_c < \beta < \pi$, where $\beta_c \approx 0.103$ is a value of $\beta$ at which (8) and (11) are equal.

Outside of the stability region of $\Gamma_P$ there are several other attractors in the plane. For example, in panel (c) of Fig. 8, to the left of the curve $H_L$ both the equilibrium solution and periodic solutions bifurcating from the curves $H_L$ and $H^1_J$ may be stable depending on exact parameter values; see sections 3 and 4. To the right of the curve $H_L$ the tori bifurcating from the set of curves $T^K_P$ are initially attracting.
Recall that we do not consider the parameter regions $b_0 < 0$ or $\lambda < \frac{1}{8}$. The coordinate transformation (17a) for $b_0$ fixes $\dot{b}_0 = b_0 = 0$ and transforms $b_0 = \infty$ to $\dot{b}_0 = 1$. The coordinate transformation (17b) for $\lambda$ works in a similar fashion, but with a shift of $-\frac{1}{3}$ so that $\lambda = \frac{1}{3}$ maps to $\lambda = 0$ and $\lambda = \infty$ maps to $\lambda = 1$. Overall, we obtain a transformed bifurcation set in the unit square $(\hat{\lambda}, \hat{b}_0) \in [0, 1] \times [0, 1]$.

Figure 10 shows the Hopf bifurcation curves $H_L$, $H_J^1$, and $H_R^1$ in the $(\hat{\lambda}, \hat{b}_0)$-plane. The top boundary of the compactified plane represents infinity in $b_0$ with vertical asymptotes for fixed values of $\lambda$. The right boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane represents infinity in $\lambda$ with horizontal asymptotes for fixed values of $b_0$. The corner point $(1,1)$ corresponds to infinity in both $\lambda$ and $b_0$ and the corner point $(0,1)$ corresponds to infinity in $b_0$ where there is a vertical asymptote at $\lambda = \frac{1}{8}$.

The shapes of the curves $H_L$, $H_J^1$ and $H_R^1$ in the $(\hat{\lambda}, \hat{b}_0)$-plane in Fig. 10 are similar to the equivalent curves before the compactification. Moreover, we can now see that the curve $H_J^1$ goes to infinity in $\lambda$ at a finite value of $\lambda$, whilst the curve $H_R^1$ goes to infinity in both $\lambda$ and $b_0$, ending at the top right corner $(1,1)$. We also show how the three families of delay-induced Hopf bifurcation curves change in the $(\hat{\lambda}, \hat{b}_0)$-plane as the feedback phase $\beta$ is increased through a period of $2\pi$. An increase or decrease of $2\pi$ in $\beta$ will result in an identical bifurcation set to that shown in Fig. 10. However, we find that, as $\beta$ is changed, the Hopf bifurcation curves transition through the plane and transform into other Hopf bifurcation curves in a quite complicated manner.

We start by showing the transitions of the curves $H_L$, $H_J^1$ and $H_R^1$ in the $(\hat{\lambda}, \hat{b}_0)$-plane. We show how the curves $H_L^1$, $H_J^1$ and $H_R^1$ are formed and move in the plane and, finally, we show how the curve $H_J^1$ is formed. The transition of each curve is shown in a multi-panel diagram, where each panel shows the $(\hat{\lambda}, \hat{b}_0)$-plane at a different phase of $\beta$. For reference, each panel shows the curve $H_P$ in green.

Figure 11 shows how the curve $H_L$ transforms into the curve $H_J^1$ as $\beta$ increases from $\beta = \frac{\pi}{2}$ to $\beta = \frac{3\pi}{4}$. Panel (a) shows the curve $H_L$, from Fig. 10, that emerges from the point $b_0^* = 0$ in the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. As $\beta$ is increased from $\frac{\pi}{2}$, the point $b_0^*$ moves up and down the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. At $\beta = [\arctan(-\frac{1}{3}) + \pi] \approx 3.24$, the point $b_0^*$ has moved to the top left corner $(0,1)$; see panel (b). Panel (c) for $\beta = 3.242$ shows that the edge points of the curve are no longer the same; one end point, the right one, has moved along the top boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane to the top right corner $(1,1)$. Thus, the Hopf bifurcation curve no longer forms a loop. Panel (d) for $\beta = 3.26$ shows that the left end point of the curve has detached from the corner $(0,1)$, and also moved along the top boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. When $\beta$ is approximately $2\pi$ the left end point of the curve has stopped moving right; panel (e) shows the left end point of the curve near its largest value of $\lambda$. The left end point of the curve then moves left along the top boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane, at $\beta = 6.3513$ it reaches the point $(0,1)$; this is shown in the enlargement of panel (f). The enlargement in panel (g) shows that at $\beta = 6.3628$, the right end point of the curve has detached from the corner point $(1,1)$. The right end point of the curve then moves left along the top boundary of the plane. At $\beta = [\arctan(-\frac{1}{3}) + 2\pi]$ the left end point of the curve starts to move down the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane; an example of this is shown in panel (h). At $\beta = \frac{9\pi}{4}$, the curve $H_J^1$ is formed, this is shown in panel (i). The left end point of the curve of is at $b_0^*$ on the left boundary of the plane and the right end point is connected to the top boundary of the plane. Thus, through a change of $2\pi$ in $\beta$, the curve $H_L$ transforms into $H_J^1$.

Figure 12 shows how the curve $H_J^1$ transforms into the curve $H_J^2$ as the feedback phase $\beta$ is increased through a period of $2\pi$. Panel (a) shows the curve $H_J^1$ as it appears in the $(\hat{\lambda}, \hat{b}_0)$-plane in Fig. 10 for $\beta = \frac{\pi}{2}$. Its left end point is at the point $b_0^*$ on the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane and the right end point is connected to the top boundary of the plane at infinity in $b_0$. Figure 12(b) for $\beta = 3.1416$ shows that the right end point of the curve has moved right along the top boundary of the plane and that there is a slight bend near the right end of the curve. Panel (c) shows that when $\beta = [\arctan(-\frac{1}{3}) + \pi] \approx 3.2328$ the right end point of the curve is attached to the corner $(1,1)$ and the left end point of the curve is attached to the corner $(0,1)$. Panel (d) shows that the left end point of the curve has detached from the corner $(0,1)$ and has moved right along the top boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. At the same time the entire Hopf bifurcation curve has shifted upwards in the plane. At $\beta \approx 2\pi$ as in panel (e), the left end point of the curve has stopped moving right along the top boundary of the plane. The left end point of the curve then starts to move left along the top boundary of the
Figure 11: Transition of Hopf bifurcation curves in the \((\hat{\lambda}, \hat{b}_0)\)-plane for values of \(\beta\) as given. The curve \(H_P\) is shown in green. The curve \(H_L\) (red) for \(\beta = \frac{\pi}{4}\) in panel (a) transforms into the curve \(H^1_J\) (red) for \(\beta = \frac{9\pi}{4}\) in panel (i). Inserts in (e), (f) and (g) show respective enlargements.

(\hat{\lambda}, \hat{b}_0)\)-plane. Panel (f) for \(\beta = 6.3548\) shows that the left end point of the curve connects to the corner (0,1). At \(\beta = \left[\arctan\left(-\frac{1}{\gamma}\right) + 2\pi\right] \approx 6.3701\), the right end point of the curve has detached from the corner (1,1) and has moved left along the top boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane; this is shown in the enlargement of panel (g). The left end point of the curve then moves down the left boundary of the plane forming the point \(b_0^*\) and, thus, the curve \(H^1_J\) is formed. This formation is shown in panels (h) and (i), with the Hopf bifurcation curve \(H^1_J\) shown in red for \(\beta = \frac{9\pi}{4}\). Thus, a change of \(2\pi\) in the phase of the feedback \(\beta\), transforms the curve \(H^1_J\) into the curve \(H^2_J\). Although not shown here, we have found that an increase of \(2\pi\) in \(\beta\) transforms the curve \(H^2_J\) into the curve \(H^3_J\) and the curve \(H^1_J\) into the curve \(H^2_J\). This is strong evidence for the natural conjecture that a increase of \(2\pi\) in the feedback phase \(\beta\) transforms the curve \(H^k_J\) into the curve \(H^{k+1}_J\).

Figure 13 shows how the curves \(H^k_R\) and \(H^k_J\) transition through the plane as \(\beta\) is changed. We start with the curve \(H^2_R\) from Fig. 4(a) but for \(\beta = -\frac{7\pi}{4} = \frac{5\pi}{4} - 2\pi\). Although in the \((\lambda, b_0)\)-plane the curves \(H^1_R\) and \(H^2_R\) are both straight lines, in the \((\hat{\lambda}, \hat{b}_0)\)-plane, \(H^1_R\) and \(H^2_R\) have differing shapes. This is because the curve \(H^2_R\) has a greater gradient than \(H^1_R\), which results in differently shaped curves when the plane is compactified by (17).

Figure 13(a) shows the Hopf bifurcation curve \(H^2_R\) for \(\beta = -\frac{7\pi}{4}\) with its right end point at the corner (1,1) and its left end point connected to the left boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane at the point \(b_0^*\). As \(\beta\) is increased the
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right end point of the curve moves left along the top boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane and the left end point of the curve moves up the left boundary of the plane; this is shown in panels (b) and (c). The curve disappears into the top left corner $(0,1)$ of the $(\hat{\lambda}, \hat{b}_0)$-plane at $\beta = [\arctan(-\frac{1}{\gamma}) - \pi]$. At $\beta \approx -0.06$ another Hopf bifurcation curve emerges from the top right corner $(1,1)$ of the $(\hat{\lambda}, \hat{b}_0)$-plane. It starts to extend downwards in the plane, with its right endpoint at $(1,1)$. The left end point of the curve starts to move left along the top boundary of the plane. Panel (d) shows the curve at $\beta = 0$; the right end point of the curve is at the corner $(1,1)$ and the left end point of the curve is connected to the top boundary of the plane. Panel (e) shows that the left end point of the curve has connected to the point $(0,1)$ at $\beta = \arctan(-\frac{1}{\gamma}) \approx 0.1$. The left end point of the curve then starts to move down the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane forming the point $b_5^*$. Panel (f) shows the Hopf bifurcation curve $H_{R_1}^1$ as it appears in Fig. 10 at $\beta = \frac{\pi}{4}$. The curve has its right end point at the corner $(1,1)$ and its left end point at the point $b_5^*$ on the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. It should be noted that the curve $H_{R_1}^2$ is formed in a similar way to the curve $H_{R_1}^1$. A curve appears from the corner $(1,1)$, the left end point of this curve firstly moves left along the top boundary of the curve and then after reaching the corner $(0,1)$ it moves down the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. The right end point of the curve remains at the corner $(1,1)$.

Figure 13(g) shows that at $\beta = 1.8151$ a second Hopf bifurcation curve has appeared from the top right
corner (1,1) of the $(\hat{\lambda}, \hat{b}_0)$-plane. The right end point of this secondary curve is at the corner (1,1) and the left end point immediately starts to move left along the top boundary of the $(\lambda, b_0)$-plane. This secondary curve is shown in blue in the enlargement in Fig. 13 panel (g). The right end points of both curves become tangent at the corner (1,1) when $\beta = 2.0833$. At this point the two Hopf bifurcation curves join; this is shown in panel (h). As shown in panel (i) for $\beta = 2.15$, the curve has moved away from the point (1,1), its right end point is connected to the top boundary of the plane and its left end point is connected to the left boundary of the plane. Panel (j) for $\beta = 2.35$ shows that the kink in the curve where it was connected to the corner (1,1) has straightened out. The right end point of the curve then starts to move left along the top boundary of the plane. Panel (k) shows the curve with its left end point connected to the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane and its right end point connected approximately midway across the top boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane. The left end point of the curve starts to move up the left boundary of the $(\hat{\lambda}, \hat{b}_0)$-plane and the right end point moves further left along the top boundary of the plane. Thus, the Hopf bifurcation curve moves into the top left corner (0,1) of the $(\hat{\lambda}, \hat{b}_0)$-plane; this is shown in Fig. 13(l) where the curve has both its right and left end points near the corner (0,1). When $\beta = [\arctan(-\frac{1}{2}) + \pi]$, the curve disappears into the top left corner (0,1) of the plane. At approximately $\beta = 4.2009$ another Hopf bifurcation curve emerges from infinity in both $\lambda$ and $b_0$ at the corner (1,1); this is shown in Fig. 14.

Panels (a) and (b) of Fig. 14 show this curve at $\beta = 4.5182$ and $\beta = 4.7143$ respectively, it has formed into a figure of eight shape that starts and ends at the corner (1,1) of the $(\hat{\lambda}, \hat{b}_0)$-plane. This curve stretches downwards into the plane. Panel (c) shows the curve at $\beta = 5.6096$, where the left end point of the curve has detached from the corner (1,1). This end point of the curve moves left along the top boundary of the plane until it reaches the corner (0,1), forming the point $b_0^*$; see panel (d). Panel (e) shows the curve at $\beta = 6.3829$, the right end point of the curve has detached from the corner (1,1) and has moved left along the top boundary of the plane. In panel (f) for $\beta = 6.3863$ the right end point of the curve joins the left end point at the corner (0,1) of the $(\hat{\lambda}, \hat{b}_0)$-plane. Both ends of the curve remain joined and move down the left boundary of the plane, the curve starts as a figure of eight shape before deforming into an enclosed loop; see panels (g) and (h). Panel (i) shows the red curve $H_I$ as it appears in Fig. 10, here for $\beta = \frac{9\pi}{4}$.

Overall, Figures 11–14 show that the movement of delay-induced Hopf bifurcation curves through the $(\hat{\lambda}, \hat{b}_0)$-plane, as $\beta$ is changed through a period of $2\pi$, is highly non-trivial. As $\beta$ is increased by $2\pi$, the curve $H_L$ transforms into the curve $H^1_L$, the curve $H^2_L$ transforms into the curve $H^3_L$ and so on. This means that although a shift of $2\pi$ in the phase $\beta$ obviously results in an identical bifurcation set, the Hopf bifurcation curves transform into different Hopf bifurcation curves rather than settling back into their original positions. We also note that the delay-induced Hopf bifurcation curves all have movements to and from infinity at the values $\beta = [\arctan(-\frac{1}{2}) + m\pi]$ (where $m \in \mathbb{Z}$). This is where the $b_0$-coordinates of the points $b_0^*$ (equation (8)) and $b_0^*$ (equation (13)) go to infinity. Also for this value of $\beta$ the curve $H^I$ no longer forms a closed loop. As we found in section 5.1, plays an important role in whether or not a domain of stability exists; see Fig. 8.

The effect of a change in $\beta$ has on the entire bifurcation set (including SNLC/torus bifurcations of periodic orbits shown in Fig. 4 and Fig. 5) is an even more complex transition than the above analysis revealed for only the delay-induced Hopf bifurcation curves. A complete analysis of how every bifurcation curve of periodic orbits transitions through the plane as $\beta$ changes is beyond the scope of this paper. However, we have shown how the bifurcation curves of periodic orbits and the double Hopf bifurcation points $b_0^*$ and $HH_0$, that bound the domain of stability of $\Gamma_P$, move as $\beta$ is changed; see Fig. 8 and Fig. 9. We conjecture that the other bifurcation curves of periodic orbits that emerge from the Hopf bifurcation curves also move in accordance with how the Hopf bifurcation curves move in the $(\lambda, b_0)$-plane.

6 The effect of the parameter $\gamma$

Previous work has shown that stabilization of $\Gamma_P$ is impossible when $\gamma = 0$ [3]. However, why stabilization fails at $\gamma = 0$ has not been fully explained geometrically. Here we show how the domain of stability for $\Gamma_P$ disappears as $\gamma$ is increased to 0. We also consider the effect of $\gamma$ on the curve $H_L$ and the two points $b_0^*$ and $HH_0$. We observe that the transition of $H_L$ in the compactified $(\hat{\lambda}, \hat{b}_0)$-plane, when $\gamma$ is increased, causes the points $b_0^*$
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Figure 13: Transition of Hopf bifurcation curves in the $(\hat{\lambda}, \hat{b}_0)$-plane for values of $\beta$ as given. The curve $H_P$ is shown in green. The curve $H_{R}^2$ (red) for $\beta = -\frac{7\pi}{4}$ is shown in panel (a) and the curve $H_{R}^1$ (red) for $\beta = \frac{\pi}{4}$ is shown in panel (f).

and $HH_0$ to cross and then stabilization of $\Gamma_P$ fails. Finally, we present the region in the $(\beta, \gamma)$-plane where a successful stabilization of the target periodic orbit $\Gamma_P$ is possible.

Figure 15 shows how the overall domain of stability (shaded) of $\Gamma_P$ for $\beta = \frac{\pi}{4}$ changes as $\gamma$ is increased. Panel (a) shows the domain of stability as in Fig. 5, bounded by the curves $H_P$, $T_{P}^0$ and TC. As $\gamma$ is increased the point $b_0^c$ moves upwards, while the point $HH_0$ is independent of $\gamma$ and so does not move. Panels (b) and (c), for $\gamma = -8$ and $\gamma = -5$, respectively, show that the point $R_1$ has moved left along the curve TC. It should
Figure 14: Transition of Hopf bifurcation curves in the $(\hat{\lambda}, \hat{b}_0)$-plane for values of $\beta$ as given. Shown are the curves $H_P$ (green) and $H_L$ (red) for $\beta = \frac{\pi}{4}$ in panel (i).

be noted that an increase in $\gamma$ shifts the left-hand boundary at $\lambda = \frac{1}{2}$ further left in the $(\lambda, b_0)$-plane. Since the curve TC starts at $b_0^c$ and ends at this boundary, along with its starting point changing, the gradient of the curve TC is also changing; see equation (10). As the point $R_1$ shifts left in the $(\lambda, b_0)$-plane the area of the domain of stability of $\Gamma_P$ increases; however, the range of stability in $b_0$ (the difference in $b_0$ between the points $HH_0$ and $b_0^c$) for $\Gamma_P$ decreases. At approximately $\gamma = -1.63$ the point $R_1$ stops moving left in the $(\lambda, b_0)$-plane; for this value of $\gamma$ the area of the domain of stability of $\Gamma_P$ is maximal. As $\gamma$ is further increased the point $R_1$ starts to move right along the curve TC, reducing the area of the domain of stability of $\Gamma_P$. Figure 15(d) for $\gamma = -1.5$ shows that the point $R_1$ has moved right in the $(\lambda, b_0)$-plane and is now close to the starting point $b_0^c$ of the curve TC; hence, the range of stability of $\Gamma_P$ in $b_0$ is much smaller than for lower values of $\gamma$. The point $b_0^c$ has moved up in the $(\lambda, b_0)$-plane and is just below the point $HH_0$. Thus, the boundary curves of the domain, $T_0^P$ and TC, are close to each other resulting in a small domain of stability.

Figure 15(e) shows that at $\gamma = -1.4244$, the points $b_0^c$ and $HH_0$ are equal, and the domain of stability of $\Gamma_P$ has disappeared. A periodic orbit bifurcating from $H_P$ below this point is unstable with one unstable Floquet multiplier. A periodic orbit that bifurcates from $H_P$ above this point is also unstable but has a complex conjugate pair of unstable Floquet multipliers. As $\gamma$ is further increased the point $b_0^c$ moves above the point $HH_0$; an example of this is shown in panel (f) for $\gamma = -1.2$. When $\gamma = -1$ the point $b_0^c$ goes to infinity, and for $\gamma > -1$ the point $b_0^c$ is negative.
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Figure 15: The overall domain of stability (shaded) in the $(\lambda, b_0)$-plane for different values of increasing $\gamma$. Panel (a) is for $\gamma = -10$ as it is in Fig. 5, and panels (b)-(f) are for the given value of $\gamma$. Each panel shows the curves $H_P$ (green) and $H_L$ (red) as well as the points $b_0^c$ (black dot) and $HH_0$ (green dot). The transcritical bifurcation curve $TC$ (purple) and the torus bifurcation curve $T_P$ (grey) meet at the point $R_1$ (light green dot).

Figure 16 shows the transition of the curve $H_L$ in the $(\hat{\lambda}, \hat{b}_0)$-plane as the parameter $\gamma$ is changed. Note that, due to the $\gamma$-dependent compactification of the $\lambda$ coordinate by (17b), the curves $H_L$ and $H_P$ are shifted right as $\gamma$ is increased from its starting value of $-10$. Thus, the last couple of panels of Fig. 16 show the Hopf bifurcation curves on the far right of the $(\hat{\lambda}, \hat{b}_0)$-plane.

Figure 16(a) shows the curve $H_L$ (red) and the curve $H_P$ (green) for $\gamma = -10$. The Hopf bifurcation curve $H_L$ forms a closed loop shape with both end points connected at the point $b_0^c$ on the left boundary of the $(\lambda, b_0)$-plane. As $\gamma$ is increased, the lower portion of the curve shifts upwards in the plane, deforming the loop structure and the curve develops a point of self-intersection; an example of this is shown in shown in panel (b) for $\gamma = -2$. When $\gamma = -1.5$ as in panel (c), the point of self intersection of the Hopf bifurcation curve has
moved right in the \((\hat{\lambda}, \hat{b}_0)\)-plane and the curve forms a more distinct figure of eight shape. The point \(b_0^*\) has shifted up the left boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane. Panel (d) for \(\gamma = -1.4244\) shows that the Hopf bifurcation curve has its point of self-intersection on the curve \(H_P\). As was shown in Fig. 15 at this value of \(\gamma\) stabilization of \(\Gamma_P\) is longer possible.

As \(\gamma\) is increased further, the self-intersection point of the curve moves to the right of \(H_P\); an example of this is shown in Fig. 16(e) for \(\gamma = -1.2\). Panel (f) shows the Hopf bifurcation curve at \(\gamma = -1.01\), where the point \(b_0^*\) has moved up the left boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane to the corner \((0,1)\), and the curve has opened up from its figure of eight shape. When \(\gamma = -1\) as in panel (g), the right end point of the curve has detached from the corner \((0,1)\) and has shifted along the top boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane to the right of the curve \(H_P\). Thus, the curve only crosses \(H_P\) once. As described earlier in this section, the point \(b_0^*\) is negative for this value of \(\gamma\) and remains negative as \(\gamma\) is increased further; see equation (8).

Figure 16(h) shows that at \(\gamma = -0.999\) the right end point of the curve has moved to the top right corner \((1,1)\) of the \((\hat{\lambda}, \hat{b}_0)\)-plane. The left end point of the curve is still connected to the corner \((0,1)\). The left end point of the curve then detaches from \((0,1)\) for \(\gamma \approx -0.995\) and starts to move right along the top boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane. Panel (i) for \(\gamma = -0.95\) shows the curve with its left end point connected to the top boundary, whilst its right end point is still connected to the top right corner \((1,1)\). The left end point of the curve continues.

Figure 16: Transition of Hopf bifurcation curves in the \((\hat{\lambda}, \hat{b}_0)\)-plane for values of \(\gamma\) as given. Shown are the curves \(H_P\) (green) and \(H_L\) (red) for \(\gamma = -10\) in panel (a).
to move right along the top boundary of the \((\hat{\lambda}, \hat{b}_0)\)-plane as the value of \(\gamma\) is increased. As previously discussed, stabilization is only possible when the \(b_0\)-coordinate of the point HH_0 is greater than that of the point \(b_0^c\); see also [3, 32]. From this relationship and the points \(b_0^c\) and HH_0 given by equations (8) and (11), we can find an overall stability boundary as the union of the curve \(\gamma = S(\beta)\) and the line \(\beta = \pi\), where \(S(\beta)\) is given by

\[
S(\beta) = -\frac{1}{\pi - \beta} - \frac{1}{\tan \beta}.
\] (18)

Equation (18) defines the stability boundary \(\gamma = S(\beta)\) (black) shown in Fig. 17 in the \((\beta, \gamma)\)-plane. When \(0 < \beta < \pi\) stabilization fails on and above this curve, for \(\pi < \beta < 2\pi\) stabilization fails on and below this curve. The vertical dotted red line in Fig. 17 is at \(\beta = \frac{\pi}{4}\), and the panels in Fig. 16 have values of \(\gamma\) on this line. In the shaded regions in the \((\beta, \gamma)\)-plane the targeted periodic orbit \(\Gamma_P\) can be stabilized by Pyragas control with an appropriate choice of \(\lambda\) and \(b_0\). The stability boundary \(S(\beta)\) defines the upper and left boundaries of the lower shaded region. It also forms the lower and right boundaries for the upper shaded region. The vertical (blue) curve at \(\beta = \pi\) marks the right boundary of the lower shaded region and the left boundary of the upper shaded region. As explained in section 5.1, stabilization is impossible when \(\beta = \pi\). For \(\pi < \beta < 2\pi\) the dynamics presented in this paper are replicated for negative \(b_0\) through the symmetry relation \((\pi - \beta, \gamma) \rightarrow (2\pi - \beta, -\gamma)\). Therefore, the shaded region on the right is a rotation of the shaded region on the left. The shaded regions extend to negative and positive infinity in \(\gamma\) respectively.

7 Conclusion

In this paper we have considered the global effects of the addition of Pyragas control to the subcritical Hopf normal form. The addition of feedback induces infinitely many Hopf bifurcation curves, which can be classified as belonging to three families \(H_L, H_K\) and \(H_R\). In particular, we have shown that, in addition to the target periodic orbit, system (3) has (a possibly infinite number of) other stable periodic orbits. Thus, system (3) may converge to a periodic orbit that is not the one targeted by Pyragas control. Furthermore, we have identified a region of bistability, where both the equilibrium solution and the target periodic orbit \(\Gamma_P\) are stable. This means that the system may not actually reach this periodic orbit if the initial condition is chosen incorrectly. We characterized the bifurcation curves that form the boundaries of the overall domains of stability of the target periodic orbit \(\Gamma_P\) and of the stable delay-induced periodic orbits. Moreover, we have shown how the domain
of stability of $\Gamma_P$ changes as the phase of the feedback $\beta$ is increased. From this it was shown that, as $\beta$ is increased or decreased from its original value of $\frac{\pi}{4}$, the degenerate Hopf bifurcation point $b_0^L$ and the double Hopf bifurcation point $HH_0$ move closer together in the $(\lambda, b_0)$-plane. As this happens both the area of the domain and the range in $b_0$ for which $\Gamma_P$ is stable are reduced. When the points $b_0^L$ and $HH_0$ are equal, the domain of stability disappears. There appear to be no remaining pockets of stability of $\Gamma_P$ in the $(\lambda, b_0)$-plane and thus, the Pyragas control scheme fails.

In addition to showing the geometry of the delay-induced Hopf bifurcations in the conventional parameter regime (where $\beta = \frac{\pi}{4}$ and $\gamma = -10$) we have shown how these curves change in the $(\lambda, b_0)$-plane when the feedback phase $\beta$ is varied. To fully understand the highly non-trivial changes in these Hopf bifurcation curves, the $(\lambda, b_0)$-plane was compactified to the $(\hat{\lambda}, b_0)$-plane. We found that, as $\beta$ is increased by $2\pi$, the curve $H_L$ transforms into the curve $H_{L}^1$, the curve $H_{L}^3$ into the curve $H_{L}^4$, the curve $H_{T}^2$ into the curve $H_{T}^3$ and so on. We also found that the Hopf bifurcation curves of the family $H_V^2$ emerge from the top right corner $(1,1)$ of the $(\lambda, b_0)$-plane at infinity in $\lambda$ and $b_0$ and, as $\beta$ is increased, they disappear into the top left corner $(0,1)$ of the $(\hat{\lambda}, b_0)$-plane.

We also considered the effect of a change in the parameter $\gamma$. We showed that as $\gamma$ is increased the point $b_0^L$ moves up in the $(\lambda, b_0)$-plane. Furthermore, the $\gamma$-dependent left-hand boundary of this plane and the gradient of the transcritical bifurcation curve $TC$ also changed as $\gamma$ was increased. Thus, the range in $b_0$ for which $\Gamma_P$ is stable starts to decrease as soon as $\gamma$ is increased. However, the area of the domain of stability of $\Gamma_P$ initially increases as $\gamma$ is increased. As $\gamma$ is increased further the area of this domain then starts to reduce. When the points $b_0^L$ and $HH_0$ are equal the domain of stability of $\Gamma_P$ disappears. Again, there are no remaining pockets of stability for $\Gamma_P$ and thus at this point the Pyragas control scheme fails. Brown et al. [3] have shown that stabilization fails for $\gamma = 0$; here we have shown that stabilization of the target periodic orbit actually fails on a stability boundary $S(\beta)$ in the $(\beta, \gamma)$-plane [32]. This stability boundary defines regions in the $(\beta, \gamma)$-plane for which stabilization is possible.

Overall, this paper shows that it is very useful to take a global point of view of a Pyragas controlled system to ascertain if, how and where the target periodic orbit is stabilized. It also highlights that care must be taken when implementing Pyragas control. Even when the delay is set as the exact parameter-dependent period, if parameters are not set carefully the system may converge to a stable delay-induced periodic orbit rather than the target periodic orbit. Moreover, the existence of bistability in part of the domain of stability of $\Gamma_P$ means that there is no guarantee that the system reaches the target state in the respective parameter regime.

We have considered the application of Pyragas control to the subcritical Hopf normal form and, therefore, conjecture that the dynamics found here will also be relevant for any systems with delay near a subcritical Hopf bifurcation. This conjecture is supported by the local results of Brown et al. [3]. In ongoing work, we consider a more general system subject to Pyragas control near a subcritical Hopf bifurcation. More specifically, we consider the Lorenz system subject to Pyragas control, which was studied locally near the subcritical Hopf bifurcation by Postlethwaite and Silber [33]. They found the same mechanism of stabilization close to the Hopf bifurcation as in the normal form. Therefore, this system is a good candidate for the investigation of its global dynamics.

\section{Parametrization of the delay-induced Hopf bifurcation curves}

A parametrization for the delay-induced Hopf bifurcations is found by entering the ansatz $z(t) = re^{i\omega t}$ into (3) and splitting the resulting equation into real and imaginary parts, yielding

\begin{align}
0 &= \lambda + r^2 + b_0(\cos(\beta - \omega \tau) - \cos \beta), \\
\omega &= 1 + \gamma r^2 + b_0(\sin(\beta - \omega \tau) - \sin \beta).
\end{align}

(19)

At a Hopf bifurcation, $r = 0$; therefore, after re-arranging, equations (19) can be written as
\[
\begin{align*}
\frac{-\lambda}{b_0} + \cos \beta &= \cos(\beta - \omega \tau), \\
\frac{\omega - 1}{b_0} + \sin \beta &= \sin(\beta - \omega \tau).
\end{align*}
\]  

Equation (20)

Squaring both equations (20) and adding them yields the expression (after simplification)

\[-2\lambda b_0 \cos \beta + \lambda^2 + (\omega - 1)^2 + 2(\omega - 1)b_0 \sin \beta = 0,
\]

Equation (21)

Solving for \(b_0\) yields the relationship (12).

**B  The double Hopf bifurcation points in the set \(HH_K\)**

To find an expression for the \(b_0\) coordinate of the set of double Hopf bifurcation points \(HH_K\) on \(H_P\), we substitute \(\lambda = 0\) (as the points in the set \(HH_K\) lie on the curve \(H_P\)) and \(\tau = 2\pi\) (the delay at \(\lambda = 0\)) into the equations (20). This gives

\[
\begin{align*}
\cos \beta &= \cos(\beta - 2\pi \omega), \\
\frac{\omega - 1}{b_0} + \sin \beta &= \sin(\beta - 2\pi \omega).
\end{align*}
\]

Equation (22)

Therefore, from (22) either \(\beta = \beta - 2\pi \omega + 2\pi k\), which gives \(\omega = k\), or \(\beta = -(\beta - 2\pi \omega) + 2\pi k\), which gives \(\omega = \frac{\beta}{2} + k\). Setting \(\omega = \frac{\beta}{2} + k\) in (22) and solving for \(b_0\) gives the set of \(b_0\)-coordinates

\[b_0 = \frac{\pi(1 + k) - \beta}{2\pi(\sin \beta)}.\]

Equation (23)

The \((\lambda, b_0)\)-coordinates of the set of double Hopf bifurcation points \(HH_K\) are thus given by

\[HH_K = \left(0, \frac{\pi(1 + k) - \beta}{2\pi(\sin \beta)}\right).\]

Equation (24)

As discussed in this paper, for stabilization to be possible, given a positive \(b_0\) the point \(b_0^*\) (8) must be below the point \(HH_0\) (11). If we consider both positive and negative values of \(b_0\), stabilization is only possible when the following inequality is satisfied

\[-\beta < -1 < \frac{\pi - \beta}{2\pi(\sin \beta)}\]

Equation (25)

Here \(b_0 \neq 0\) and the expression \(-\frac{\beta}{2\pi(\sin \beta)}\) is the point \(HH_{-1}\), which is the first negative point of the set \(HH_K\). This inequality cannot be satisfied when \(\gamma = 0\), therefore, at this value of the parameter stabilization is impossible [3,32]. Rearranging the right-hand inequality of (25) gives the function (18) along which stabilization fails for a positive \(b_0\).

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**References**


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