A Survey of Weighted Polynomial Approximation with Exponential Weights

D. S. Lubinsky

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Abstract

Let $W : \mathbb{R} \to (0, 1]$ be continuous. Bernstein's approximation problem, posed in 1924, deals with approximation by polynomials in the weighted uniform norm $f \to \|fW\|_{L_\infty(\mathbb{R})}$. The qualitative form of this problem was solved by Achieser, Mergelyan, and Pollard, in the 1950's. Quantitative forms of the problem were actively investigated starting from the 1960's. We survey old and recent aspects of this topic, including the Bernstein problem, weighted Jackson and Bernstein Theorems, Markov–Bernstein and Nikolskii inequalities, orthogonal expansions and Lagrange interpolation. We present the main ideas used in many of the proofs, and different techniques of proof, though not the full proofs. The class of weights we consider is typically even, and supported on the whole real line, so we exclude Laguerre type weights on $[0, \infty)$. Nor do we discuss Saff's weighted approximation problem, nor the asymptotics of orthogonal polynomials.

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1 Bernstein's Approximation Problem

The first quarter of the twentieth century was a great period for approximation theory. In that time, Dunham Jackson and Sergei Bernstein completed their great works on the degree of approximation. Müntz proved his theorem on approximation by powers $\left\{x^{\lambda_j}\right\}_{j=0}^{\infty}$, solving a problem of Bernstein, Faber introduced Faber polynomials and Faber series, and Szegő was developing the theory of orthogonal polynomials on the unit circle. Right at the end of that quarter, in 1924, Bernstein [4] came up with a problem that became known as *Bernstein's approximation problem*, and whose ramifications continue to be explored to this day.

One can speculate that one day Bernstein must have felt confined in approximating on bounded intervals, and so asked: by Weierstrass, we know that we can uniformly approximate any continuous function on a compact interval by polynomials. Are there analogues on the whole real line? The first thing to deal with is the unboundedness of polynomials on unbounded intervals. Clearly we need to damp the growth of a polynomial at infinity, by multiplying by a weight. For example, consider

$$P(x)\exp(-x^2), \qquad x \in \mathbb{R},$$

where P is a polynomial, or more generally,

$$P(x)W(x).$$

Here W must decay sufficiently fast at $\pm\infty$ to counteract the growth of every polynomial. That is,

$$\lim_{|x|\to\infty} x^n W(x) = 0, \qquad n = 0, 1, 2, \dots \ . \tag{1.1}$$

What can be approximated, and in what sense? This problem is known as **Bernstein's approximation problem**. A more precise statement is as follows: let $W : \mathbb{R} \to [0, 1]$ be measurable. When is it true that for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} (fW)(x) = 0,$$

there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \|(f - P_n)W\|_{L_\infty(\mathbb{R})} = 0?$$

If true we then say that the polynomials are **dense**, or that Bernstein's problem has a positive solution. The restriction that fW has limit 0 at $\pm\infty$ is essential: if $x^k W(x)$ is bounded on the real line for every non-negative k, then $x^k W(x)$ has limit 0 at $\pm\infty$ for every such k, and so the same is true of every weighted polynomial PW. So we could not hope to approximate, in the uniform norm, any function f for which fW does not have limit 0 at $\pm\infty$.

When W vanishes on a set of positive measure, or is not continuous, extra complications ensue. So in the sequel, we shall assume that W is both *positive and continuous*. The more general case is surveyed at length in [67]. The case where we approximate only on a countable set of points is included in that study.

In his early works on the problem, Bernstein often assumed that $1/W$ is the restriction to the real line of an even entire function with positive (even order) Maclaurin series coefficients. Other major contributors were N. I. Akhiezer, K. I. Babenko, L. de Branges, L. Carleson, M. M. Dzrbasjan, T. Hall, S. Izumi, T. Kawata, S. N. Mergelyan, and V. S. Videnskii. Yes, that's Lennart Carleson, and the same Mergelyan of Mergelyan Theorem fame (on uniform approximation by polynomials on compact subsets of the plane).

Bernstein's approximation problem was solved independently by Achieser, Mergelyan, and Pollard, in the 1950's. Mergelyan [67, p. 147] introduced a regularization of the weight

$$\Omega(z) = \sup\left\{ |P(z)| : P \text{ a polynomial and } \sup_{t\in\mathbb{R}} \frac{|P(t)W(t)|}{\sqrt{1+t^2}} \le 1 \right\}.$$

Theorem 1.1 (Mergelyan) *Let $W : \mathbb{R} \to (0,1]$ be continuous and satisfy (1.1). There is a positive answer to Bernstein's problem iff*

$$\int_{-\infty}^{\infty} \frac{\log \Omega(t)}{1+t^2} dt = \infty.$$

In another formulation, there is a positive answer iff

$$\Omega(z) = \infty$$

for at least one non-real z (and then $\Omega(z) = \infty$ for all non-real z). We shall outline the proof of this in the next section.

Akhiezer (and perhaps Bernstein?) [67, p. 158] used instead the regularization

$$W_*(z) = \sup \left\{ |P(z)| : P \text{ a polynomial with } \|PW\|_{L_\infty(\mathbb{R})} \leq 1 \right\}.$$

Theorem 1.2 (Akhiezer) *Let $W : \mathbb{R} \to (0,1]$ be continuous and satisfy (1.1). There is a positive answer to Bernstein's problem iff*

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2} dt = \infty.$$

Finally, Pollard [161], [67, p. 164] showed:

Theorem 1.3 (Pollard) *Let $W : \mathbb{R} \to (0,1]$ be continuous and satisfy (1.1). There is a positive answer to Bernstein's problem iff both*

$$\int_{-\infty}^{\infty} \frac{\log(1/W(t))}{1+t^2} dt = \infty$$

and there exists a sequence of polynomials $\{P_n\}$ such that for each x,

$$\lim_{n\to\infty} P_n(x) W(x) = 1,$$

while

$$\sup_{n\geq 1} \|P_n W\|_{L_\infty(\mathbb{R})} < \infty.$$

Pollard later [162] reformulated this as

$$\sup \left\{ \int_{-\infty}^{\infty} \frac{\log |P(x)|}{1+x^2} dx : P \text{ a polynomial with } \|PW\|_{L_\infty(\mathbb{R})} \leq 1 \right\} = \infty.$$

Of course, these are not very transparent criteria. When the weight is in some sense regular, simplifications are possible.

Theorem 1.4 *Let W be even, and $\log(1/W(e^x))$ be convex. There is a positive answer to Bernstein's problem iff*

$$\int_0^{\infty} \frac{\log(1/W(x))}{1+x^2} dx = \infty. \tag{1.2}$$

This result was proved by Lennart Carleson in 1951 [10], although Misha Sodin pointed out to the author that it appeared in a 1937 paper of Izumi and Kawata [51]. On perusing the latter paper, I agree that it is clearly implicit in the results there, though not explicitly stated. I have also seen the result attributed to M. Dzrbasjan. The reader should conclude that the "history" presented in this survey is by no means authoritative, but merely what could be deduced (often from secondary sources) in the available time.

Corollary 1.5 *Let $\alpha > 0$ and*

$$W_\alpha(x) = \exp(-|x|^\alpha). \tag{1.3}$$

There is a positive answer to Bernstein's problem iff $\alpha \geq 1$.

As regards necessary conditions, Hall showed that (1.2) is necessary for density. When density fails, only a limited class of entire functions can be approximated [70]. A comprehensive treatment of this topic is given in Koosis' book [67]. A concise elegant exposition appears in [86, p. 28 ff.]. A more abstract solution to Bernstein's problem was given by Louis de Branges (of Bieberbach fame) in 1959 [29]. The ideas in that paper have had several recent ramifications [174], [175].

What about extensions to L_p? At least when W is continuous, the answer is the same:

Theorem 1.6 *Let $W : \mathbb{R} \to (0, 1]$ be continuous and satisfy (1.1). Let $p \geq 1$. Then any of the conditions of Achieser, Mergelyan, or Pollard is necessary and sufficient so that for every measurable $f : \mathbb{R} \to \mathbb{R}$ with $\|fW\|_{L_p(\mathbb{R})} < \infty$, there exist polynomials $\{P_n\}$ with*

$$\lim_{n\to\infty} \|(f - P_n)W\|_{L_p(\mathbb{R})} = 0.$$

There is even a generalization of this theorem to weighted Müntz polynomials [188]. As regards the proof of the L_p case, let us quote Koosis [67, p. 211]: "In general, in the *kind* of approximation problem considered here, (that of the *density* of a certain simple class of functions in the whole space), *it makes very little difference which L_p norm is chosen.* If the proofs vary in difficulty, they are hardest for the L_1 norm or the uniform norm." Indeed, in his 1955 paper [162], Pollard added about two pages to deal with the L_p case.

In the special case of L_2, there are connections to the *classical moment problem.* Let $\{s_k\}$ be a sequence of real numbers. If there is a positive measure σ on the real line such that

$$s_k = \int_{-\infty}^{\infty} t^k d\sigma(t), \qquad k = 0, 1, 2, \ldots\ , \tag{1.4}$$

then we say that the Hamburger Moment Problem for $\{s_k\}$ has a solution. If there is only one solution σ, of the equations (1.4), we say the moment problem is **determinate**. Existence of σ is equivalent to positivity of certain determinants. The determinacy of the moment problem is often a deeper and more difficult issue. If we define

$$\Sigma(z) = \sup\left\{|P(z)|^2 : P \text{ a polynomial with } \int_{-\infty}^{\infty} |P|^2 d\sigma \leq 1\right\},$$

then a necessary and sufficient condition for determinacy is [67, p. 141]

$$\int_{-\infty}^{\infty} \frac{\log^+ \Sigma(t)}{1 + t^2} dt = \infty.$$

Here $\log^+ x := \max\{0, \log x\}$. In particular, for the measure

$$d\sigma(t) = \exp(-|t|^\alpha)dt,$$

one can use this to show that the corresponding moment problem is determinate iff $\alpha \geq 1$. Recall that Bernstein's approximation problem also had a positive solution iff $\alpha \geq 1$.

There are still closer connections to Bernstein's problem [38, Thm. 4.3, p. 77]. It is noteworthy that this was obtained by M. Riesz in 1923 even before Bernstein's problem was formulated.

Theorem 1.7 *Suppose that σ is a positive measure on the real line, with finite moments $\{s_k\}$, as in (1.4). Suppose, moreover, that σ has no point masses, so that $x \mapsto \int_{-\infty}^{x} d\sigma$ is a continuous function of x. Then the moment problem associated with $\{s_k\}$ is determinate iff for every σ-measurable function f with $\int f^2 d\sigma$ finite, and for every $\varepsilon > 0$, there exists a polynomial P such that*

$$\int_{-\infty}^{\infty} (f - P)^2 d\sigma < \varepsilon.$$

Without assuming the continuity of $\int d\sigma$, determinacy still implies density of the polynomials [38, p. 74 ff.]. There is also an important result on one-sided approximation [38, Theorem 3.3, p. 73]:

Theorem 1.8 *Suppose that σ is a positive measure on the real line, with finite moments $\{s_k\}$, as in (1.4). Suppose moreover, that the moment problem associated with $\{s_k\}$ is determinate. Let $\varepsilon > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a function that is Riemann–Stieltjes integrable against $d\sigma$ over every finite interval, and improperly Riemann integrable over the whole real line, and of polynomial growth at ∞. Then there exist polynomials R and S such that*

$$R \leq f \leq S \quad \text{in } \mathbb{R}$$

and

$$\int_{-\infty}^{\infty} (S - R)d\sigma < \varepsilon.$$

Some recent work related to Bernstein's approximation problem and its L_2 analogue appear in [6], [7], [8], [79], [155], [157], [163], [164], [165], [173], [174], [175], [188].

2 Some Ideas for the Resolution of Bernstein's Problem

In this section, we present some of the ideas used by Akhiezer, Mergelyan, Pollard, to resolve Bernstein's Approximation problem. It's relatively easy to derive the necessary parts of the conditions. We follow [67, p. 147 ff.] and [86, p. 28 ff.]. Recall Mergelyan's regularization of W :

$$\Omega(z) = \sup\left\{|P(z)| : P \text{ a polynomial and } \sup_{t\in\mathbb{R}} \frac{|P(t)W(t)|}{\sqrt{1+t^2}} \leq 1\right\}.$$

In the sequel, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t and polynomials P of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively show that C depends on α, or does not depend on α. We use the notation $\sim$ in the following sense: given sequences of real numbers $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if for some positive constants C_1, C_2 independent of n, we have

$$C_1 \leq c_n/d_n \leq C_2.$$

Lemma 2.1 *In order that Bernstein's approximation problem has a positive solution, it is necessary that for all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\Omega(z) = \infty. \tag{2.1}$$

Proof. Suppose that Bernstein's problem has a positive solution. Fix $z \in \mathbb{C}\backslash\mathbb{R}$. Since the function $f(t) = (t-z)^{-1}$ is continuous, and fW has limit 0 at ∞, we can find polynomials $\{R_n\}$ such that

$$\delta_n = \|(f - R_n)W\|_{L_\infty(\mathbb{R})} \to 0, \tag{2.2}$$

as $n \to \infty$. Let

$$P_n(t) = \frac{1 - (t-z)R_n(t)}{\delta_n} = \frac{t-z}{\delta_n}(f - R_n)(t).$$

Then for all $t \in \mathbb{R}$,

$$\left|\frac{P_n(t)}{(t-z)}W(t)\right| = \frac{1}{\delta_n}|f - R_n|(t)W(t) \leq 1.$$

Next, for some constant C independent of t, we have

$$\left|\frac{t-z}{t-i}\right| \leq C, \qquad t \in \mathbb{R}.$$

Then

$$\sup_{t\in\mathbb{R}} \frac{|P_n(t)W(t)|}{\sqrt{1+t^2}} = \sup_{t\in\mathbb{R}} \left|\frac{t-z}{t-i}\right| \left|\frac{P_n(t)}{(t-z)}W(t)\right| \leq C.$$

It follows that P_n/C is one of the polynomials considered in forming the sup in $\Omega(z)$, so

$$\Omega(z) \geq \left|\frac{P_n(z)}{C}\right| = \frac{1}{\delta_n C} \to \infty,$$

$n \to \infty$. $\square$

Let's think how we can reverse this to show the condition is also sufficient. If $\Omega(z) = \infty$, then reversing the above argument, we see that there are polynomials $\{R_n\}$ satisfying (2.2), with $f(t) = (t-z)^{-1}$. If $\Omega(z) = \infty$ for all z, we can approximate linear combinations

$$f(t) = \sum_{j=1}^{m} \frac{c_j}{t - z_j}$$

by polynomials in the norm $\|\cdot W\|_{L_\infty(\mathbb{R})}$. For small ε, we can then approximate

$$\frac{1}{2\varepsilon}\left[\frac{1}{(t-z)-\varepsilon} - \frac{1}{(t-z)+\varepsilon}\right] = \frac{1}{(t-z)^2 - \varepsilon^2}$$

and hence also $1/(t-z)^2$. Iterating this, we can approximate $1/(t-z)^m$ for any non-real z, and $m \geq 1$. We can then approximate polynomials in $1/(t-z)$. The latter can in turn uniformly approximate on the real line, any continuous function $f(t)$ that has limit 0 at ∞. (Use Weierstrass' Theorem and make a transformation $x = 1/(t-c)$.) Since $W \leq 1$, we then obtain a positive solution to Bernstein's problem, for continuous functions that have limit 0 at ∞. This class of continuous functions is big enough to approximate arbitrary ones for the Bernstein problem. For full details, see [67, p. 148 ff.].

Next, we show:

Lemma 2.2 *In order that Bernstein's approximation problem has a positive solution, it is necessary that*

$$\int_{-\infty}^{\infty} \frac{\log(1/W)(t)}{1+t^2} dt = \infty. \tag{2.3}$$

Proof. The basic tool is the inequality

$$\log|P(i)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log|P(t)|}{1+t^2} dt, \tag{2.4}$$

valid for all polynomials P. To prove this, suppose first that P has no zeros in the closed upper half-plane. We can then choose an analytic branch of $\log P$ there, and the residue theorem gives

$$\log P(i) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log P(t)}{1+t^2} dt.$$

Taking real parts, we obtain (2.4), with equality instead of inequality. When P has no zeros in the upper half-plane, but possibly has zeros on the real axis, a continuity argument shows that (2.4) persists, but with equality. Finally when $P(z)$ contains factors $(z-a)$ with a in the upper half-plane, we use

$$\log|a-i| \leq \log|\overline{a}-i| = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log|\overline{a}-t|}{1+t^2} dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log|a-t|}{1+t^2} dt$$

and divide such factors out from P. So we have (2.4). Now if

$$\sup_{t \in \mathbb{R}} \frac{|P(t)W(t)|}{\sqrt{1+t^2}} \leq 1, \tag{2.5}$$

we obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log|P(t)|}{1+t^2} dt \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(1/W(t))}{1+t^2} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log\left(1+t^2\right)}{1+t^2} dt$$

and hence

$$\log|P(i)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(1/W(t))}{1+t^2} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log\left(1+t^2\right)}{1+t^2} dt.$$

Taking sup's over all P satisfying (2.5) gives

$$\log\Omega(i) \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(1/W(t))}{1+t^2} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log\left(1+t^2\right)}{1+t^2} dt.$$

As we assumed Bernstein's approximation problem has a positive solution, Lemma 2.1 shows that $\Omega(i) = \infty$, and hence (2.3) follows. $\square$

With a little more work, this proof also gives

$$\int_{-\infty}^{\infty} \frac{\log \Omega(t)}{1+t^2} dt = \infty,$$

see [67, p. 153].

3 Weighted Jackson and Bernstein Theorems

In the 1950's the search began for quantitative estimates, rather than theorems establishing the possibility of approximation. Bernstein and Jackson had provided quantitative forms of Weierstrass' Theorem in 1911 and 1912, two wonderful years for approximation theory. Recall first the classical (unweighted) case. Jackson and Bernstein independently proved that

$$E_n\left[f\right]_\infty := \inf_{\deg(P)\leq n} \|f-P\|_{L_\infty[-1,1]} \leq \frac{C}{n}\|f'\|_{L_\infty[-1,1]}, \tag{3.1}$$

with C independent of f and n, and the inf being over (algebraic) polynomials of degree at most n. The rate is best possible amongst absolutely continuous functions f on $[-1,1]$ whose derivative is bounded. More generally, if f has a bounded kth derivative, then the rate is $O(n^{-k})$. In addition, Jackson obtained general results involving moduli of continuity: for example, if f is continuous, and its modulus of continuity is

$$\omega\left(f;\delta\right) = \sup\left\{|f(x)-f(y)| : x,y\in[-1,1] \text{ and } |x-y|\leq\delta\right\},$$

then

$$E_n\left[f\right]_\infty \leq C\omega\left(f;\frac{1}{n}\right),$$

where C is independent of f and n.

Bernstein also obtained remarkable converse theorems, which show that the rate (or degree) of approximation is determined by the smoothness of f. These are most elegantly stated for trigonometric polynomial approximation. Let

$$\mathcal{E}_n\left[g\right] := \inf_{\deg(R)\leq n} \|g-R\|_{L_\infty[0,2\pi]}$$

denote the distance from a periodic function g to the set of all trigonometric polynomials R of degree $\leq n$. Let $0<\alpha<1$. Bernstein showed that

$$\mathcal{E}_n\left[g\right] = O(n^{-\alpha}),\quad n\to\infty \quad\Longleftrightarrow\quad \omega\left(g;t\right) = O(t^\alpha),\qquad t\to 0+,$$

where $\omega\left(g;\cdot\right)$ is the modulus of continuity of g on $[0,2\pi]$, defined much as above. That is, the error of approximation of a 2π-periodic function g on $[0,2\pi]$ by trigonometric polynomials of degree at most n decays with rate $n^{-\alpha}$ iff g satisfies a Lipschitz condition of order α. Moreover, if $k\geq 1$,

$$\mathcal{E}_n\left[g\right] = O(n^{-k-\alpha}),\quad n\to\infty \quad\Longleftrightarrow\quad \omega(g^{(k)};t) = O(t^\alpha),\qquad t\to 0+.$$

Here in the converse implication, the existence and continuity of the kth derivative $g^{(k)}$ is assured. Bernstein never resolved the exact smoothness required for a rate of decay of n^{-1}, or more generally

$O(n^{-k})$. The case $k = 1$ was solved much later in 1945 by A. Zygmund, the father of the Chicago school of harmonic analysis, and author of the classic "Trigonometric Series" [193]. Zygmund used a second order modulus of continuity.

For approximation by algebraic polynomials, converse theorems are more complicated, as better approximation is possible near the endpoints of the interval of approximation. Only in the 1980's were complete characterizations obtained, with the aid of the Ditzian–Totik modulus of continuity [36]. An earlier alternative approach is that of Brudnyi–Dzadyk–Timan [31]. We shall discuss only the Ditzian–Totik approach, since that has been adopted in weighted polynomial approximation. Define the symmetric differences

$$\begin{aligned}
\Delta_h f(x) &= f(x+\frac{h}{2}) - f(x-\frac{h}{2}); \\
\Delta_h^2 f(x) &= \Delta_h \left(\Delta_h f(x)\right); \\
&\vdots \\
\Delta_h^k f(x) &= \Delta_h \left(\Delta_h^{k-1} f(x)\right)
\end{aligned}$$

so that

$$\Delta_h^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i f(x + k\frac{h}{2} - ih). \tag{3.2}$$

If any of the arguments of f lies outside the interval of approximation ($[-1,1]$ in this setting), we adopt the convention that the difference is 0. The rth order Ditzian–Totik modulus of continuity in $L_p[-1,1]$ is

$$\omega_\varphi^r(f;h)_p = \sup_{0<h\leq t} \|\Delta^r_{h\sqrt{1-x^2}} f(x)\|_{L_p[-1,1]}.$$

Note the factor

$$\varphi(x) = \sqrt{1-x^2}$$

multiplying the increment h. This forces a smaller increment near the endpoints ± 1 of $[-1,1]$, reflecting the possibility of better approximation rates there.

For $1 \leq p \leq \infty$, Ditzian and Totik [36, Thm. 7.2.1, p. 79] proved the estimate

$$E_n[f]_p := \inf_{\deg(P)\leq n} \|f - P\|_{L_p[-1,1]} \leq C\omega_\varphi^r(f;\frac{1}{n})_p,$$

with C independent of f and n. This implies the Jackson (or Jackson–Favard) estimate [31, p. 260]

$$E_n[f]_p \leq Cn^{-r} \|\varphi^r f^{(r)}\|_{L_p[-1,1]},$$

$n \geq r$, provided $f^{(r-1)}$ is absolutely continuous, and the norm on the right-hand side is finite. Moreover, they showed that if $0 < \alpha < r$, then [31, p. 265]

$$E_n[f]_p = O(n^{-\alpha}), \qquad n \to \infty, \tag{3.3}$$

iff

$$\omega_\varphi^r(f;h)_p = O(h^\alpha), \qquad h \to 0+.$$

For example, if (3.3) holds with $\alpha = 3\frac{1}{2}$, this implies that f has 3 continuous derivatives inside $(-1,1)$ and f''' satisfies a Lipschitz condition of order $1/2$ in each compact subinterval of $(-1,1)$.

This equivalence is easily deduced from the Jackson inequality above and the general converse inequality [36, Theorem 7.2.4, p. 83]

$$\omega_\varphi^r(f;t)_p \le Mt^r \sum_{0<n<\frac{1}{t}} (n+1)^{r-1} E_n[f]_p, \qquad t \in (0,1).$$

The constant M depends on r, but is independent of f and t. Of course, this subject has a long and rich history, and all we are attempting here is to set the background for developments in weighted approximation. Please forgive the many themes omitted!

For weights on the whole real line, the first attempts at general Jackson theorems seem due to Dzrbasjan [37]. He obtained the correct weighted rates, but only when restricting the approximated function to a finite interval. In the 1960's and 1970's, Freud and Nevai made major strides in this topic [150]. That 1986 survey of Paul Nevai is still relevant, and a very readable introduction to the subject.

Let us review some of the fundamental features discovered by Freud, in the case of the weight $W_\alpha(x) = \exp(-|x|^\alpha), \alpha > 1$. A little calculus shows that the weighted monomial $x^n W_\alpha(x)$ attains its maximum modulus on the real line at **Freud's number**

$$q_n = (n/\alpha)^{1/\alpha}.$$

Thereafter it decays quickly to zero. With this in mind, Freud and Nevai proved that there are constants C_1 and C_2 such that for all polynomials P_n of degree at most n,

$$\|P_n W_\alpha\|_{L_p(\mathbb{R})} \le C_2 \|P_n W_\alpha\|_{L_p\left[-C_1 n^{1/\alpha}, C_1 n^{1/\alpha}\right]}. \tag{3.4}$$

The constants C_1 and C_2 can be taken independent of n, P_n and even the L_p parameter $p \in [1,\infty]$. Outside the interval $\left[-C_1 n^{1/\alpha}, C_1 n^{1/\alpha}\right]$, $P_n W_\alpha$ decays quickly to zero. This meant that one cannot hope to approximate fW by $P_n W$ outside $\left[-C_1 n^{1/\alpha}, C_1 n^{1/\alpha}\right]$. So either a "tail term" $\|fW_\alpha\|_{L_p\left[|x| \ge C_1 n^{1/\alpha}\right]}$ must appear in the error estimate, or be handled some other way. Inequalities of the form (3.4) are called **restricted range inequalities**, or **infinite-finite range inequalities**.

The sharp form of these was found later by Mhaskar and Saff, using potential theory [136], [138], [139], [140], [167]. Let $W = \exp(-Q)$, where Q is even, and $xQ'(x)$ is positive and increasing in $(0,\infty)$, with limits 0 and ∞ at 0 and ∞, respectively. For $n \ge 1$, let $a_n = a_n(Q)$ denote the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt. \tag{3.5}$$

We call a_n the nth **Mhaskar–Rakhmanov–Saff number**. For example, if $\alpha > 0$, and $W_\alpha(x) = \exp(-|x|^\alpha)$,

$$a_n = \left\{ 2^{\alpha-2} \frac{\Gamma(\alpha/2)^2}{\Gamma(\alpha)} \right\}^{1/\alpha} n^{1/\alpha}.$$

Mhaskar and Saff established the **Mhaskar–Saff identity**: for polynomials P_n of degree at most n,

$$\|P_n W\|_{L_\infty(\mathbb{R})} = \|P_n W\|_{L_\infty[-a_n, a_n]}. \tag{3.6}$$

Moreover, if P_n is not the zero polynomial,

$$\|P_n W\|_{L_\infty(\mathbb{R} \setminus [-a_n, a_n])} < \|P_n W\|_{L_\infty[-a_n, a_n]}$$

and a_n is asymptotically the "smallest" such number.

There are L_p analogues, valid for all $p > 0$. For example, if $\varepsilon > 0$, there exists $C > 0$ such that for $n \geq 1$ and polynomials P_n of degree $\leq n$,

$$\|P_n W\|_{L_p(\mathbb{R}\setminus[-a_n(1+\varepsilon),a_n(1+\varepsilon)])} \leq e^{-Cn}\|P_n W\|_{L_p[-a_n,a_n]}. \tag{3.7}$$

We shall discuss these more in Section 4.6.

The next task is to determine what happens on $[-Ca_n, Ca_n]$. Now if we had to approximate in the unweighted setting on this interval, a scale change in the Jackson–Bernstein estimate (3.1) gives

$$\inf_{\deg(P)\leq n} \|f - P\|_{L_\infty[-Ca_n,Ca_n]} \leq CC_1 \frac{a_n}{n} \|f'\|_{L_\infty[-Ca_n,Ca_n]}.$$

Remarkably, the same is true when we insert the weight W_α in both norms:

$$\inf_{\deg(P)\leq n} \|(f - P)W_\alpha\|_{L_\infty[-Ca_n,Ca_n]} \leq C_3 \frac{a_n}{n} \|f' W_\alpha\|_{L_\infty[-Ca_n,Ca_n]}. \tag{3.8}$$

Very roughly, this works for the following reason: it seems that if C is small enough, we can approximate $1/W_\alpha$ on $[-Ca_n, Ca_n]$ by a polynomial $R_{n/2}$ of degree $\leq n/2$, and then use the remaining part $n/2$ degree polynomial in P to approximate fW_α itself on $[-Ca_n, Ca_n]$. In real terms, this approach works only for a small class of weights. Nevertheless, it at least indicated the form that general results should take. To obtain an estimate over the whole real line, Freud then proved a "tail inequality", such as

$$\|fW_\alpha\|_{L_p[|x|\geq Ca_n]} \leq C_4 \frac{a_n}{n} \|f' W_\alpha\|_{L_p(\mathbb{R})}, \tag{3.9}$$

with C_4 independent of f and n. Combining (3.8), (3.9), and that suitable weighted polynomials are tiny outside $[-C_1 a_n, C_1 a_n]$ yielded the following:

Theorem 3.1 *Let $1 \leq p \leq \infty$, $\alpha > 1$ and $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, with $\|f' W_\alpha\|_{L_p(\mathbb{R})} < \infty$. Then*

$$E_n [f; W_\alpha]_p := \inf_{\deg(P)\leq n} \|(f - P)W_\alpha\|_{L_p(\mathbb{R})} \leq C_5 \frac{a_n}{n} \|f' W_\alpha\|_{L_p(\mathbb{R})}, \tag{3.10}$$

with C_5 independent of f and n.

While this might illustrate some of the ideas, we emphasize that the technical details underlying proper proofs of this Jackson (or Jackson–Favard) inequality are formidable. Some of these ideas will be illustrated in the next section. Freud and Nevai developed an original theory of orthogonal polynomials for the weights W_α^2 partly for use in this approximation theory.

We note that Freud proved (3.10) for W_α for $\alpha \geq 2$ [41], [42]. The technical estimates required to extend this to the case $1 < \alpha < 2$ were provided by Eli Levin and the author [80]. What about $\alpha \leq 1$? Well, recall that the polynomials are only dense if $\alpha \geq 1$, so there is no point in considering $\alpha < 1$. But $\alpha = 1$ is still worth consideration, and we shall discuss that below in Section 5.

One consequence of (3.10) is an estimate of the rate of weighted polynomial approximation of f in terms of that of f'. Indeed, if P_n is any polynomial of degree $\leq n-1$, then

$$E_n [f; W_\alpha]_p = E_n [f - P_n; W_\alpha]_p \leq C_5 \frac{a_n}{n} \| (f - P_n)' W_\alpha\|_{L_p(\mathbb{R})},$$

and since P_n' may be any polynomial of degree $\leq n-1$, we obtain the Favard or Jackson–Favard inequality

$$E_n\left[f;W_\alpha\right]_p \leq C_5\frac{a_n}{n}E_{n-1}\left[f';W_\alpha\right]_p. \tag{3.11}$$

This can be iterated:

Corollary 3.2 *Let $1\leq p\leq\infty$, $r\geq 1$, $\alpha>1$ and $f:\mathbb{R}\to\mathbb{R}$ have $r-1$ continuous derivatives. Assume, moreover, that $f^{(r)}$ is absolutely continuous, with $\|f^{(r)}W_\alpha\|_{L_p(\mathbb{R})}<\infty$. Then for some C independent of f and n,*

$$E_n\left[f;W_\alpha\right]_p \leq C_5\left(\frac{a_n}{n}\right)^r\|f^{(r)}W_\alpha\|_{L_p(\mathbb{R})}. \tag{3.12}$$

Freud also obtained estimates involving moduli of continuity. Here one cannot avoid the tail term, and has to build it directly into the modulus. Partly for this reason, there are many forms of the modulus, and more than one way to decide which interval is the principal interval, and over what interval we take the tail. However it is done, it is awkward. We shall follow essentially the modulus used by Ditzian and Totik [36], Ditzian and the author [33], and Mhaskar [136]. All these owe a great deal to earlier work of Freud.

The first order modulus for the weight W_α has the form

$$\omega_{1,p}(f,W_\alpha,t)=\sup_{0<h\leq t}\|W_\alpha\left(\Delta_h f\right)\|_{L_p[-h^{\frac{1}{1-\alpha}},h^{\frac{1}{1-\alpha}}]}+\inf_{c\in\mathbb{R}}\|(f-c)W_\alpha\|_{L_p\left(\mathbb{R}\setminus[-t^{\frac{1}{1-\alpha}},t^{\frac{1}{1-\alpha}}]\right)}.$$

Why the inf over the constant c in the tail term? It ensures that if f is constant, then the modulus vanishes identically, as one expects from a first order modulus. Why the strange interval $[-h^{\frac{1}{1-\alpha}},h^{\frac{1}{1-\alpha}}]$? It ensures that when we substitute

$$h=\frac{n^{1/\alpha}}{n}=n^{-1+1/\alpha},$$

then

$$[-h^{\frac{1}{1-\alpha}},h^{\frac{1}{1-\alpha}}]=\left[-n^{1/\alpha},n^{1/\alpha}\right]=\left[-C_1a_n,C_1a_n\right],$$

for an appropriate constant C_1 (independent of n). More generally if $r\geq 1$, the rth order modulus is

$$\begin{aligned}\omega_{r,p}(f,W_\alpha,t) &= \sup_{0<h\leq t}\|W_\alpha\left(\Delta_h^r f\right)\|_{L_p[-h^{\frac{1}{1-\alpha}},h^{\frac{1}{1-\alpha}}]}\\ &+ \inf_{\deg(P)\leq r-1}\|(f-P)W_\alpha\|_{L_p\left(\mathbb{R}\setminus[-t^{\frac{1}{1-\alpha}},t^{\frac{1}{1-\alpha}}]\right)}.\end{aligned} \tag{3.13}$$

Again the inf in the tail term ensures that if f is a polynomial of degree $\leq r-1$, then the modulus of continuity vanishes identically, as is expected from an rth order modulus. The Jackson theorem takes the form

$$E_n\left[f;W_\alpha\right]_p\leq C\omega_{r,p}(f,W_\alpha,\frac{a_n}{n}). \tag{3.14}$$

This is valid for $1\leq p\leq\infty$, and the constant C is independent of f and n (but depends on p and W_α).

One can consider more general weights than W_α. Almost invariably the weight considered has the form $W = \exp(-Q)$, and the rate of growth of Q has a major impact on the form of the modulus. Let us suppose, for example, that Q is of polynomial growth at ∞, the so-called **Freud case**. The most general class of such weights for which a Jackson theorem is known is the following. It includes $W_\alpha, \alpha > 1$, but excludes W_1.

Definition 3.3 (Freud Weights) *Let $W = \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, Q' exists and is positive in $(0, \infty)$. Moreover, assume that $xQ'(x)$ is strictly increasing, with right limit 0 at 0, and for some $\lambda, A, B > 1, C > 0$,*

$$A \leq \frac{Q'(\lambda x)}{Q'(x)} \leq B, \qquad x \geq C. \tag{3.15}$$

Then we write $W \in \mathcal{F}$.

For such W, we take a_n to be the positive root of the equation (3.5) (the existence and uniqueness of a_n is guaranteed by the strict monotonicity of $xQ'(x)$). To replace the function $t^{\frac{1}{1-\alpha}}$, we can use the decreasing function of t,

$$\sigma(t) := \inf \left\{ a_n : \frac{a_n}{n} \leq t \right\}, \qquad t > 0. \tag{3.16}$$

The modulus of continuity becomes

$$\begin{aligned} \omega_{r,p}(f, W, t) &= \sup_{0<h\leq t} \| W \left(\Delta_h^r f \right) \|_{L_p[-\sigma(h), \sigma(h)]} + \\ &+ \inf_{\deg(P)\leq r-1} \|(f - P)W\|_{L_p(\mathbb{R}\setminus[-\sigma(t), \sigma(t)])}. \end{aligned} \tag{3.17}$$

The reader new to this subject will be encouraged to hear that this strange looking creature has all the main properties of more familiar moduli of continuity [33], [35], [36], [48], [78], [136]:

Theorem 3.4 (Properties of $\omega_{r,p}$) *Let $W \in \mathcal{F}$, $r \geq 1, 0 < p \leq \infty$. Let $f : \mathbb{R} \to \mathbb{R}$, and if $p < \infty$, assume that $fW \in L_p(\mathbb{R})$. If $p = \infty$, assume in addition that f is continuous and that fW has limit 0 at $\pm\infty$.*

(a) *$\omega_{r,p}(f, W, t)$ is an increasing function of $t \geq 0$.*

(b)

$$\lim_{t\to 0+} \omega_{r,p}(f, W, t) = 0.$$

(c) *Assume that $p \geq 1$, or assume that W admits a Markov–Bernstein inequality*

$$\|P'W\|_{L_p(\mathbb{R})} \leq C \frac{n}{a_n} \|PW\|_{L_p(\mathbb{R})}, \tag{3.18}$$

valid for $n \geq 1$, and polynomials P of degree $\leq n$, where $C \neq C(n, P)$. Then there exists $C_1 \neq C_1(t, f)$ such that

$$\omega_{r,p}(f, W, 2t) \leq C_1 \omega_{r,p}(f, W, t), \qquad t \geq 0.$$

(d) *Assume that $p \geq 1$. Then for $t \geq 0$,*

$$\omega_{r,p}(f, W, t) \leq C t^r \|f^{(r)} W\|_{L_p(\mathbb{R})},$$

provided $f^{(r-1)}$ is absolutely continuous and the norm on the right-hand side is finite.

(e) *Let $r > 1$. There exists $C \neq C(f, t)$, such that*

$$\omega_{r,p}(f, W, t) \leq C \omega_{r-1,p}(f, W, t), \qquad t \geq 0. \tag{3.19}$$

(f) *Assume the Markov–Bernstein inequality (3.18). Then if $q = \min\{1, p\}$, there is the Marchaud inequality*

$$\omega_{r,p}(f, W, t) \leq C_1 t^r \left\{ \int_t^{C_2} \frac{\omega_{r+1}^q(f, W, u)}{u^{rq+1}} du + \|fW\|_{L_p(\mathbb{R})}^q \right\}^{1/q},$$

where $C_j \neq C_j(f, t)$, $j = 1, 2$.

(g) *Assume that $\beta > 1$ and $0 < p < q < \infty$. Then there is the Ulyanov type inequality*

$$\omega_{r,q}(f, W_\beta, t) \leq C \left\{ \int_0^t \left[u^{\frac{1}{q} - \frac{1}{p}} \omega_{r,p}(f, W_\beta, u) \right]^q \frac{du}{u} \right\}^{1/q}, \qquad t > 0,$$

where $C \neq C(f, t)$.

(h) *Assume that $p \geq 1$ and for some $0 < \alpha < r$,*

$$\omega_{r,p}(f, W, t) = O(t^\alpha), \qquad t \to 0+. \tag{3.20}$$

Let $k = \lfloor \alpha \rfloor$, the integer part of α. Then $f^{(k)}$ exists a.e. in the real line, and

$$\omega_{r-k,p}(f^{(k)}, W, t) = O(t^{\alpha-k}), \qquad t \to 0+. \tag{3.21}$$

If $p = \infty$, $f^{(k)}$ is continuous on the real line.

Proof. (a) This is immediate as $\sigma(t)$ is a decreasing function of t.

(b) This follows as for classical moduli of continuity. One first establishes it for suitably restricted continuous functions, and then approximates an arbitrary function by a continuous one.

(c) This is part of Theorem 1.4 in [33, p. 104].

(d) This is part of Corollary 1.8 in [33, p. 105].

(e) This follows easily from the definition (3.13) and the recursive definition of the symmetric differences.

(f) This is Corollary 1.7 in [33, p. 105].

(g) This is part of Theorem 9.1 in [35, p. 133].

(h) See [36, pp. 62–64] for the analogous proofs on a finite interval. □

Note all the strictures for $p \leq 1$. Fundamentally these arise because we cannot bound norms of functions in terms of their derivatives when $p < 1$. At least the Jackson theorem is the obvious analogue of the result for finite intervals [33, Theorem 1.2, p. 102]:

Theorem 3.5 *Let $W \in \mathcal{F}$, $r \geq 1$, and $0 < p \leq \infty$. Let $fW \in L_p(\mathbb{R})$. If $p = \infty$, we also require f to be continuous and fW to have limit 0 at $\pm\infty$. Then for $n \geq r - 1$,*

$$E_n [f; W]_p \leq C_1 \omega_{r,p}(f, W, C_2 \frac{a_n}{n}), \tag{3.22}$$

where C_1 and C_2 are independent of f and n.

In the case where $p \geq 1$, or W admits the Markov–Bernstein inequality (3.18), one can omit the constant C_2 inside the modulus. This follows directly from Theorem 3.4(c). We shall say much more about Markov–Bernstein inequalities in Section 7.

Corollary 3.6 *Let $1 \leq p \leq \infty$, $r \geq 1$, $W \in \mathcal{F}$ and $f : \mathbb{R} \to \mathbb{R}$ have $r - 1$ continuous derivatives. Assume, moreover, that $f^{(r)}$ is absolutely continuous, with $\|f^{(r)}W\|_{L_p(\mathbb{R})} < \infty$. Then for some C independent of f and n,*

$$E_n [f; W]_p \leq C_5 \left(\frac{a_n}{n}\right)^r \|f^{(r)}W\|_{L_p(\mathbb{R})}. \tag{3.23}$$

The converse inequality, which can be interpreted as a Bernstein type converse theorem, has the form [33, p. 105]:

Theorem 3.7 *Let $0 < p \leq \infty$, $r \geq 1$, $W \in \mathcal{F}$. Assume that W admits the Markov–Bernstein inequality (3.18). Let $q = \min\{1, p\}$. For $t \leq a_1$, define the positive integer $n = n(t)$ by*

$$n := n(t) := \inf \left\{k : \frac{a_k}{k} \leq t\right\}. \tag{3.24}$$

Then for some $C \neq C(t, f)$,

$$\omega_{r,p}(f, W, t)^q \leq C \left(\frac{a_n}{n}\right)^{rq} \sum_{j=-1}^{[\log_2 n]} \left(\frac{2^j}{a_{2^j}}\right)^{rq} E_{2^j} [f; W]_p^q, \tag{3.25}$$

where we define $E_{2^{-1}} := E_0$ and $\lfloor \log_2 n \rfloor$ denotes the largest integer $\leq \log_2 n$.

From this one readily deduces:

Corollary 3.8 *Assume the hypotheses of the previous theorem, and let $0 < \alpha < r$. Then*

$$\omega_{r,p}(f, W, t) = O(t^\alpha), \quad t \to 0+ \quad \iff \quad E_n [f; W]_p = O\left(\left(\frac{a_n}{n}\right)^\alpha\right), \quad n \to \infty.$$

A related smoothness theorem is given by Damelin [14].

One of the important tools in establishing this is K-functionals and the concept of realization. This is a topic on its own. In the setting of weighted polynomial approximation, it has been explored by Freud and Mhaskar, and later Ditzian and Totik, Damelin and the author. See [11], [27], [33], [44], [45], [136], [137]. In our context, an appropriate K-functional is

$$K_{r,p}(f, W, t^r) := \inf_g \left\{ \|(f-g)W\|_{L_p(\mathbb{R})} + t^r \|g^{(r)}W\|_{L_p(\mathbb{R})} \right\}, \qquad t \geq 0,$$

where the inf is taken over all g whose $(r-1)$st derivative is locally absolutely continuous. It works only for $p \geq 1$, again because of the problems of estimating functions in L_p, $p < 1$, in terms of their derivatives. The K-functional is equivalent to the modulus of continuity in the following sense [33, Thm. 1.4, Cor. 1.9, pp. 104–105]:

Theorem 3.9 *Let $W \in \mathcal{F}$ and $p \geq 1$. Then for some $C_1, C_2 > 0$ independent of f, t,*

$$C_1 \omega_{r,p}(f, W, t) \leq K_{r,p}(f, W, t^r) \leq C_2 \omega_{r,p}(f, W, t).$$

The appearance of fairly general functions g in the inf defining $K_{r,p}$ helps to explain its usefulness. In fact, many of the properties of the modulus described above, go via the K-functional. When $p < 1$, the K-functional is identically zero, so instead we use the **realization functional**, introduced by Hristov and Ivanov [50] and analyzed by those authors and Ditzian [32]:

$$\overline{K}_{r,p}(f, W, t^r) := \inf_P \left\{ \|(f-P)W\|_{L_p(\mathbb{R})} + t^r \|P^{(r)}W\|_{L_p(\mathbb{R})} \right\}, \qquad t > 0.$$

Here the inf is taken over all polynomials P of degree $\leq n(t)$ and $n(t)$ is defined by (3.24). For $p \geq 1$, $K_{r,p}$ and $\overline{K}_{r,p}$ are equivalent [32]. For all p, one can prove [33, Thm. 1.4, p. 104]:

Theorem 3.10 *Assume $W \in \mathcal{F}$ and W admits the Markov–Bernstein inequality* (3.18). *Then for some $C_1, C_2 > 0$ independent of f, t,*

$$C_1 \omega_{r,p}(f, W, t) \leq \overline{K}_{r,p}(f, W, t^r) \leq C_2 \omega_{r,p}(f, W, t).$$

Observe that if we choose $t = a_n/n$, the above result actually gives more than the Jackson estimate Theorem 3.5, at least when a_k/k decreases strictly with k. We obtain

$$\inf_{\deg(P) \leq n} \left\{ \|(f-P)W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \|P^{(r)}W\|_{L_p(\mathbb{R})} \right\} \leq C_2 \omega_{r,p}(f, W, \frac{a_n}{n}),$$

so there is an automatic bound on the rth derivatives of best approximating polynomials.

The Freud weights above have the form $W = \exp(-Q)$, where Q is of polynomial growth at ∞, with $Q(x)$ growing at least as fast as $|x|^\alpha$ for some $\alpha > 1$. For the special weight $\exp(-|x|)$, the polynomials are still dense, but we have not established anything about the degree of approximation. We shall devote a separate section to this. The case where Q is of faster than polynomial growth is often called the **Erdős case**, and also has received some attention. The main difference is that the modulus of continuity becomes more complicated. Here is a suitable class of Erdős weights:

Definition 3.11 (Erdős Weights) *Let $W = \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, Q' exists and is positive in $(0, \infty)$. Assume that $xQ'(x)$ is strictly increasing, with right limit 0 at 0, and the function*

$$T(x) := \frac{xQ'(x)}{Q(x)} \tag{3.26}$$

is quasi-increasing in the sense that for some $C > 0$,

$$0 \leq x < y \;\Rightarrow\; T(x) \leq CT(y), \tag{3.27}$$

while

$$\lim_{x\to\infty} T(x) = \infty.$$

Assume, moreover, that for some C_1, C_2 *and* $C_3 > 0$,

$$\frac{yQ'(y)}{xQ'(x)} \leq C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2}, \qquad y \geq x \geq C_3.$$

Then we write $W \in \mathcal{E}$.

Examples of weights in this class include

$$W(x) = \exp(-\exp_k(|x|^\alpha) + \exp_k(0)),$$

where $\alpha > 0$ and for $k \geq 1$,

$$\exp_k = \exp(\exp(\cdots\exp()))$$

denotes the kth iterated exponential. We set $\exp_0(x) = x$. For this weight,

$$T(x) \sim \alpha x^\alpha \prod_{j=1}^{k-1} \exp_j\left(x^\alpha\right),$$

for large x [85, p. 9], while the Mhaskar–Rakhmanov–Saff number has the asymptotic [85, p. 29]

$$a_n = \left\{\log_{k-1}\left(\log n - \frac{1}{2}\sum_{j=1}^{k+1}\log_j n + O(1)\right)\right\}^{1/\alpha} = \left(\log_k n\right)^{1/\alpha}\left(1+o(1)\right),$$

$n \to \infty$; $\log_k$ denotes the kth iterated logarithm.

For general $W \in \mathcal{E}$, the modulus of continuity involves the function

$$\Phi_t(x) := \sqrt{1 - \frac{|x|}{\sigma(t)}} + T(\sigma(t))^{-1/2},$$

where $\sigma(t)$ is as in (3.16). One may think of this as an analogue of the function $\sqrt{1-x^2}$ which appears in the Ditzian–Totik modulus of continuity, and it appears for the same reason: in approximating by polynomials with Erdős weights, the rate of approximation improves towards the endpoints of the **Mhaskar–Rakhmanov–Saff interval**. The modulus is

$$\begin{aligned} \omega_{r,p}(f, W, t) &= \sup_{0<h\leq t} \left\| W(x)\left(\Delta^r_{h\Phi_t(x)} f(x)\right)\right\|_{L_p[-\sigma(2t),\sigma(2t)]} \\ &+ \inf_{\deg(P)\leq r-1} \|(f-P)W\|_{L_p(\mathbb{R}\setminus[-\sigma(4t),\sigma(4t)])}. \end{aligned} \tag{3.28}$$

Once one has this modulus, the Jackson estimate goes through [27, Thm. 1.2, p. 337]:

Theorem 3.12 *Let $W \in \mathcal{E}$, $r \geq 1$, and $0 < p \leq \infty$. Let $fW \in L_p(\mathbb{R})$. If $p = \infty$, we also require f to be continuous and fW to have limit 0 at $\pm\infty$. Then for $n \geq 1$,*

$$E_n [f; W]_p \leq C_1 \omega_{r,p}(f, W, C_2 \frac{a_n}{n}), \tag{3.29}$$

where C_1 and C_2 are independent of f and n.

There are also converse estimates [11], [16], [27]. The details are more difficult than for Freud weights, because of the more complicated modulus. Analogous results for exponential weights on $(-1, 1)$ are given in [15], [92], [93]. For exponential weights multiplied by a generalized Jacobi weight or other factor having singularities, see [49], [120], [121], [122], [178]. For Laguerre and other exponential weights on $(0, \infty)$, see [28], [62], [115]. Geometric rates of approximation in the weighted setting have been explored in [70], [127], [128], [135], [136].

4 Methods for Proving Weighted Jackson Theorems

In this section, we shall outline various methods to prove weighted Jackson Theorems, but will not provide complete expositions. In the unweighted case, on $[-1, 1]$, many of the elegant methods involve convolution operators. However, unfortunately these depend heavily on translation invariance, so fail for the weighted case. We begin with two of the oldest, used by Freud and Nevai.

4.1 Freud and Nevai's One-sided L_1 Method

The L_1 method is primarily a tool to obtain estimates on the degree of approximation for special functions such as characteristic functions. Once one has it, one can use duality and other tricks to go to L_∞, and then interpolation for $1 < p < \infty$, and this will be done in the next section. The method is based on the theory of orthogonal polynomials, and Gauss quadratures. Under the tutelage of Géza Freud and Paul Nevai, the two subjects of weighted approximation and orthogonal polynomials for weights on the real line, developed in tandem throughout the 1970's and 1980's. Mhaskar's monograph [136] provides an excellent treatment of the material in this and the next section.

Corresponding to the weight W, we define orthonormal polynomials

$$p_n(x) = p_n(W^2, x) = \gamma_n x^n + \cdots,$$

where $\gamma_n > 0$, satisfying

$$\int_{-\infty}^{\infty} p_n p_m W^2 = \delta_{mn.}$$

Note that the weight is W^2, not W. This convention simplifies some formulations later on. Let us denote the zeros of p_n by

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

The nth Christoffel function for W^2 is

$$\lambda_n(W^2, x) = \inf_{\deg(P) \leq n-1} \frac{\int_{-\infty}^{\infty} (PW)^2}{P^2(x)}. \tag{4.1}$$

It also satisfies

$$\lambda_n(W^2, x) = 1 \Big/ \sum_{j=0}^{n-1} p_j^2(x).$$

Try this as an exercise: expand an arbitrary polynomial of degree $\leq n-1$ in $\{p_j\}_{j=0}^{n-1}$, and then use Cauchy–Schwarz, and orthonormality.

Many readers unfamiliar with the detailed theory of orthogonal polynomials, will nevertheless have seen the Christoffel functions in the Gauss quadrature formula:

$$\int_{-\infty}^{\infty} PW^2 = \sum_{j=1}^{n} \lambda_n(W^2, x_{jn}) P(x_{jn}),$$

valid for all polynomials P of degree $\leq 2n-1$. For fixed ξ, one can also develop similar quadrature formulae, based on the zeros of

$$p_n(x)p_{n-1}(\xi) - p_{n-1}(x)p_n(\xi).$$

These zeros are real and simple and include ξ. There are important inequalities, the Posse–Markov–Stieltjes inequalities, that are used to analyze these quadratures. In the simple Gauss case, they assert that if f is a function with its first $2n-1$ derivatives positive in $(-\infty, x_{kn})$, then

$$\sum_{j=k+1}^{n} \lambda_n(W^2, x_{jn}) f(x_{jn}) \leq \int_{-\infty}^{x_{kn}} fW^2 \leq \sum_{j=k}^{n} \lambda_n(W^2, x_{jn}) f(x_{jn}).$$

See [38, p. 33], [136, p. 13]. This may be used to prove [136, p. 17]:

Lemma 4.1 *Let $n \geq 1$, and $\xi \in (x_{k+1,n}, x_{kn}]$. Then there exist upper and lower polynomials R_ξ and r_ξ such that*

$$r_\xi \leq \chi_{(-\infty,\xi]} \leq R_\xi \quad \text{in } \mathbb{R}$$

and

$$\int_{-\infty}^{\infty} (R_\xi - r_\xi) W^2 \leq \lambda_n(W^2, x_{kn}) + \lambda_n(W^2, x_{k+1,n}).$$

Thus once we have upper estimates on the Christoffel functions, we have bounds on the error of one-sided polynomial approximation. One could write a survey on methods to estimate Christoffel functions, there are so many. Paul Nevai paid homage to them and their applications in his still relevant survey article [150], as well as in his earlier memoir [148]. We shall present a very simple method of Freud in a very special case:

Lemma 4.2 *Assume that $W = \exp(-Q) \in \mathcal{F}$, and in addition that Q is convex. Then there exists $C_1, C_2 > 0$ such that for $n \geq 1$ and $|\xi| \leq C_1 a_n$,*

$$\lambda_n(W^2, \xi) \leq C_2 \frac{a_n}{n} W^2(\xi). \tag{4.2}$$

Proof. Let

$$e_m(x) = \sum_{j=0}^{m} \frac{x^j}{j!}$$

denote the mth partial sum of the exponential function. Fix ξ, n and let

$$R(x) = W^{-1}(\xi) e_{\lfloor n/2 \rfloor} \left((x-\xi) Q'(\xi) \right).$$

Here, $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$. Now

$$\left| e_{\lfloor n/2 \rfloor}(u) \right| \le C_2 \exp(u),$$

for $|u| \le \frac{n}{8}$. Moreover, given $\varepsilon > 0$, we have if $|x| \le 2a_n$, and $|\xi| \le \varepsilon a_n$,

$$\left| (x-\xi) Q'(\xi) \right| \le 3 a_n Q'(\varepsilon a_n) \le \frac{n}{8}, \tag{4.3}$$

if only ε is small enough. To see this, we use the definition of a_n and the monotonicity of $t \mapsto tQ'(t)$:

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt \ge \frac{2}{\pi} \int_{1/2}^1 \frac{dt}{\sqrt{1-t^2}} \frac{a_n}{2} Q'\left(\frac{a_n}{2}\right).$$

Thus

$$a_n Q'\left(\frac{a_n}{2}\right) \le C_1 n.$$

Using the lower bound in (3.15) of Definition 3.3, we obtain for $m \ge 1$,

$$a_n Q'\left(\lambda^{-m} \frac{a_n}{2}\right) \le A^{-m} a_n Q'\left(\frac{a_n}{2}\right) \le C_1 A^{-m} n.$$

There we had $\lambda, A > 1$, so choosing $\varepsilon = \lambda^{-m}/2$ with large enough m, gives (4.3). Next,

$$|R(x)| \le C_2 W^{-1}(\xi) \exp((x-\xi)Q'(\xi)) = C_2 \exp(Q(\xi) + (x-\xi)Q'(\xi)) \le C_2 \exp(Q(x)),$$

by convexity of Q. Thus

$$|R(x)| W(x) \le C_2 \text{ for all } |x| \le 2a_n.$$

Moreover,

$$(RW)(\xi) = 1.$$

Now, we use the extremal property (4.1) of Christoffel functions. We set $P = R\rho$, there, where ρ is a polynomial of degree $< \lfloor n/2 \rfloor$. We see that

$$\begin{aligned}
\lambda_n(W^2, \xi)/W^2(\xi) &= \inf_{\deg(P) < n} \left[\int_{-\infty}^{\infty} (PW)^2 \right] / (PW)^2(\xi) \\
&\le \inf_{\deg(\rho) < \lfloor n/2 \rfloor} \left[\int_{-\infty}^{\infty} (\rho R W)^2 \right] / (\rho R W)^2(\xi) \\
&\le C \inf_{\deg(\rho) < \lfloor n/2 \rfloor} \left[\int_{-2a_n}^{2a_n} (\rho R W)^2 \right] / (\rho R W)^2(\xi),
\end{aligned}$$

by the restricted range inequality (3.7). Now we use the upper bound on RW and $(RW)(\xi) = 1$ to continue this as

$$\leq CC_2^2 \inf_{\deg(\rho)<\lfloor n/2\rfloor} \left(\int_{-2a_n}^{2a_n} \rho^2 \right) / \rho^2(\xi) = CC_2^2 2a_n \inf_{\deg(S)<\lfloor n/2\rfloor} \left(\int_{-1}^{1} S^2 \right) / S^2 \left(\frac{\xi}{2a_n} \right).$$

The latter is the Christoffel function of order $\lfloor n/2\rfloor$ for the Legendre weight on $[-1, 1]$, evaluated at $\xi/(2a_n)$. Using classical estimates for these [38, p. 103], we continue this as

$$\lambda_n(W^2, \xi)/W^2(\xi) \leq C\frac{a_n}{n}. \qquad \square$$

Now we present a very special case of the L_1 approximation:

Theorem 4.3 *Assume the hypotheses of Lemma 4.2 on W. Assume that f' is continuous, and $f'W^2 \in L_1(\mathbb{R})$, while f is of polynomial growth at ∞. Then there exist upper and lower polynomials S_n and s_n such that*

$$s_n \leq f \leq S_n \text{ in } \mathbb{R} \tag{4.4}$$

and

$$\int_{-\infty}^{\infty} (S_n - s_n)W^2 \leq C\frac{a_n}{n} \left(\int_{-\infty}^{\infty} |f'| \, W^2 + \|f'W^2\|_{L_\infty(|x|\geq Ca_n)} \right).$$

Main idea of the proof. We assume that $f' \geq 0$, and that $f' = 0$ outside (x_{nn}, x_{1n}) to simplify the proof. Recall that x_{nn} and x_{1n} are the largest and smallest zeros of $p_n(W^2, x)$. We write the fundamental theorem of calculus

$$f(x) = f(0) + \int_0^x f'(\xi)d\xi$$

in the form

$$f(x) = f(0) + \int_0^{\infty} (1 - \chi_{(-\infty,\xi]}(x))f'(\xi)d\xi - \int_{-\infty}^{0} \chi_{(-\infty,\xi]}(x)f'(\xi)d\xi.$$

To check this, consider separately $x \geq 0$ and $x < 0$. We use the upper and lower polynomials R_ξ and r_ξ of Lemma 4.1 and define

$$S_n(x) = f(0) + \int_0^{\infty} (1 - r_\xi(x))f'(\xi)d\xi - \int_{-\infty}^{0} r_\xi(x)f'(\xi)d\xi$$

and

$$s_n(x) = f(0) + \int_0^{\infty} (1 - R_\xi(x))f'(\xi)d\xi - \int_{-\infty}^{0} R_\xi(x)f'(\xi)d\xi.$$

As $f' \geq 0$ and $r_\xi \leq \chi_{(-\infty,\xi]} \leq R_\xi$, we obtain (4.4). Moreover,

$$(S_n - s_n)(x) = \int_{-\infty}^{\infty} (R_\xi - r_\xi)(x)f'(\xi)d\xi$$

so

$$\int_{-\infty}^{\infty} (S_n - s_n) W^2 \le \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} (R_\xi - r_\xi)(x) W^2(x) dx \right] f'(\xi) d\xi.$$

Now by Lemmas 4.1 and 4.2, and some other minor considerations,

$$\int_{-\infty}^{\infty} (R_\xi - r_\xi)(x) W^2(x) dx \le C \frac{a_n}{n} W^2(\xi),$$

at least for $|\xi| \le Ca_n$. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} (S_n - s_n) W^2 &\le C \frac{a_n}{n} \int_{-Ca_n}^{Ca_n} f' W^2 \\ &+ \|f' W^2\|_{L_\infty(|x| \ge Ca_n)} \int_{|\xi| \ge Ca_n} \left[W^{-2}(\xi) \int_{-\infty}^{\infty} (R_\xi - r_\xi)(x) W^2(x) dx \right] d\xi. \end{aligned}$$

The first term on the right-hand side has the correct form. The second tail term does not. One can still use estimates for Christoffel functions, and some other tricks. See [40], [46] or [136, p. 83 ff.] for more details. □

This is what Freud proved in 1974 [40, p. 297], using these ideas:

Theorem 4.4 *Let Q be an even and convex function on the real line. Assume that Q is increasing and Q' exists in $(0, \infty)$, and that for some $C_1, C_2 > 0$,*

$$1 + C_1 < \frac{Q'(2x)}{Q'(x)} < 1 + C_2.$$

Let $r \ge 0$ and $f^{(r)}$ be of bounded variation over every finite interval, and of polynomial growth at ∞, satisfying for some $A, B > 0$ and integer m,

$$|f(x)| \le A + Bx^{2m}.$$

Then there exist upper and lower polynomials S_n and s_n of degree $\le n$ such that (4.4) holds and

$$\int_{-\infty}^{\infty} (S_n - s_n) W^2 \le C \left(\frac{a_n}{n} \right)^{r+1} \left(\int_{-\infty}^{\infty} W^2 |df^{(r)}| + A + B \right).$$

The constant C is independent of f, n, A, B.

While this is a one-sided L_1 result, it also implies

$$E_n[f; W^2]_1 \le C \left(\frac{a_n}{n} \right)^{r+1} \left(\int_{-\infty}^{\infty} W^2 |df^{(r)}| + A + B \right).$$

This can be extended to other L_p spaces, using duality. See [120] for an extension to weights with inner singularities.

4.2 Freud's Method involving de la Vallée Poussin Means

This is based on orthogonal expansions, and for the finite interval, was used before Freud. Freud was the first to make it work on infinite intervals [41], [42]. For a measurable function for which $\int_{-\infty}^{\infty}(fW)^2$ is finite, we can form the orthonormal expansion

$$f \sim \sum_{j=0}^{\infty} c_j p_j,$$

where

$$c_j = \int_{-\infty}^{\infty} f p_j W^2, \qquad j \geq 0.$$

Define the partial sums of the orthonormal expansion,

$$S_n[f] = \sum_{j=0}^{n-1} c_j p_j.$$

The classic minimum property asserts that

$$\|(f - S_n[f])\, W\|_{L_2(\mathbb{R})} = E_{n-1}[f; W]_2 = \inf_{\deg(P) \leq n-1} \|(f-P)W\|_{L_2(\mathbb{R})},$$

and the classic Bessel's inequality asserts that

$$\|S_n[f]\, W\|_{L_2(\mathbb{R})} = \left(\sum_{j=0}^{n-1} |c_j|^2\right)^{1/2} \leq \|fW\|_{L_2(\mathbb{R})}.$$

In particular, $\{S_n\}$ is a uniformly bounded sequence of operators in a weighted L_2 space.

Sometimes it is more convenient to use the de la Vallée Poussin operators

$$V_n[f] = \frac{1}{n} \sum_{j=n+1}^{2n} S_j[f].$$

We see that still

$$\|V_n[f]\, W\|_{L_2(\mathbb{R})} \leq \|fW\|_{L_2(\mathbb{R})}$$

so the $\{V_n\}$ are uniformly bounded in this weighted L_2 setting. They also reproduce polynomials:

$$V_n[P] = P$$

if P is a polynomial of degree $\leq n-1$. However, they are not projection operators, as $V_n[P]$ is a polynomial of degree $2n-1$ in general. The real advantage is that in spaces other than L_2, $\{V_n\}$ is often uniformly bounded when $\{S_n\}$ is not.

Another crucial ingredient are estimates for the Christoffel functions defined by (4.1). Let us outline some of the main ideas of Freud's method to prove the Jackson–Favard inequality

$$E_n[f; W]_p \leq C \frac{a_n}{n} E_{n-1}[f'; W]_p. \tag{4.5}$$

(a) We show that $\{V_n\}$ is a uniformly bounded sequence of linear operators, first in L_∞, and then in L_1 (using duality) and then in $L_p, 1 < p < \infty$ (using interpolation).

(b) We approximate a very special function, something like a characteristic function, in L_1, using the method of the previous section.

(c) We prove the estimate in L_∞, using $\{V_n\}$ and the special function.

(d) We extend to L_p, $1 \leq p < \infty$.

In our outline, we shall do this only for $p = \infty$.

The first step is to show, for some $C \neq C(n, f)$, that

$$E_{2n}\left[f; W\right]_\infty \leq \left\|\left(f - V_n\left[f\right]\right) W\right\|_{L_\infty(\mathbb{R})} \leq C E_n\left[f; W\right]_\infty . \tag{4.6}$$

Lemma 4.5 *The inequality* (4.6) *holds, provided we assume the estimate* (4.13) *below.*

Proof. The left inequality here is immediate, as $V_n[f]$ is a polynomial of degree $\leq 2n - 1$. It is the right-hand inequality that requires work. We follow Freud [42]. The idea goes back at least to Torsten Carleman. The partial sum $s_m\left[f\right]$ admits the representation

$$s_m\left[f\right](x) = \sum_{j=0}^{m-1} c_j p_j(x) = \int_{-\infty}^{\infty} f(t) K_m(x, t) W^2(t) dt, \tag{4.7}$$

where

$$K_m(x, t) := \sum_{j=0}^{m-1} p_j(x) p_j(t).$$

The Christoffel–Darboux formula [38] asserts that

$$K_m(x, t) = \frac{\gamma_{m-1}}{\gamma_m} \frac{p_m(x) p_{m-1}(t) - p_{m-1}(x) p_m(t)}{x - t}. \tag{4.8}$$

We now fix x and n, and define

$$g = f(x) \chi_{\left(x - \frac{a_n}{n}, x + \frac{a_n}{n}\right)}$$

and

$$h = f - g = f \chi_{\mathbb{R} \setminus \left(x - \frac{a_n}{n}, x + \frac{a_n}{n}\right)}.$$

Here χ_S denotes the characteristic function of a set S. Thus $g = f$ in an interval of radius $\frac{a_n}{n}$ about x, and is 0 elsewhere, while h is the "rest" of f. We have

$$s_m\left[f\right](x) = s_m\left[g\right](x) + s_m\left[h\right](x). \tag{4.9}$$

Now

$$\begin{aligned}
|s_m[g]|(x) &= \left| \int_{x-\frac{a_n}{n}}^{x+\frac{a_n}{n}} f(t) K_m(x,t) W^2(t) dt \right| \leq \|fW\|_{L_\infty(\mathbb{R})} \int_{x-\frac{a_n}{n}}^{x+\frac{a_n}{n}} |K_m(x,t)| W(t) dt \\
&\leq \|fW\|_{L_\infty(\mathbb{R})} \left(\frac{2a_n}{n} \right)^{1/2} \left(\int_{x-\frac{a_n}{n}}^{x+\frac{a_n}{n}} K_m^2(x,t) W^2(t) dt \right)^{1/2},
\end{aligned}$$

by the Cauchy–Schwarz inequality. Here, using orthogonality,

$$\begin{aligned}
\int_{x-\frac{a_n}{n}}^{x+\frac{a_n}{n}} K_m^2(x,t) W^2(t) dt &\leq \int_{-\infty}^{\infty} K_m^2(x,t) W^2(t) dt \\
&= \sum_{j=0}^{m-1} p_j^2(x) \leq \sum_{j=0}^{2n-1} p_j^2(x) = \lambda_{2n}^{-1}(W^2, x),
\end{aligned}$$

provided $m \leq 2n$. Thus for $m \leq 2n$,

$$|s_m[g](x)| \leq \|fW\|_{L_\infty(\mathbb{R})} \left(\frac{2a_n}{n} \lambda_{2n}^{-1}(W^2, x) \right)^{1/2},$$

and hence, averaging over s_m, $n+1 \leq m \leq 2n$,

$$|V_n[g](x)| \leq \|fW\|_{L_\infty(\mathbb{R})} \left(\frac{2a_n}{n} \lambda_{2n}^{-1}(W^2, x) \right)^{1/2}. \tag{4.10}$$

Now comes the clever idea. Let

$$H(t) = \frac{h(t)}{x-t}, \qquad t \in \mathbb{R},$$

and denote its Fourier coefficients with respect to $\{p_j\}$ by $\{c_j[H]\}$, so that

$$H \sim \sum_{j=0}^{\infty} c_j[H] p_j.$$

We use the Christoffel–Darboux formula (4.8) to write

$$\begin{aligned}
|s_m[h](x)| &= \frac{\gamma_{m-1}}{\gamma_m} \left| p_m(x) \int_{-\infty}^{\infty} H(t) p_{m-1}(t) W^2(t) dt - p_{m-1}(x) \int_{-\infty}^{\infty} H(t) p_m(t) W^2(t) dt \right| \\
&= \frac{\gamma_{m-1}}{\gamma_m} |p_m(x) c_{m-1}[H] - p_{m-1}(x) c_m[H]|.
\end{aligned}$$

Then summing and using Cauchy–Schwarz,

$$|V_n[h](x)| \leq \frac{1}{n} \left[\max_{m \leq 2n} \frac{\gamma_{m-1}}{\gamma_m} \right] 2 \left(\sum_{m=0}^{2n} p_m^2(x) \right)^{1/2} \left(\sum_{m=0}^{2n} c_m^2(H) \right)^{1/2}. \tag{4.11}$$

Yet another clever idea: use Bessel's inequality on the Fourier coefficients of H (recall, this holds for any orthogonal system in any inner product space),

$$\sum_{m=0}^{2n} c_m^2(H) \leq \sum_{m=0}^{\infty} c_m^2(H) \leq \int_{-\infty}^{\infty} (HW)^2 = \int_{\left\{t:|t-x|\geq \frac{a_n}{n}\right\}} \left(\frac{f(t)W(t)}{t-x}\right)^2 dt \leq \|fW\|^2_{L_\infty(\mathbb{R})} 2\frac{n}{a_n}.$$

Then (4.11) becomes

$$|V_n[h](x)| \leq \frac{2\sqrt{2}}{n}\left[\max_{m\leq 2n} \frac{\gamma_{m-1}}{\gamma_m}\right]\left(\frac{n}{a_n}\lambda_{2n+1}^{-1}(W^2,x)\right)^{1/2}.$$

Finally, combining this with (4.9) and (4.10) gives

$$|V_n[f](x)| \leq \|fW\|_{L_\infty(\mathbb{R})} \left(\lambda_{2n+1}(W^2,x)\right)^{-1/2} \left\{\left(\frac{2a_n}{n}\right)^{1/2} + \frac{1}{n}\left[\max_{m\leq 2n}\frac{\gamma_{m-1}}{\gamma_m}\right] 2\sqrt{2}\left(\frac{n}{a_n}\right)^{1/2}\right\}. \tag{4.12}$$

Up to this stage, we have not used any properties of the weight, it's completely general. But now we use lower bounds for the Christoffel function

$$\lambda_n(W^2,x) \geq C\frac{a_n}{n}W^2(x), \qquad x\in\mathbb{R},\ n\geq 1, \tag{4.13}$$

and an upper bound

$$\max_{m\leq 2n}\frac{\gamma_{m-1}}{\gamma_m} \leq Ca_n \tag{4.14}$$

to deduce that uniformly for $n \geq 1$, and $x \in \mathbb{R}$,

$$|V_n[f](x)|W(x) \leq C\|fW\|_{L_\infty(\mathbb{R})}.$$

Thus, for some $C_* \neq C(n,f)$,

$$\|V_n[f]W\|_{L_\infty(\mathbb{R})} \leq C_*\|fW\|_{L_\infty(\mathbb{R})}.$$

Then for any polynomial P of degree $\leq n$, the reproducing property of V_n gives

$$\begin{aligned}\|(f-V_n[f])W\|_{L_\infty(\mathbb{R})} &= \|(f-P-V_n[f-P])W\|_{L_\infty(\mathbb{R})}\\ &\leq \|(f-P)W\|_{L_\infty(\mathbb{R})} + \|V_n[f-P]W\|_{L_\infty(\mathbb{R})}\\ &\leq (1+C_*)\|(f-P)W\|_{L_\infty(\mathbb{R})}.\end{aligned}$$

Taking the inf over all P of degree $\leq n$ gives the right-hand inequality in (4.6).

We turn to the discussion of (4.13) and (4.14). For the latter, we can just use restricted range inequalities: if $m \leq 2n$

$$\begin{aligned}\frac{\gamma_{m-1}}{\gamma_m} &= \int_{-\infty}^{\infty} xp_{m-1}(x)p_m(x)W^2(x)dx\\ &\leq C\int_{-2a_m}^{2a_m} |xp_{m-1}(x)p_m(x)|\,W^2(x)dx \leq C2a_m,\end{aligned}$$

by Cauchy–Schwarz. The lower bound (4.13) is more difficult; see for example [83], [136], [150]. □

As an aside, let us see how duality also gives this bound in L_1. We use duality of L_p norms:

$$\|V_n[f]W\|_{L_1(\mathbb{R})} = \sup_g \int_{-\infty}^{\infty} V_n[f]\,gW^2, \tag{4.15}$$

where the sup is over all measurable g with $\|gW\|_{L_\infty(\mathbb{R})} \leq 1$. (Sorry, we used g above in a different sense.) Now we use the self-adjointness of V_n :

$$\int_{-\infty}^{\infty} V_n[f]\,gW^2 = \int_{-\infty}^{\infty} fV_n[g]\,W^2. \tag{4.16}$$

This follows easily, once we prove the self-adjointness of s_m :

$$\int_{-\infty}^{\infty} s_m[f]\,gW^2 = \int_{-\infty}^{\infty} fs_m[g]\,W^2.$$

We leave the latter as an exercise (just substitute in the definitions of $s_m[f]$ and $s_m[g]$). Combining (4.15) and (4.16) gives

$$\begin{aligned}\|V_n[f]W\|_{L_1(\mathbb{R})} &= \sup_g \int_{-\infty}^{\infty} fV_n[g]\,W^2 \leq \sup_g \|fW\|_{L_1(\mathbb{R})} \|V_n[g]W\|_{L_\infty(\mathbb{R})} \\ &\leq C\|fW\|_{L_1(\mathbb{R})}, \end{aligned}\tag{4.17}$$

using the case $p=\infty$. With this bound, we can finish off the rest of the proof as we did for the case $p=\infty$. For $1<p<\infty$, one can use interpolation of operators. It is possible to improve the estimate (4.6) to include factors that vanish near $\pm a_n$, reflecting improved approximation there, see [101]. A partially successful attempt to extend (4.5) to general exponential weights was given in [102].

An obvious question is for which weights, we have the lower bound (4.13) for the Christoffel functions. They have not been established for the class $\mathcal{F}$ of Definition 3.3. However, they were established in [83] for the following class:

Definition 4.6 *Let $W=\exp(-Q)$, where Q'' exists and is positive in $(0,\infty)$, while Q' is positive there, with limit 0 at 0, and for some $A,B>1$,*

$$A-1 \leq \frac{xQ''(x)}{Q'(x)} \leq B-1, \qquad x\in(0,\infty). \tag{4.18}$$

Then we write $W\in\mathcal{F}^*$.

For weaker (but difficult to formulate) hypotheses, these estimates were proved in [85]. An excellent exposition is given in [136, Chapter 3]. The Christoffel function bound (4.13) there is proved assuming Q'' is increasing. This is true for $W=W_\alpha$ if $\alpha\geq 2$, while (4.18) holds for $\alpha>1$.

The next step is an inequality that is a distant relative of Hardy's inequality:

Lemma 4.7 *Let $n\geq 1$ and g be a function such that*

$$\int_{-\infty}^{\infty} gPW^2 = 0, \tag{4.19}$$

for all polynomials P of degree $\leq n$. Then there exists $C\neq C(n,g)$ such that

$$\sup_{x\in\mathbb{R}} W(x)\left|\int_0^x g\right| \leq C\frac{a_n}{n}\|gW\|_{L_\infty(\mathbb{R})}. \tag{4.20}$$

Main idea of the proof. Fix $x > 0$, and let

$$\phi_x(t) = W^{-2}(t)\chi_{[0,x]}(t).$$

For g as above, and any polynomial P of degree $\leq n$,

$$\begin{aligned} \left|\int_0^x g\right| &= \left|\int_{-\infty}^{\infty} g(t)\phi_x(t)W^2(t)dt\right| \\ &= \left|\int_{-\infty}^{\infty} g(t)\left[\phi_x(t) - P(t)\right]W^2(t)dt\right| \\ &\leq \|gW\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} |\phi_x - P|\, W. \end{aligned}$$

Taking the inf over all such P gives

$$\left|\int_0^x g\right| \leq \|gW\|_{L_\infty(\mathbb{R})} E_n\left[\phi_x; W\right]_1.$$

Once we have the estimate

$$W(x)E_n\left[\phi_x; W\right]_1 \leq C\frac{a_n}{n}, \tag{4.21}$$

for some $C \neq C(n, x)$, the result follows. To prove this, one can use one-sided approximation as in Section 4.1. See, for example, [42, p. 34]. □

Now we can give the Jackson–Favard inequality in the case $p = \infty$:

Theorem 4.8 *Let $W \in \mathcal{F}^*$. Let f be absolutely continuous in each finite interval, with $\|f'W\|_{L_\infty(\mathbb{R})}$ finite. Then for some $C \neq C(n, f)$,*

$$E_n\left[f; W\right]_\infty \leq C\frac{a_n}{n}\left\|f'W\right\|_{L_\infty(\mathbb{R})} \tag{4.22}$$

and

$$E_n\left[f; W\right]_\infty \leq C\frac{a_n}{n}E_{n-1}\left[f'; W\right]_\infty. \tag{4.23}$$

Proof. Let

$$g(x) = f'(x) - V_n\left[f'\right](x).$$

This does satisfy (4.19). Indeed, if P is a polynomial of degree $\leq n$, self-adjointness of V_n gives

$$\begin{aligned} \int_{-\infty}^{\infty} gPW^2 &= \int_{-\infty}^{\infty} f'PW^2 - \int_{-\infty}^{\infty} V_n\left[f'\right]PW^2 \\ &= \int_{-\infty}^{\infty} f'PW^2 - \int_{-\infty}^{\infty} f'V_n\left[P\right]W^2 = 0, \end{aligned}$$

as $V_n\left[P\right] = P$. Next, let

$$U_n(x) = f(0) + \int_0^x V_n\left[f'\right](t)dt,$$

so that

$$(f-U_n)(x)=\int_0^x \left(f'-V_n\left[f'\right]\right)=\int_0^x g.$$

Then

$$\begin{aligned}\|W(f-U_n)\|_{L_\infty(\mathbb{R})} &= \left\|W\int_0^x g\right\|_{L_\infty(\mathbb{R})}\leq C\frac{a_n}{n}\|gW\|_{L_\infty(\mathbb{R})}\\ &= C\frac{a_n}{n}\left\|\left(f'-V_n\left[f'\right]\right)W\right\|_{L_\infty(\mathbb{R})}\leq \frac{a_n}{n}E_n\left[f';W\right]_\infty.\end{aligned}$$

Here we applied Lemma 4.7 and then Lemma 4.5. This also gives, as U_n has degree $\leq 2n$,

$$E_{2n}\left[f;W\right]_\infty \leq C\frac{a_n}{n}E_n\left[f';W\right]_\infty \leq C\frac{a_n}{n}\left\|f'W\right\|_{L_\infty(\mathbb{R})}.$$

Replacing n by $n/2$, and using the fact that $a_{n/2}\leq a_n$, we obtain

$$E_n\left[f;W\right]_\infty \leq C\frac{a_n}{n}\left\|f'W\right\|_{L_\infty(\mathbb{R})}.$$

Finally, we observe that for any polynomial P of degree $\leq n$,

$$E_n\left[f;W\right]_\infty = E_n\left[f-P;W\right]_\infty \leq C\frac{a_n}{n}\left\|(f-P)'W\right\|_{L_\infty(\mathbb{R})}.$$

Choosing P' suitably gives (4.23). $\square$

For the extension to all $1\leq p\leq\infty$, see [136, Chapter 4].

4.3 The Kroó–Szabados Method

The idea here [69] is to make use of the alternation/equioscillation for best polynomial approximants, together with some clever tricks. On a finite interval, the idea was used by Bojanov [5], Babenko and Shalaev. While it works quite generally, it does not yield the correct Jackson rate. Maybe some clever tweaking can repair that?

Let us fix a function f, and $n\geq 1$, and let P_n^* be its best polynomial approximation of degree $\leq n$ in the weighted uniform norm. Thus

$$\|(f-P_n^*)W\|_{L_\infty(\mathbb{R})}=\inf_{\deg(P)\leq n}\|(f-P)W\|_{L_\infty(\mathbb{R})}=E_n\left[f;W\right]_\infty.$$

Then there exist equioscillation points $\{y_j\}_{j=0}^{n+1}$ such that for $0\leq j\leq n+1$,

$$\left[(f-P_n^*)W\right](y_j)=\varepsilon(-1)^jE_n\left[f;W\right]_\infty.$$

The number $\varepsilon\in\{-1,1\}$ is independent of j. In terms of these, there is the determinant expression

$$E_n\left[f;W\right]_\infty=\left|\frac{U\begin{pmatrix} f & 1 & \dots & x^n\\ y_0 & y_1 & \dots & y_{n+1}\end{pmatrix}}{U\begin{pmatrix} g & 1 & \dots & x^n\\ y_0 & y_1 & \dots & y_{n+1}\end{pmatrix}}\right|,$$

where for functions $\{\phi_j\}$

$$U\begin{pmatrix} \phi_0 & \phi_1 & \dots & \phi_{n+1} \\ y_0 & y_1 & \dots & y_{n+1} \end{pmatrix} := \det\left(\phi_i\left(y_j\right)\right)_{i,j=0}^{n+1}$$

and g is a function such that

$$(gW)\left(y_j\right) = (-1)^j, \qquad 0 \le j \le n+1.$$

A proof of this formula may be found in [170, p. 28]. By some elementary determinantal manipulations, one can show that

$$U\begin{pmatrix} f & 1 & \dots & x^n \\ y_0 & y_1 & \dots & y_{n+1} \end{pmatrix} = (-1)^n \sum_{k=0}^{n} (-1)^k \left[f\left(y_{k+1}\right) - f\left(y_k\right)\right] B_k$$

and

$$U\begin{pmatrix} g & 1 & \dots & x^n \\ y_0 & y_1 & \dots & y_{n+1} \end{pmatrix} = (-1)^{n+1} \sum_{k=0}^{n} \left[W^{-1}\left(y_{k+1}\right) + W^{-1}\left(y_k\right)\right] B_k.$$

Here B_k is the determinant of a matrix with entries involving only $\{y_j\}$. Then one obtains

$$E_n\left[f; W\right]_\infty = \left| \sum_{k=0}^{n} (-1)^k \left[f\left(y_{k+1}\right) - f\left(y_k\right)\right] d_k \right|, \tag{4.24}$$

where

$$d_k = \frac{B_k}{\sum_{j=0}^{n} \left[W^{-1}\left(y_{j+1}\right) + W^{-1}\left(y_j\right)\right] B_j}.$$

Suppose we define the (unusual!) modulus

$$\omega_\gamma(f; t) = \sup_{x,y \in \mathbb{R},\ |x-y| \le t} \frac{|f(x) - f(y)|}{W(x)^{-\gamma} + W(y)^{-\gamma}},$$

where $\gamma \in (0, 1)$. Then from (4.24),

$$E_n\left[f; W\right]_\infty \le \sum_{k=0}^{n} |d_k| \left(W\left(y_{k+1}\right)^{-\gamma} + W\left(y_k\right)^{-\gamma}\right) \omega_\gamma\left(f; |y_{k+1} - y_k|\right).$$

Now one uses properties of the modulus ω_γ and then has to estimate the $\{d_k\}$. This involves tricks such as needle polynomials. Here is a sample of what can be achieved:

Theorem 4.9 *Let $Q : \mathbb{R} \to \mathbb{R}$ be an even continuous function, which is positive and differentiable for large x, and with*

$$0 < \liminf_{x \to \infty} \frac{xQ'(x)}{Q(x)} \le \limsup_{x \to \infty} \frac{xQ'(x)}{Q(x)} < \infty. \tag{4.25}$$

Let $Q^{[-1]}$ denote the inverse of Q, defined for sufficiently large positive x. For small n, take $Q^{[-1]}(n)$ to be 1. Assume that $0 < \gamma < 1$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, with

$$\lim_{|x| \to \infty} f(x) W^\gamma(x) = 0.$$

(a) *Then for some* $C_j \neq C_j(f,n)$,

$$E_n[f;W]_\infty \leq C_1\omega_\gamma(f;\frac{Q^{[-1]}(n)\log n}{(1-\gamma)n}) + e^{-C_2 n}\|fW^\gamma\|_{L_\infty(\mathbb{R})}. \tag{4.26}$$

(b) *Let* $0 < \varepsilon < \frac{1-\gamma}{2-\gamma}$. *Then for some* $C \neq C(f,n)$,

$$E_n[f;W]_\infty \leq C_1\omega_\gamma(f;I_n^{-\frac{1-\gamma}{2-\gamma}+\varepsilon}), \tag{4.27}$$

where for large enough n,

$$I_n := \int_1^{Q^{[-1]}(n)} \frac{Q(t)}{t^2}dt. \tag{4.28}$$

Note that (4.25) requires less than we required for $W \in \mathcal{F}$, but the restrictions on f are more severe than in Theorem 3.5, and instead of $\frac{a_n}{n}$ inside the modulus, we obtain essentially $\frac{a_n \log n}{n}$. The boundary case $Q(x) = |x|$ satisfies the above conditions, and in this case we obtain

$$E_n[f;W]_\infty \leq C_1\omega_\gamma(f;(\log n)^{-\frac{1-\gamma}{2-\gamma}+\varepsilon}).$$

Here by choosing γ small enough, the exponent of $\log n$ can be made arbitrarily close to -1.

This method is interesting, and general. The challenge is how to tweak it, if possible, to get the correct Jackson rate.

4.4 The Piecewise Polynomial Method

This is undoubtedly the most direct and general method, and Ditzian and the author used it to prove Theorem 3.5. However, it does pose substantial technical challenges. For finite intervals, it has been used for a long time, and in spirit goes back to Lebesgue's proof of Weierstrass' Theorem. Lebesgue first approximated by a piecewise linear function, and then polynomials. It has served as a powerful tool on finite intervals, for example in investigating shape preserving polynomial approximation, the degree of spline approximation, and even approximation by rational functions [30], [156].

The function f is first approximated by a piecewise polynomial (or spline). Each of the piecewise polynomials is generated via Whitney's Theorem. Then special polynomials that approximate characteristic functions are used to turn the spline approximation into a polynomial approximation. We illustrate the method as it is used to prove Theorem 3.5 for Freud weights.

Step 1: Partition $[-a_n, a_n]$. Recall that our modulus $\omega_{r,p}$ involves a tail piece that will take care of the behavior of $f(x)$ for very large x. So we fix n and concentrate on approximation on the Mhaskar–Rakhmanov–Saff interval $[-a_n, a_n]$. We partition this interval into small intervals, all of length $\frac{a_n}{n}$:

$$-a_n = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{2n} = a_n.$$

Set

$$I_j := [\tau_j, \tau_{j+1}], \qquad 0 \leq j \leq 2n-1.$$

Step 2: Use Whitney's Theorem to develop a piecewise polynomial approximation. We fix r — this will be the order of the modulus. We approximate f on I_j by a polynomial p_j of degree $\leq r$, and then form the piecewise polynomial

$$S[f] := p_0 + \sum_{j=1}^{2n-1} (p_j - p_{j-1})\, \chi_{[\tau_j, a_n]},$$

where, as usual, $\chi_{[\tau_j,a_n]}$ denotes the characteristic function of the interval $[\tau_j, a_n]$. When restricted to I_k, the sum becomes a telescopic sum: we see that in the interior of I_k, namely in (τ_k, τ_{k+1}),

$$S[f] = p_0 + \sum_{j=1}^{k} (p_j - p_{j-1}) = p_k.$$

Then, if $p < \infty$,

$$\|(f - S[f])\, W\|^p_{L_p[-a_n,a_n]} = \sum_{j=0}^{2n-1} \int_{I_j} |(f - p_j)\, W|^p \quad \leq \quad C \sum_{j=0}^{2n-1} W(\tau_j)^p \, \|f - p_j\|^p_{L_p(I_j)}. \tag{4.29}$$

In this step, we use the fact that because the intervals I_j have length $\frac{a_n}{n}$, W does not grow or decay by more than a constant. The idea is that for $x, y \in I_k$,

$$W(x)/W(y) = \exp(Q(y) - Q(x)) = \exp(Q'(\xi)(x-y)) \leq \exp\left(Q'(\xi)\frac{a_n}{n}\right) \leq C, \tag{4.30}$$

since Q' is bounded by $C\frac{n}{a_n}$ throughout $[-a_n, a_n]$. (This is true for Freud weights in the class $\mathcal{F}^*$ but fails for Erdős weights.) Now comes the application of Whitney. Let

$$I_j^* = I_j \cup I_{j+1}.$$

By Whitney's Theorem on the interval I_j^* [156, p. 195, p. 191], we can choose p_j of degree $\leq r$ such that

$$\|f - p_j\|^p_{L_p(I_j^*)} \leq C \frac{n}{a_n} \int_0^{a_n/n} \int_{I_j^*} |\Delta_s^r f(x)|^p \, dx \; ds =: C\Omega_j^p.$$

Here C is independent of f, n, j. The strange creature on the right is really the pth power of an rth order integral modulus of continuity on the interval I_j^*. In forming the rth difference in this integral, one uses the convention that the difference is taken as 0 if any of the arguments of the function are outside the interval I_j^*. Substituting this in (4.29) and using (4.30) gives

$$\begin{aligned} \|(f - S[f])\, W\|^p_{L_p[-a_n,a_n]} &\leq C \frac{n}{a_n} \int_0^{a_n/n} \int_{-a_n}^{a_n} |W \Delta_s^r f(x)|^p \, dx \; ds \\ &\leq C \sup_{0 < h \leq \frac{a_n}{n}} \|W \Delta_h^r f\|^p_{L_p[-a_n,a_n]}. \end{aligned} \tag{4.31}$$

Step 3: Approximate the characteristic function $\chi_{[\tau_j,a_n]}$. Now comes the difficult part. We approximate $\chi_{[\tau_j,a_n]}$ by a polynomial R_j of degree $\leq Ln$ giving the polynomial

$$P[f] = p_0 + \sum_{j=1}^{2n-1} (p_j - p_{j-1})\, R_j,$$

which is of degree at most $Ln + r$. Here the constant L is independent of n, and arises in the challenging task of generating R_j. We see that then

$$S(f) - P[f] = \sum_{j=1}^{2n-1} (p_j - p_{j-1}) \left(\chi_{[\tau_j, a_n]} - R_j \right). \tag{4.32}$$

To estimate this, we compare p_j and p_{j-1} on the common interval where they approximate f, namely I_j. (That is why we used Whitney on $I_j^* = I_j \cup I_{j+1}$.) We obtain, even for $p < 1$,

$$\|p_j - p_{j-1}\|_{L_p(I_j)} \le C\left(\Omega_{j-1} + \Omega_j\right).$$

Now we use Nikolskii inequalities, which compare the norms of polynomials in L_p and L_q, and the Bernstein–Walsh inequality, which bounds the growth of polynomials outside an interval, once we know their size on the interval. Together they yield for all real x,

$$|p_j - p_{j-1}|(x) \le C\left(\frac{n}{a_n}\right)^{1/p} \left(1 + \frac{n}{a_n}|x - \tau_j|\right)^r \left(\Omega_{j-1} + \Omega_j\right). \tag{4.33}$$

Again, the constant C does not depend on x, or n, or f. It does however depend on r, which crucially remains fixed. The $\frac{n}{a_n}$ factor arises from the length of I_j. Suppose now that for a given ℓ,

$$\left|\chi_{[\tau_j, a_n]} - R_j\right|(x) \frac{W(x)}{W(\tau_j)} \le C\left(1 + \frac{n}{a_n}|x - \tau_j|\right)^{-\ell}, \qquad x \in \mathbb{R}, \tag{4.34}$$

where C is independent of f, n, j, x. Then substituting this and (4.33) into (4.32) gives

$$|S[f] - P[f]|(x) W(x) \ \le C\left(\frac{n}{a_n}\right)^{1/p} \sum_{j=1}^{2n-1} \left(1 + \frac{n}{a_n}|x - \tau_j|\right)^{r-\ell} W(\tau_j)\left(\Omega_{j-1} + \Omega_j\right).$$

From here on, we need to proceed a little differently for $p \le 1$ and $p > 1$. Let us suppose $p > 1$. By Hölder's inequality, with $q = \frac{p}{p-1}$,

$$\begin{aligned} |S[f] - P[f]|^p (x) W^p(x) \ &\le \ C\frac{n}{a_n}\left[\sum_{j=1}^{2n-1}\left(1 + \frac{n}{a_n}|x - \tau_j|\right)^{(r-\ell)p/2} W^p(\tau_j)\Omega_j^p\right] \times \\ &\quad \times \left[\sum_{j=1}^{2n-1}\left(1 + \frac{n}{a_n}|x - \tau_j|\right)^{(r-\ell)q/2}\right]^{p/q}. \end{aligned} \tag{4.35}$$

We also use the fact that $1 + \frac{n}{a_n}|x - \tau_j|$ is bounded by a constant times $1 + \frac{n}{a_n}|x - \tau_{j-1}|$ throughout the real line. Next, if $\ell > r$, the function $u \mapsto \left(1 + \frac{n}{a_n}|x - u|\right)^{(r-\ell)q/2}$ is increasing in $(-\infty, x)$ and decreasing in (x, ∞), so we can bound the second sum by an integral:

$$\sum_{j=1}^{2n-1}\left(1 + \frac{n}{a_n}|x - \tau_j|\right)^{(r-\ell)q/2} \ \le \ 2\frac{n}{a_n}\int_{-\infty}^{\infty}\left(1 + \frac{n}{a_n}|x - u|\right)^{(r-\ell)q/2} du + 1 \le C,$$

with $C \neq C(n,x)$, provided only $(r-\ell)q/2 < -1$. So all we need is that ℓ is large enough. Now we integrate (4.35):

$$\begin{aligned} \|(S[f]-P[f])W\|^p_{L_p(\mathbb{R})} &\leq C\frac{n}{a_n}\sum_{j=1}^{2n-1} W^p(\tau_j)\,\Omega_j^p \int_{-\infty}^{\infty}\left(1+\frac{n}{a_n}|x-\tau_j|\right)^{(r-\ell)p/2} dx \\ &\leq C\sum_{j=1}^{2n-1} W^p(\tau_j)\,\Omega_j^p, \end{aligned} \tag{4.36}$$

again, provided ℓ is so large that $(r-\ell)p/2 < -1$. Finally,

$$\begin{aligned} \sum_{j=1}^{2n-1} W^p(\tau_j)\,\Omega_j^p &= \frac{n}{a_n}\int_0^{a_n/n}\left[\sum_{j=1}^{2n-1} W^p(\tau_j)\int_{I_j^*}|\Delta_s^r f(x)|^p\,dx\right] ds \\ &\leq C\sup_{0<h\leq\frac{a_n}{n}}\int_{-a_n}^{a_n}|W(x)\Delta_h^r f(x)|^p\,dx = C\sup_{0<h\leq\frac{a_n}{n}}\|W\Delta_h^r f\|^p_{L_p[-a_n,a_n]}. \end{aligned}$$

Combining this, (4.31), and (4.36), gives

$$\|(f-P[f])W\|^p_{L_p(\mathbb{R})} \leq C\left\{\sup_{0<h\leq\frac{a_n}{n}}\|W\Delta_h^r f\|^p_{L_p[-a_n,a_n]} + \|fW\|^p_{L_p(|x|\geq a_n)}\right\}.$$

That's it! Reformulating this in terms of the modulus of continuity is relatively straightforward. Of course we assumed:

Step 4: Construction of the $\{R_j\}$. Recall we want R_j of degree $\leq Ln$, satisfying

$$\left|\chi_{[\tau,a_n]}-R_j\right|(x)\frac{W(x)}{W(\tau)} \leq C\left(1+\frac{n}{a_n}|x-\tau|\right)^{-\ell}, \quad x\in\mathbb{R},\ \tau\in[-a_n,a_n], \tag{4.37}$$

with constants independent of τ, n, x. The problem here is the $W(\tau)$ in the denominator. It's tiny for τ close to a_n, and we want R_j to approximate 1 in $[\tau, a_n]$, and to approximate 0 elsewhere in $\mathbb{R}$. One starts with an even entire function

$$G(x) = \sum_{j=0}^{\infty} g_{2j}x^{2j}$$

with all $g_{2j}\geq 0$ such that

$$C_1 \leq (GW)(x) \leq C_2, \qquad x\in\mathbb{R}.$$

Such functions were constructed in [87]. We let G_n denote the nth partial sum. One can show that

$$(G_nW)(x) \leq C_1, \qquad x\in\mathbb{R},$$

and

$$C_1 \leq (G_nW)(x) \leq C_2, \qquad |x| \leq C_1 a_n.$$

Next, we need needle or peaking polynomials $V_{n,\xi}$, built from Chebyshev polynomials, satisfying

$$\|V_{n,\xi}\|_{L_\infty[-1,1]} = V_{n,\xi}(\xi) = 1;$$

$$|V_{n,\xi}(t)| \leq \frac{B\sqrt{1-|\xi|}}{n\,|t-\xi|}, \qquad t \in [-1,1] \setminus \{\xi\}\,;$$

and

$$V_{n,\xi}(t) \geq \frac{1}{2}, \qquad |t-\xi| \leq C\frac{\sqrt{1-|\xi|}}{n}.$$

The constants B, C are independent of n, ξ, x. We define [33, p. 121]

$$R_j(x) = \frac{\int_0^x G_{Ln/4}(s) V_{n,\xi}(s/a_{2Ln})^L ds}{\int_0^{\tau^*} G_{Ln/4}(s) V_{n,\xi}(s/a_{2Ln})^L ds}$$

where L and τ^* are appropriately chosen, and $\xi = \tau/a_{Ln}$. To prove this works, we split the integral into various pieces, consider several ranges of x, and reduce other ranges to the main range, where $\tau \in [S, a_n]$, for some fixed S. All the details appear in [33].

We note that for Erdős weights, or exponential weights on $[-1,1]$, the technical details are still more complicated — the subintervals I_j of the $[-a_n, a_n]$ are no longer of equal length. See [27], [92], [93].

5 Weights Close to $\exp(-|x|)$

The weight $\exp(-|x|)$ sits "on" the boundary of the class of weights admitting a positive solution to Bernstein's problem. That boundary is fuzzy, but if you recall that $W_\alpha(x) = \exp(-|x|^\alpha)$ admits a positive solution iff $\alpha \geq 1$, this makes sense. So it is not surprising that results like Jackson's theorem, tend to take a different form. Freud was interested in this boundary case right throughout his research on weighted polynomial approximation. In 1978, Freud, Giroux and Rahman [43, p. 360] proved that

$$E_n[f;W_1]_1 = \inf_{\deg(P)\leq n} \|(f-P)W_1\|_{L_1(\mathbb{R})} \leq C\left[\omega\left(f, \frac{1}{\log n}\right) + \int_{|x|\geq\sqrt{n}} |fW_1|(x)dx\right],$$

where

$$\omega(f,\varepsilon) = \sup_{|h|\leq\varepsilon} \int_{-\infty}^{\infty} |(fW_1)(x+h) - (fW_1)(x)|\,dx + \varepsilon\int_{-\infty}^{\infty} |fW_1|\,.$$

Here C is independent of f and n, and $\sqrt{n}$ could be replaced by $n^{1-\delta}$ for any fixed $\delta \in (0,1)$. Compare the $\frac{1}{\log n}$ inside the modulus to the $n^{-1+1/\alpha}$ we obtained for $W_\alpha, \alpha > 1$. This suggests that

$$\lim_{\alpha\to 1+} n^{-1+1/\alpha} = \frac{1}{\log n}, \tag{5.1}$$

at least in the sense of Jackson rates! Ditzian, the author, Nevai and Totik [34] later extended this result to a characterization in L_1. The technique used by Freud, Giroux and Rahman was essentially an L_1 technique, using the relation between one-sided weighted approximation, Gauss quadratures, and Christoffel functions — as we discussed in Section 4.1.

Only recently has it been possible to establish the analogous results in L_p, $p > 1$ [98]. The author modified the spline method discussed in Section 4.4 above. The challenge is that however you tweak the construction of $\{R_j\}$ there, it does not work. So a new idea was needed. As the peaking polynomials used there do not work for W_1, they were replaced by the reproducing kernel for orthogonal polynomials for W_1^2, and in the proofs, the author needed bounds for these orthogonal polynomials, implied by recent work of Kriecherbauer and McLaughlin [68].

If we examine the modulus used in (3.13) for $W_\alpha, \alpha > 1$, we see that the interval $[-h^{\frac{1}{1-\alpha}}, h^{\frac{1}{1-\alpha}}]$ is no longer meaningful for $\alpha = 1$. It turns out to be replaced by $\left[-\exp\left(\frac{1-\varepsilon}{h}\right), \exp\left(\frac{1-\varepsilon}{h}\right)\right]$, for some fixed $\varepsilon \in (0,1)$. The modulus becomes

$$\begin{aligned} \omega_{r,p}(f, W_1, t) &= \sup_{0<h\le t} \|W_1 \left(\Delta_h^r f\right)\|_{L_p\left[-\exp\left(\frac{1-\varepsilon}{t}\right), \exp\left(\frac{1-\varepsilon}{t}\right)\right]} \\ &+ \inf_{\deg(P)\le r-1} \|(f-P)W_1\|_{L_p\left(\mathbb{R}\setminus\left[-\exp\left(\frac{1-\varepsilon}{t}\right)+1, \exp\left(\frac{1-\varepsilon}{t}\right)-1\right]\right)}. \end{aligned} \tag{5.2}$$

The author proved [98]:

Theorem 5.1 *For $0 < p \le \infty$, and $n \ge C_3$,*

$$E_n [f; W_1]_p \le C_1 \omega_{r,p}(f, W_1, \frac{1}{\log (C_2 n)}). \tag{5.3}$$

Here C_1, C_2 are independent of f and n.

While this may be a technical achievement, it is scarcely surprising, given that Freud, Giroux and Rahman already had the rate $O\left(\frac{1}{\log n}\right)$ in the L_1 case. What about a Jackson–Favard inequality? The "limit" (5.1) suggests that an analogue of (3.10) should have the form

$$E_n [f; W_1]_p \le \frac{C}{\log n} \|f' W_1\|_{L_p(\mathbb{R})}.$$

Remarkably enough this is false, and there is no Jackson–Favard inequality for W_1, not even if we replace $\frac{1}{\log n}$ by a sequence decreasing arbitrarily slowly to 0. More generally, we answered in [99] the question: which weights admit a Jackson type theorem, of the form (3.10), with $\{a_n/n\}_{n=1}^\infty$ replaced by some sequence $\{\eta_n\}_{n=1}^\infty$ with limit 0? We proved [99]:

Theorem 5.2 *Let $W : \mathbb{R} \to (0, \infty)$ be continuous. The following are equivalent:*

(a) *There exists a sequence $\{\eta_n\}_{n=1}^\infty$ of positive numbers with limit 0 and with the following property. For each $1 \le p \le \infty$, and for all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, we have*

$$\inf_{\deg(P)\le n} \|(f-P)W\|_{L_p(\mathbb{R})} \le \eta_n \|f'W\|_{L_p(\mathbb{R})}, \qquad n \ge 1. \tag{5.4}$$

(b) *Both*

$$\lim_{x\to\infty} W(x) \int_0^x W^{-1} = 0 \tag{5.5}$$

and

$$\lim_{x\to\infty} W(x)^{-1} \int_x^\infty W = 0, \tag{5.6}$$

with analogous limits as $x \to -\infty$.

Two fairly direct corollaries of this are:

Corollary 5.3 *Let $W : \mathbb{R} \to (0, \infty)$ be continuous, with $W = e^{-Q}$, where $Q(x)$ is differentiable for large $|x|$, and*

$$\lim_{x\to\infty} Q'(x) = \infty \text{ and } \lim_{x\to-\infty} Q'(x) = -\infty. \tag{5.7}$$

Then there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 such that for each $1 \leq p \leq \infty$, and for all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, we have (5.4).

Corollary 5.4 *Let $W : \mathbb{R} \to (0, \infty)$ be continuous, with $W = e^{-Q}$, where $Q(x)$ is differentiable for large $|x|$, and $Q'(x)$ is bounded for large $|x|$. Then for both $p = 1$ and $p = \infty$, there does not exist a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 satisfying (5.4) for all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite.*

In particular for W_1, there is no Jackson–Favard inequality, since both (5.5) and (5.6) are false. Thus there is a real difference between density of weighted polynomials, and weighted Jackson–Favard theorems. It is possible to have the former without the latter.

A key ingredient in the above theorem is an estimate for tails [99]:

Theorem 5.5 *Assume that $W : \mathbb{R} \to (0, \infty)$ is continuous.*

(a) *Assume W satisfies (5.5) and (5.6), with analogous limits at $-\infty$. Then there exists a decreasing positive function $\eta : [0, \infty) \to (0, \infty)$ with limit 0 at ∞ such that for $1 \leq p \leq \infty$ and $\lambda \geq 0$,*

$$\|fW\|_{L_p(\mathbb{R}\setminus[-\lambda,\lambda])} \leq \eta(\lambda) \|f'W\|_{L_p(\mathbb{R})} \tag{5.8}$$

for all absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(0) = 0$ and the right-hand side is finite.

(b) *Conversely assume that (5.8) holds for $p = 1$ and for $p = \infty$, for large enough λ. Then the limits (5.5) and (5.6) in Theorem 1.1 are valid, with analogous limits at $-\infty$.*

The above results deal with L_p for all $1 \leq p \leq \infty$. What happens if we focus on a single L_p space? We recently proved [100]:

Theorem 5.6 *Let $W : \mathbb{R} \to (0, \infty)$ be continuous and let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:*

(a) *There exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 and with the following property. For all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, we have*

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})}, \qquad n \geq 1. \tag{5.9}$$

(b)

$$\lim_{x\to\infty} \left\|W^{-1}\right\|_{L_q[0,x]} \|W\|_{L_p[x,\infty)} = 0, \tag{5.10}$$

with analogous limits as $x \to -\infty$.

As a consequence one can construct weights that admit a Jackson theorem in L_p but not in L_r for any $1 \le p, r \le \infty$ with $p \ne r$:

Theorem 5.7 *Let $1 \le p, r \le \infty$ with $p \ne r$. There exists $W : \mathbb{R} \to (0, \infty)$ such that*

$$\frac{1}{1+x^2} \le W(x)/\exp(-x^2) \le 1 + x^2, \qquad x \in \mathbb{R},$$

and W admits an L_r Jackson theorem , but not an L_p Jackson theorem. That is, there exists $\{\eta_n\}_{n=1}^{\infty}$ with limit 0 at ∞ satisfying (5.9) in the L_r norm, but there does not exist such a sequence satisfying (5.9) in the L_p norm.

This is a highly unusual occurrence in weighted approximation — in fact the first occurrence of this phenomenon known to this author. Density of polynomials, and the degree of approximation is almost invariably the same for any L_p space (suitably weighted of course). Recall Koosis' remark quoted after Theorem 1.6 [67, pp. 210–211].

6 Restricted Range Inequalities

We have already seen the role played by *restricted range inequalities* (often called **infinite-finite range inequalities**) in weighted Jackson theorems. Paul Nevai rated Freud's discovery of these as one of his most important contributions to weighted approximation theory. One is reminded of their import when one recalls that Dzrbasjan did not have these, and could only prove estimates in a fixed finite interval. This should not detract from admiration for the generality of Dzrbasjan's results, and the sophistication of his ideas, which are still being used.

Let's recap a little. Recall that Freud and Nevai proved that there are constants C_1 and C_2 such that for all polynomials P_n of degree at most n,

$$\|P_n W_\alpha\|_{L_p(\mathbb{R})} \le C_1 \|P_n W_\alpha\|_{L_p\left[-C_1 n^{1/\alpha}, C_1 n^{1/\alpha}\right]}. \tag{6.1}$$

The constants C_1 and C_2 can be taken independent of n, P_n and even the L_p parameter $p \in [1, \infty]$. Outside the interval $\left[-C_1 n^{1/\alpha}, C_1 n^{1/\alpha}\right]$, $P_n W_\alpha$ decays quickly to zero. For more general $W = \exp(-Q)$, one replaces $n^{1/\alpha}$, with Freud's number q_n, the positive root of the equation

$$n = q_n Q'(q_n).$$

If $xQ'(x)$ is positive, continuous, and strictly increasing in $(0, \infty)$, with limit 0 at 0, and ∞ at ∞, then q_n exists and is unique.

Theorem 6.1 *Assume that $xQ'(x)$ is positive, continuous, and strictly increasing in $(0, \infty)$, with limit 0 at 0, and ∞ at ∞. Then for $n \ge 1$ and polynomials P of degree $\le n$,*

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-4q_{2n}, 4q_{2n}]}. \tag{6.2}$$

Proof. This type of proof was used many times by Freud and Nevai; see also the monograph [136, p. 66]. Recall that if T_n is the classical Chebyshev polynomial, then for $|x| \geq 1$ and polynomials P of degree $\leq n$,

$$|P(x)| \leq T_n(|x|) \|P\|_{L_\infty[-1,1]} \leq (2|x|)^n \|P\|_{L_\infty[-1,1]} . \tag{6.3}$$

Scaling this to $[-q_{2n}, q_{2n}]$, and using the fact that Q is increasing in $(0, \infty)$ gives

$$|P(x)| \leq \left(\frac{2|x|}{q_{2n}}\right)^n \|P\|_{L_\infty[-q_{2n},q_{2n}]} \leq \left(\frac{2|x|}{q_{2n}}\right)^n W^{-1}(q_{2n}) \|PW\|_{L_\infty[-q_{2n},q_{2n}]} .$$

So

$$|PW|(x) \leq 2^n \frac{|x|^n W(x)}{q_{2n}^n W(q_{2n})} \|PW\|_{L_\infty[-q_{2n},q_{2n}]} .$$

Now if $x \geq 4q_{2n}$,

$$\log \frac{x^n W(x)}{q_{2n}^n W(q_{2n})} = \int_{q_{2n}}^{x} \frac{n - uQ'(u)}{u} du \leq -n \int_{q_{2n}}^{4q_{2n}} \frac{du}{u} = -n \log 4,$$

since $u \geq q_{2n} \Rightarrow uQ'(u) \geq 2n$. Substituting in above, gives for $x \geq 4q_{2n}$,

$$|PW|(x) \leq 2^{-n} \|PW\|_{L_\infty[-q_{2n},q_{2n}]} .$$

This certainly implies (6.2). $\square$

The proof clearly shows the exponential decay of PW outside the interval $[-q_{2n}, q_{2n}]$, which depends only on n. In L_p, any $p > 0$, a typical analogue is

$$\|PW\|_{L_p(\mathbb{R})} \leq (1 + e^{-Cn}) \|PW\|_{L_p[-Bq_{2n}, Bq_{2n}]}$$

where $C > 0$, and B is large enough.

These inequalities are sufficient for weighted Jackson and Bernstein theorems, but not for some of the deeper results such as Bernstein inequalities with endpoint effects, or convergence of orthogonal expansions, or Lagrange interpolation, let alone asymptotics of orthogonal polynomials. With such questions in mind, Mhaskar and Saff asked in the early 1980's: *where does the sup or L_p norm of a weighted polynomial really live?* Clearly it is inside the intervals $[-Bq_{2n}, B_{q_{2n}}]$, with appropriate B, but what is the best B? In seminal papers [138], [139], [140], they used potential theory to obtain the answer. In another seminal development, E. A. Rakhmanov a little earlier [166] developed the same potential theory in order to investigate asymptotics of orthogonal polynomials.

One of the best ways to understand their work is to recall Bernstein's bound for growth of polynomials in the complex plane. For $n \geq 1$, polynomials P of degree $\leq n$, and $z \notin [-1, 1]$,

$$|P(z)| \leq \left|z + \sqrt{z^2 - 1}\right|^n \|P\|_{L_\infty[-1,1]} .$$

Thus once we have a bound on P on an interval, we can estimate its growth outside. Of course, it is related to the inequality involving T_n that we used above, but that works only for real x.

Now let us look for a weighted analogue for even weights $W = \exp(-Q)$, where, say, $x \mapsto xQ'(x)$ is increasing, positive, and continuous in $(0, \infty)$. Suppose that $a > 0$, and we have a function $G = G_{n,a}$ with the following properties:

(I) G is analytic and non-vanishing in $\mathbb{C}\backslash[-a,a]$;

(II) $G(z)z^n$ has a finite limit as $|z|\to\infty$;

(III) $|G(z)|\to W(x)$ as $z\to x\in(-a,a)$.

Then given a polynomial P of degree $\leq n$, the function PG is analytic in $\mathbb{C}\backslash[-a,a]$, including at ∞, where it has a finite limit. Moreover, for $x\in(-a,a)$,

$$\lim_{z\to x}|PG|(z)=|PW|(x)\leq\|PW\|_{L_\infty[-a,a]}.$$

By the maximum modulus principle, we then obtain

$$|PG|(z)\leq\|PW\|_{L_\infty[-a,a]},\qquad z\notin[-a,a].$$

In particular, for real x with $|x|>a$,

$$|PW|(x)\leq\|PW\|_{L_\infty[-a,a]}W(x)/G_{n,a}(x).$$

For a given n, Mhaskar and Saff found the smallest $a=a_n$ for which

$$W(x)/G_{n,a}(x)<1\quad\text{ for all }|x|>a. \tag{6.4}$$

Clearly, we then have

$$|PW|(x)<\|PW\|_{L_\infty[-a_n,a_n]},\qquad |x|>a_n,$$

and the *Mhaskar–Saff identity*

$$\|PW\|_{L_\infty(\mathbb{R})}=\|PW\|_{L_\infty[-a_n,a_n]}.$$

Recall that a_n is called the *Mhaskar–Rakhmanov–Saff* number, and is the positive root of the equation

$$n=\frac{2}{\pi}\int_0^1\frac{a_ntQ'(a_nt)}{\sqrt{1-t^2}}dt. \tag{6.5}$$

For the weight $W_\alpha(x)=\exp(-|x|^\alpha),\alpha>0$,

$$a_n=\left\{2^{\alpha-2}\frac{\Gamma(\alpha/2)^2}{\Gamma(\alpha)}\right\}^{1/\alpha}n^{1/\alpha}.$$

How does this arise and how does one find the function $G_{n,a}$? We solve the integral equation

$$\int_{-1}^1\log|x-t|\,\mu_{n,a}(t)dt=\frac{Q(ax)}{n}+c_{n,a},\qquad x\in[-1,1], \tag{6.6}$$

for the function $\mu_{n,a}$, and some constant $c_{n,a}$, subject to the condition

$$\int_{-a}^a\mu_{n,a}=1.$$

By (formal) differentiation, we see that this is really equivalent to solving a singular integral equation

$$\int_{-1}^{1} \frac{\mu_{n,a}(t)}{x-t} dt = \frac{aQ'(ax)}{n}, \qquad x \in (-a,a).$$

The integral on the left is a Cauchy principal value integral. There is a well developed theory of such equations [47], [167]. One can show that [111, p. 37], [136, p. 124 ff.]

$$\mu_{n,a}(x) = \frac{2}{\pi^2} \int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1-s^2}} \frac{asQ'(as) - axQ'(ax)}{n(s^2-x^2)} ds + \frac{B_{n,a}}{\pi\sqrt{1-x^2}}, \tag{6.7}$$

where

$$B_{n,a} = 1 - \frac{2}{n\pi} \int_0^1 \frac{atQ'(at)}{\sqrt{1-t^2}} dt. \tag{6.8}$$

Then the function $G_{n,a}$ is given by

$$G_{n,a}(z) = \exp\left(-n \int_{-1}^{1} \log\left(\frac{z}{a} - t\right) \mu_{n,a}(t) dt + nc_{n,a}\right)$$

so that for $x \in [-a,a]$,

$$|G_{n,a}(x)| = \exp\left(-n \int_{-1}^{1} \log\left|\frac{x}{a} - t\right| \mu_{n,a}(t) dt + nc_{n,a}\right) = W(x),$$

by (6.6). So at least we have verified (III) above. It turns out that whenever $B_{n,a} > 0$, then $W/G_{n,a} > 1$ in some right neighborhood of 1. Thus we look for the smallest a for which $B_{n,a} = 0$. We see from (6.8) that this requires

$$1 - \frac{2}{n\pi} \int_0^1 \frac{atQ'(at)}{\sqrt{1-t^2}} dt = 0 \quad \Longrightarrow \quad a = a_n.$$

This scratches the surface of an extensive theory. The monograph of Saff and Totik [167] contains a detailed and deep development. A more elementary treatment is provided in [136], though that will be more than sufficient for those interested only in the topics of this survey. For general exponential weights, restricted range inequalities in quite precise form are investigated in [85]. Here is a typical result [85, Theorem 1.8, p. 15], [136, Thm. 6.2.4, p. 142]:

Theorem 6.2 *Let $Q : \mathbb{R} \to [0,\infty)$ be even and convex, with limit ∞ at ∞ and*

$$0 = Q(0) < Q(x), \qquad x \neq 0.$$

(a) *For not identically zero polynomials P of degree $\leq n$,*

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n,a_n]}$$

and

$$\|PW\|_{L_\infty(\mathbb{R}\setminus[-a_n,a_n])} < \|PW\|_{L_\infty[-a_n,a_n]}.$$

(b) *Let $0 < p < \infty$ and P be a not identically zero polynomial of degree $\leq n - \frac{2}{p}$. Then*

$$\|PW\|_{L_p(\mathbb{R})} < 2^{1/p} \|PW\|_{L_p[-a_n,a_n]}$$

and

$$\|PW\|_{L_p(\mathbb{R}\setminus[-a_n,a_n])} < \|PW\|_{L_p[-a_n,a_n]} \,.$$

Quite often, we need a smaller estimate for the tail, and this is possible provided we omit a little more than the Mhaskar–Saff interval. Here is what can be achieved [85, Thm. 1.9, p. 15] for the class of weights $\mathcal{F}^*$, specified in Definition 4.6:

Theorem 6.3 *Let $W \in \mathcal{F}^*$ and $0 < p \leq \infty$. For $n \geq 1, \kappa \in (0,1]$ and polynomials P of degree $\leq n$,*

$$\|PW\|_{L_p(\mathbb{R}\setminus[-a_n(1+\kappa),a_n(1+\kappa)])} \leq C_1 \exp(-nC_2\kappa^{3/2}) \|PW\|_{L_p[-a_n,a_n]} \,.$$

Here C_1 and C_2 are independent of n, P, κ.

More generally, it was proved there

Theorem 6.4 *Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0,\infty)$ is even and convex, with Q having limit ∞ at ∞, and*

$$T(t) = \frac{tQ'(t)}{Q(t)}$$

is quasi-increasing in the sense of (3.27) of Definition 3.11. Assume furthermore that for some $\Lambda > 1$, $T \geq \Lambda$ in $(0,\infty)$, while

$$T(y) \sim T\left(y\left[1 - \frac{1}{T(y)}\right]\right), \qquad y \in (0,\infty)\,. \tag{6.9}$$

Then for $n \geq 1, \kappa \in (0, \frac{1}{T(a_n)}]$ and polynomials P of degree $\leq n$,

$$\|PW\|_{L_p(\mathbb{R}\setminus[-a_n(1+\kappa),a_n(1+\kappa)])} \leq C_1 \exp(-C_2 nT(a_n)\kappa^{3/2}) \|PW\|_{L_p[-a_n,a_n]} \,. \tag{6.10}$$

Here C_1 and C_2 are independent of n, P, κ.

Finally, we can go back a little inside the Mhaskar–Saff interval if we allow a cruder estimate on the tail [85, Theorem 4.2(a), p. 96]:

Theorem 6.5 *Assume in addition to the hypotheses of Theorem 6.4 that Q'' exists and is positive in $(0,\infty)$, while for some $C > 0$, and large enough x,*

$$\frac{Q''(x)}{Q'(x)} \leq C\frac{Q'(x)}{Q(x)}.$$

Let $0 < p \leq \infty$ and $\lambda > 0$. Then there exists n_0 and C such that for $n \geq n_0$ and polynomials P of degree $\leq n$,

$$\|PW\|_{L_p(\mathbb{R})} \leq C \|PW\|_{[-a_n(1-\lambda\eta_n),a_n(1-\lambda\eta_n)]} \,, \tag{6.11}$$

where

$$\eta_n = (nT(a_n))^{-2/3} \,. \tag{6.12}$$

See [57], [61], [85], [134], [136], [145], [167] for further discussion of restricted range inequalities.

7 Markov–Bernstein Inequalities

Markov–Bernstein inequalities have already been mentioned above. They are the main ingredient of converse theorems of approximation, but also enter in many other contexts. For the unweighted case on $[-1,1]$, the classical Markov inequality asserts that

$$\left\|P'\right\|_{L_\infty[-1,1]} \le n^2 \left\|P\right\|_{L_\infty[-1,1]}, \tag{7.1}$$

for $n \ge 1$ and all polynomials P of degree $\le n$. The Bernstein inequality improves this as long as we stay away from the endpoints:

$$\left|P'(x)\right| \le \frac{n}{\sqrt{1-x^2}} \left\|P\right\|_{L_\infty[-1,1]}, \qquad x \in (-1,1), \tag{7.2}$$

for $n \ge 1$ and all polynomials P of degree $\le n$.

The weights $W_\alpha(x) = \exp(-|x|^\alpha), \alpha > 0$, already provide a lot of insight into the weighted case on the real line. The analogue of the Markov inequality is

$$\left\|P_n' W_\alpha\right\|_{L_p(\mathbb{R})} \le C \left\|P_n W_\alpha\right\|_{L_p(\mathbb{R})} \begin{cases} n^{1-1/\alpha}, & \alpha > 1 \\ \log(n+1), & \alpha = 1 \\ 1, & \alpha < 1 \end{cases}. \tag{7.3}$$

This is valid for all $n \ge 1$ and polynomials P_n of degree $\le n$. The constant C depends on $p \in (0,\infty]$ and α, but not on n or P. It's no accident that the factors arising are similar to those in the Jackson rates, and for $\alpha < 1$, the factor is bounded independent of n. For $\alpha \ge 2$, these inequalities were proved by Freud; for $1 < \alpha < 2$, by Eli Levin and the author; and for $\alpha \le 1$, by Paul Nevai and Vili Totik.

In order to generalize this for arbitrary Q, we should try to cast this estimate in a unified form. For $\alpha > 1$, we see that

$$n^{1-1/\alpha} = C_1 \frac{n}{a_n} = C_2 Q'(a_n),$$

where $Q(x) = |x|^\alpha$, and recall, $a_n = C_1 n^{1/\alpha}$. As we shall see, $Q'(a_n)$ is the correct factor whenever $W = \exp(-Q)$ and Q is even and grows faster than $|x|^\alpha$ for some $\alpha > 1$. For Freud weights, where Q is of polynomial growth, but still grows faster than $|x|^\alpha$, for some $\alpha > 1$, we can use $\frac{n}{a_n}$. However, for all polynomial rates of growth, including $|x|^\alpha, \alpha \le 1$, we can use

$$\int_1^{Cn} \frac{ds}{Q^{[-1]}(s)},$$

where if Q is strictly increasing and continuous on $[0,\infty)$, $Q^{[-1]}$ denotes its inverse. The most general L_∞ result for Q of polynomial growth, with this factor, is due to Kroó and Szabados [69, p. 48]:

Theorem 7.1 *Let $W = \exp(-Q)$, where Q is even, continuous, increasing in $(0,\infty)$, and twice differentiable for large enough x, with*

$$\liminf_{x\to\infty} \frac{xQ'(x)}{Q(x)} > 0, \tag{7.4}$$

and

$$\limsup_{x\to\infty} \frac{xQ''(x)}{Q'(x)} < \infty. \tag{7.5}$$

Then for $n \geq 1$ and all polynomials P of degree $\leq n$,

$$\left\|P'W\right\|_{L_\infty(\mathbb{R})} \leq \int_1^{Cn} \frac{ds}{Q^{[-1]}(s)} \left\|PW\right\|_{L_\infty(\mathbb{R})}.$$

Here $C \neq C(n,P)$.

The author and Eli Levin used the slightly more restrictive condition that $W \in \mathcal{F}^*$, so (4.18) holds. This implies, like (7.4) and (7.5), that Q grows faster than some positive power of x, but slower than some other positive power of x. Strangely enough, there does not seem to be a published form of this result in L_p. Indeed, extra difficulties arise in L_p — we shall discuss techniques for proving Markov–Bernstein inequalities in the next section. However, there are results that apply to L_p *separately* for the case where Q grows faster, or slower, than $|x|$. First, we record the former case [84, p. 231]:

Theorem 7.2 *Let $W \in \mathcal{F}^*$. Let $1 \leq p < \infty$. Then for $n \geq 1$ and all polynomials P of degree $\leq n$,*

$$\left\|P'W\right\|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n} \left\|PW\right\|_{L_p(\mathbb{R})}.$$

Here $C \neq C(n,P)$.

This was later extended to the case $0 < p < 1$ [85, Corollary 1.16, p. 21]. For weights growing slower than $|x|$, Nevai and Totik [151, Thm. 2, p. 122] used a beautiful method to prove:

Theorem 7.3 *Let $W = \exp(-Q)$, where Q is even, increasing, and concave in $(0,\infty)$, with*

$$\int_0^\infty \frac{Q(x)}{1+x^2}dx < \infty.$$

Then for $n \geq 1$ and all polynomials P of degree $\leq n$,

$$\left\|P'W\right\|_{L_\infty(\mathbb{R})} \leq C \left\|PW\right\|_{L_\infty(\mathbb{R})}.$$

Here $C \neq C(n,P)$. If in addition,

$$\lim_{x\to\infty} \frac{2Q(x) - Q(2x)}{\log x} = \infty,$$

then, given $p > 0$, we have for $n \geq 1$ and all polynomials P of degree $\leq n$,

$$\left\|P'W\right\|_{L_p(\mathbb{R})} \leq C \left\|PW\right\|_{L_p(\mathbb{R})}.$$

Here $C \neq C(n,P)$.

The above are all Markov inequalities, analogues of (7.3). There are also Bernstein inequalities, which reflect the opposite feature of the finite interval case. The growth in n *decreases* towards the end of the Mhaskar–Rakhmanov–Saff interval. To state these, we need the function

$$\phi_n(x) := \max\left\{1 - \frac{|x|}{a_n}, n^{-2/3}\right\}, \qquad x \in \mathbb{R}.$$

Eli Levin and the author proved [84, Thm. 1.1, p. 231]:

Theorem 7.4 *Let $W \in \mathcal{F}^*$ and let $1 \le p \le \infty$. Then for $n \ge 1$ and all polynomials P of degree $\le n$,*

$$\|(PW)'\phi_n^{-1/2}\|_{L_p(\mathbb{R})} \le C\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})}.$$

Here $C \ne C(n, P)$.

For $p = \infty$, this was proved in [82], and for $1 \le p < \infty$, in [84]. If $p = \infty$, we see that this has the consequence

$$\left|(PW)'(a_n)\right| \le C\frac{n^{2/3}}{a_n}\|PW\|_{L_\infty(\mathbb{R})},$$

when P has degree $\le n$. There is in [84, Thm. 1.2, p. 233] an analogue of this for weights satisfying (4.18) in Definition 4.6 with $A > 0$ only (such as $\exp(-|x|^\alpha), \alpha \le 1$). One fixes $\eta \in (0,1)$, and proves that

$$\|(PW)'\phi_n^{-1/2}\|_{L_p(|x| \ge \eta a_n)} \le C\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})}.$$

Note that in the Bernstein inequalities, it is essential that we estimate $(PW)'$, and not $P'W$. There is no improvement for the latter, at least in general, near $\pm a_n$.

Analogues of Markov and Bernstein inequalities have also been obtained for Erdős weights, where Q is of faster than polynomial growth, as well as for exponential weights on $(-1,1)$. They are also available for non-even exponential weights on a possibly asymmetric finite or infinite interval [85]. For simplicity, we quote only the even case. First, we define a suitable class of weights, which includes both the Freud and Erdős weights in the real line, and exponential weights on $(-1,1)$:

Definition 7.5 *Let $I = (-d, d)$ where $0 < d \le \infty$. Let $Q : I \to [0,\infty)$ be an even function with the following properties:*

(a) *Q' is continuous and positive in I and $Q(0) = 0$;*

(b) *Q'' exists and is positive in $(0, d)$;*

(c)

$$\lim_{t \to d-} Q(t) = \infty;$$

(d) *The function*

$$T(t) = \frac{tQ'(t)}{Q(t)}$$

is quasi-increasing in $(0, d)$, in the sense that

$$0 \le x < y < d \implies T(x) \le CT(y).$$

Moreover, for some $\Lambda > 1$,

$$T(t) \geq \Lambda > 1, \qquad t \in (0, d).$$

(e) *There exists* $C_1 > 0$ *such that*

$$\frac{|Q''(x)|}{Q'(x)} \leq C_1 \frac{Q'(x)}{Q(x)}, \qquad x \in (0, d).$$

Then we write $W \in \mathcal{F}_{even}\left(C^2\right)$. *If also, there exists* $c \in (0, d)$ *such that*

$$\frac{|Q''(x)|}{Q'(x)} \geq C_2 \frac{Q'(x)}{Q(x)}, \qquad x \in (c, d),$$

then we write $W \in \mathcal{F}_{even}\left(C^2+\right)$.

Examples of weights satisfying this are $W_\alpha, \alpha > 1$, as well as $W = \exp(-Q)$, where

$$Q(x) = \exp_\ell\left(|x|^\alpha\right) - \exp_\ell(0), \qquad x \in (-\infty, \infty), \tag{7.6}$$

with $\ell \geq 1, \alpha > 1$, and

$$\exp_\ell := \exp((\cdots \exp()))$$

denoting the ℓth iterated exponential. We set $\exp_0(x) := x$. Others include

$$Q(x) = \exp_\ell\left(\left(1 - x^2\right)^{-\alpha}\right) - \exp_\ell(1), \qquad x \in (-1, 1), \tag{7.7}$$

where now $\ell \geq 0$ and $\alpha > 0$.

Theorem 7.6 *Let* $0 < p \leq \infty$ *and* $W = e^{-Q} \in \mathcal{F}_{even}\left(C^2\right)$. *For* $n \geq 1$, *let*

$$\varphi_n(x) = \frac{a_n}{n} \frac{\left|1 - \frac{|x|}{a_{2n}}\right|}{\sqrt{\left|1 - \frac{|x|}{a_n}\right| + \left(nT\left(a_n\right)\right)^{-2/3}}}. \tag{7.8}$$

Then for $n \geq 1$ *and polynomials* P_n *of degree* $\leq n$,

$$\left\|(PW)'\varphi_n\right\|_{L_p(\mathbb{R})} \leq C \left\|PW\right\|_{L_p(\mathbb{R})}.$$

Moreover,

$$\left\|P'W\right\|_{L_p(\mathbb{R})} \leq C \frac{n}{a_n} T(a_n)^{1/2} \left\|PW\right\|_{L_p(\mathbb{R})}.$$

This result is a special case of Theorem 1.15 and Corollary 1.16 in [85, p. 21]. There the restrictions on Q are weaker, but the definition of the classes is more implicit, so we restrict ourselves to the smallest even case considered there. The main feature is the extra factor $T(a_n)^{1/2}$, which really is there — weighted extremal polynomials attain it, see [85].

As examples, Q of (7.6) has Markov factor [85, p. 30]

$$\frac{n}{a_n} T(a_n)^{1/2} \sim \frac{n}{\left(\log_\ell n\right)^{1/\alpha}} \left(\prod_{j=1}^{\ell} \log_j n\right)^{1/2}.$$

Here

$$\log_j := \log(\log(\cdots \log()))$$

denotes the jth iterated logarithm. For Q of (7.7) and $\ell = 0$, the Markov factor is [85, p. 32]

$$\frac{n}{a_n} T(a_n)^{1/2} \sim n^{\frac{2\alpha+2}{2\alpha+1}}$$

while if $\ell \geq 1$, it is [85, p. 34]

$$\frac{n}{a_n} T(a_n)^{1/2} \sim n \left(\log_\ell n\right)^{\frac{1}{2}\left(1+\frac{1}{\alpha}\right)} \left(\prod_{j=1}^{\ell-1} \log_j n \right)^{1/2}.$$

8 Methods to Prove Markov–Bernstein Inequalities

In this section, we outline some of the methods that have been used to prove Markov–Bernstein inequalities for exponential weights. An extensive treatment of Markov–Bernstein inequalities for both weighted and unweighted cases is given in [9], [143]. We begin with

8.1 Freud's Method via de la Vallée Poussin Means

Recall that in Lemma 4.5 we proved

$$\left\| V_n[f] W \right\|_{L_\infty(\mathbb{R})} \leq C \left\| fW \right\|_{L_\infty(\mathbb{R})},$$

where $C \neq C(n, f)$. Thereafter, at (4.17), we showed how duality can be used to prove

$$\left\| V_n[f] W \right\|_{L_1(\mathbb{R})} \leq C \left\| fW \right\|_{L_1(\mathbb{R})},$$

where $C \neq C(n, f)$. In much the same way, one can prove that when $W \in \mathcal{F}^*$,

$$\left\| V_n'[f] W \right\|_{L_\infty(\mathbb{R})} \leq C \frac{n}{a_n} \left\| fW \right\|_{L_\infty(\mathbb{R})}. \tag{8.1}$$

The main new technical ingredient required is the estimate

$$1 \Big/ \sum_{k=0}^{n-1} p_k'(\xi)^2 \geq C \left(\frac{a_n}{n} \right)^3 W^2(\xi), \qquad \xi \in \mathbb{R}. \tag{8.2}$$

This is a cousin of the Christoffel function estimate (4.13) and can be proved using much the same ideas. Freud did this in [42]. A proof is also given in [136, p. 64]. Now if P is a polynomial of degree $\leq n$, then

$$V_n[P] = P.$$

So (8.1) immediately gives the L_∞ Markov–Bernstein inequality

$$\left\| P'W \right\|_{L_\infty(\mathbb{R})} \leq C \frac{n}{a_n} \left\| PW \right\|_{L_\infty(\mathbb{R})}.$$

To extend this to L_1, we use duality on V_n':

$$\left\| V_n'[f]\, W \right\|_{L_1(\mathbb{R})} = \sup \int_{-\infty}^{\infty} V_n'[f]\, g W^2, \tag{8.3}$$

where the sup is taken over all measurable functions g with $\|gW\|_{L_\infty(\mathbb{R})} \le 1$. Next, we use the fact that if P is a polynomial of degree $\le 2n$, then

$$\int_{-\infty}^{\infty} (g - V_{4n}[g])\, PW^2 = 0.$$

Indeed, we can prove this by considering the special case $P = p_j$, $0 \le j \le 2n$, and recalling that the first $2n$ Fourier series coefficients (with respect to $\{p_j\}$) of $V_{4n}[g]$ are the same as those of g. The above considerations and an integration by parts give

$$\begin{aligned} \int_{-\infty}^{\infty} V_n'[f]\, g W^2 &= \int_{-\infty}^{\infty} V_n'[f]\, V_{4n}[g]\, W^2 \\ &= -\int_{-\infty}^{\infty} V_n[f]\, V_{4n}'[g]\, W^2 + 2\int_{-\infty}^{\infty} V_n[f]\, V_{4n}[g]\, Q'W^2. \end{aligned} \tag{8.4}$$

Here

$$\begin{aligned} \left| \int_{-\infty}^{\infty} V_n[f]\, V_{4n}'[g]\, W^2 \right| &\le \left\| V_{4n}'[g]\, W \right\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} |V_n[f]|\, W \\ &\le C\frac{n}{a_n} \|gW\|_{L_\infty(\mathbb{R})} \|fW\|_{L_1(\mathbb{R})} \le C\frac{n}{a_n} \|fW\|_{L_1(\mathbb{R})}. \end{aligned} \tag{8.5}$$

In the second last line, we used our L_1 bound (4.17) for V_n, and the L_∞ bound (8.1) for V_n', and in the last line we used our bound on the sup norm of gW. To bound the second term in (8.4), one needs

Lemma 8.1 *Let h be absolutely continuous with $h(0) = 0$. Then*

$$\left\| Q'hW \right\|_{L_\infty(\mathbb{R})} \le C \left\| h'W \right\|_{L_\infty(\mathbb{R})}. \tag{8.6}$$

Proof. Observe that if $x > 0$,

$$\begin{aligned} \left| Q'(x)h(x)W(x) \right| &= \left| Q'(x)W(x) \int_0^x h'(t)dt \right| \\ &\le \left\| h'W \right\|_{L_\infty(\mathbb{R})} \left| Q'(x)W(x) \int_0^x W^{-1}(t)dt \right|. \end{aligned} \tag{8.7}$$

We now assume that Q' is increasing, and

$$\lim_{t\to\infty} \frac{Q''(t)}{Q'(t)^2} = 0.$$

This latter condition is true for regularly behaved Freud and Erdős weights such as $\mathcal{F}^*, \mathcal{E}$. Choose $A > 0$ such that

$$\frac{Q''(t)}{Q'(t)^2} \le \frac{1}{2}, \qquad t \ge A.$$

If $x \geq A$, an integration by parts gives

$$\begin{aligned}\int_A^x W^{-1}(t)dt &= \int_A^x Q'(t)^{-1}Q'(t)W^{-1}(t)dt \\ &= Q'(t)^{-1}W^{-1}(t)\Big|_{t=A}^{t=x} + \int_A^x \frac{Q''(t)}{Q'(t)^2}W^{-1}(t)dt \\ &\leq Q'(x)^{-1}W^{-1}(x) + \frac{1}{2}\int_A^x W^{-1}(t)dt.\end{aligned}$$

Then

$$\int_A^x W^{-1}(t)\ dt \leq 2Q'(x)^{-1}W^{-1}(x)$$

and

$$x \geq A \Rightarrow Q'(x)W(x)\int_A^x W^{-1}(t)\ dt \leq 2.$$

As $Q'W$ is bounded in $(0,\infty)$, we obtain

$$x \geq 0 \Rightarrow Q'(x)W(x)\int_0^x W^{-1}(t)\ dt \leq C.$$

A similar bound holds for $x < 0$, and then we obtain (8.6) from (8.7). $\square$

Next, we apply (8.6) to the second term in the right-hand side of (8.4), with $h = V_{4n}[g] - V_{4n}[g](0)$. We obtain

$$\begin{aligned}\left|V_{4n}[g]\,Q'W\right|(x) &\leq |V_{4n}[g](x) - V_{4n}[g](0)|\left|Q'(x)\right|W(x) + V_{4n}[g](0)\left|Q'(x)\right|W(x) \\ &\leq C\|V'_{4n}[g]\,W\|_{L_\infty(\mathbb{R})} + C\,|V_{4n}[g](0)| \leq C\frac{n}{a_n}\|gW\|_{L_\infty(\mathbb{R})} \leq C\frac{n}{a_n},\end{aligned}$$

so

$$\left|\int_{-\infty}^{\infty} V_n[f]\,V_{4n}[g]\,Q'W^2\right| \leq C\frac{n}{a_n}\left|\int_{-\infty}^{\infty} V_n[f]\,W\right| \leq C\frac{n}{a_n}\|fW\|_{L_1(\mathbb{R})}. \tag{8.8}$$

Combining (8.4), (8.5), and (8.8) gives

$$\|V'_n[f]\,W\|_{L_1(\mathbb{R})} \leq C\frac{n}{a_n}\|fW\|_{L_1(\mathbb{R})}.$$

Thus, recalling (8.1), we have that for both $p = 1$ and $p = \infty$,

$$\|V'_n[f]\,W\|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n}\|fW\|_{L_p(\mathbb{R})}.$$

By interpolation, we obtain

$$\|V'_n[f]\,W\|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n}\|fW\|_{L_p(\mathbb{R})},$$

for all $1 \leq p \leq \infty$, where $C \neq C(n, P)$. Applying this with $f = P$, a polynomial of degree $\leq n$, gives

$$\|P'W\|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})}.$$

This method is elegant, and yields more than just Markov–Bernstein inequalities. As we have seen, we also obtain a proof of a Jackson–Favard type inequality, bounds on Cesàro means of orthogonal expansions, and other useful information. Our main technical ingredients were bounds on Christoffel functions and their derivative analogue (8.2), and some technical estimates involving Q'.

8.2 Replacing the Weight by a Polynomial

This method is very simple, but applies only to a limited class of weights. Suppose that for some fixed $K > 0$, we have polynomials S_n of degree $\leq Kn$, with

$$C_1 \leq S_n/W \leq C_2 \text{ in } [-2a_n, 2a_n], \tag{8.9}$$

and

$$\left|S_n'\right|/W \leq C_3\frac{n}{a_n} \text{ in } [-a_n, a_n]. \tag{8.10}$$

These polynomials enable us to reduce weighted Bernstein inequalities to classical unweighted Bernstein inequalities. If P is a polynomial of degree $\leq n$, then our restricted range inequalities give

$$\begin{aligned}
\left\|P'W\right\|_{L_p(\mathbb{R})} &\leq C\left\|P'W\right\|_{L_p[-a_n,a_n]} \leq CC_1^{-1}\left\|P'S_n\right\|_{L_p[-a_n,a_n]} \\
&\leq CC_1^{-1}\left[\left\|(PS_n)'\right\|_{L_p[-a_n,a_n]} + \left\|PS_n'\right\|_{L_p[-a_n,a_n]}\right] \\
&\leq CC_1^{-1}\left[\frac{2n}{a_n}\left\|PS_n\right\|_{L_p[-2a_n,2a_n]} + \frac{n}{a_n}\left\|PW\right\|_{L_p[-a_n,a_n]}\right],
\end{aligned}$$

by our hypotheses on S_n', and the classical Bernstein inequality, scaled from $[-1, 1]$ to $[-2a_n, 2a_n]$. Using (8.9) again, we obtain

$$\|P'W\|_{L_p(\mathbb{R})} \leq C\left\|PW\right\|_{L_p(\mathbb{R})}.$$

This works in any L_p, $0 < p \leq \infty$. If $p = \infty$, we can weaken the requirements on S_n, which can be made different for each x. We only need the upper bound on S_n' at a given x, and the lower bound on S_n at that x (but we still need the upper bound on S_n throughout $[-2a_n, 2a_n]$).

Freud and Nevai used this for weights like $W_{2m}(x) = \exp(-x^{2m})$, where $m \geq 1$ is a positive integer. The partial sums of these entire weights can be used for $\{S_n\}$. For $W_\alpha, \alpha > 1$, Eli Levin and the author used canonical products of Weierstrass primary factors, such as

$$\prod_{n=1}^{\infty} E\left(-x/n^{1/\alpha}; \ell\right),$$

to generate these polynomials [80], [81]. This method can provide quick easy proofs in special cases, which would be useful for teaching a course on weighted approximation.

8.3 Nevai–Totik's Method

This involves fast decreasing polynomials, a topic initiated by Kamen Ivanov, Paul Nevai and Vili Totik, and works well for slowly decreasing weights such as $W_\alpha, \alpha \le 1$. Let us suppose that for $n \ge 1$, and some $K > 0$, we have polynomials $S_n^{\#}$ that satisfy

$$S_n^{\#}(0) = 1, \qquad S_n^{\#\prime}(0) = 0, \tag{8.11}$$

and

$$\left|S_n^{\#}(x)\right| \le Ke^{-Q(x)}, \qquad x \in [-a_n, a_n]\,. \tag{8.12}$$

Typically, $S_n^{\#}$ has a unique maximum in $[-1,1]$ at 0, and decreases rapidly in $(0,1)$. Nevai and Totik [151] used a construction of Marchenko to generate such polynomials. One starts with an entire function of the form

$$B(z) = \prod_{k\ge 1} \frac{\sin^2\left(\frac{\pi}{2}\sqrt{1+\left(\frac{z}{t_k}\right)^2}\right)}{1+\left(\frac{z}{t_k}\right)^2}.$$

The product may be finite or infinite, and the $\{t_k\}$ are positive numbers with

$$T = \sum_k \frac{1}{t_k} < \infty.$$

Assuming that Q is even, increasing on $(0,\infty)$, and

$$\int_0^\infty \frac{Q(x)}{1+x^2}dx < \infty, \tag{8.13}$$

one can choose $\{t_k\}$ such that $T \le 1/\pi$ and

$$|B(x)| \le K\exp(-Q(x)), \qquad x \in \mathbb{R}.$$

In this case B is an entire function of exponential type ≤ 1. Assuming (8.13), Nevai and Totik used the partial sums of B to construct polynomials P_n of degree $\le n$, with a local maximum at 0, and

$$P_n(0) = 1 \text{ and } |P_n(x)| \le K\exp(-Q(nx)), \qquad x \in [-1,1]\,. \tag{8.14}$$

Then

$$S_n^{\#}(x) = P_{[a_n]}\left(\frac{x}{[a_n]}\right)$$

satisfies (8.11) and (8.12), and is of degree $\le [a_n]$. It is easy to derive the Markov inequality at 0:

$$\left|(P'W)(0)\right| \;=\; \left|P'(0)\right| = \left|(PS_n^{\#})'(0)\right| \le \frac{n+\lfloor a_n \rfloor}{a_n}\|PS_n^{\#}\|_{L_\infty[-a_n,a_n]},$$

by the usual unweighted Bernstein inequality. We continue this as

$$\left|(P'W)(0)\right| \le C\,\|PW\|_{L_\infty[-a_n,a_n]}\,, \tag{8.15}$$

using the property (8.12), and the fact that the convergence (8.13) implies $n = O(a_n)$, which we shall not prove. To extend this to general $x \geq 0$, we use the evenness and concavity of Q, which implies that for $x, y \in \mathbb{R}$,

$$W(x)W(y) \leq W(x+y).$$

Now apply (8.15) to the polynomial $R(y) = P(x+y)W(x)$, for fixed x:

$$\begin{aligned} \left|(P'W)(x)\right| &= \left|R'(0)W(0)\right| \leq C \left\|RW\right\|_{L_\infty(\mathbb{R})} = C \sup_{y\in\mathbb{R}} \left|P(x+y)W(x)W(y)\right| \\ &\leq C \sup_{y\in\mathbb{R}} \left|P(x+y)W(x+y)\right| = C \left\|PW\right\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

This method can be modified to give the correct Markov inequality for W_1. Nevai and Totik were the first to do so.

8.4 Dzrbasjan/Kroó–Szabados' Method

Dzrbasjan was apparently the first researcher to investigate the degree of approximation for general exponential weights, in his 1955 paper [37]. Although his approximation estimates worked only on a finite interval, he nevertheless came up with a great many ideas. Kroó and Szabados subsequently used Dzrbasjan's method, and restricted range inequalities to establish Theorem 7.1.

We start with Cauchy's integral formula for derivatives (or, if you prefer, Cauchy's estimates):

$$\left|P'(x)\right| = \left|\frac{1}{2\pi i}\int_{|t-x|=\varepsilon} \frac{P(t)}{(t-x)^2}dt\right| \leq \frac{1}{\varepsilon} \sup\left\{\left|P(t)\right| : |t-x| = \varepsilon\right\}.$$

The number $1/\varepsilon$ is invariably chosen as the size of the Markov–Bernstein factor. To estimate $P(t)$ in terms of the values of PW on the real line, we write $t = u + iv$, and use an inequality that often arises in the theory of functions analytic in the upper half-plane. We already used one form of this in the proof of Lemma 2.2.

$$\begin{aligned} \log\left|P(u+iv)\right| &\leq \frac{|v|}{\pi}\int_{-\infty}^{\infty} \frac{\log\left|P(s)\right|}{(s-u)^2+v^2}ds \leq \frac{|v|}{\pi}\int_{-\infty}^{\infty} \frac{\log M + Q(s)}{(s-u)^2+v^2}ds \\ &= \log M + \frac{|v|}{\pi}\int_{-\infty}^{\infty} \frac{Q(s)}{(s-u)^2+v^2}ds, \qquad (8.16) \end{aligned}$$

where

$$M := \left\|PW\right\|_{L_\infty(\mathbb{R})}.$$

Of course, we need to assume the integral converges, which is very restrictive. We shall assume more precisely that

$$\int_0^\infty \frac{Q(t)}{1+t^2}dt < \infty,$$

and that for $s \geq 1$,

$$Q(s) \sim sQ'(s) \sim s^2 Q''(s).$$

Then the above considerations show that

$$\log\left|P'W\right|(x) \leq \log\frac{1}{\varepsilon} + \log M + \sup_v \frac{|v|}{\pi}\int_{-\infty}^{\infty} \frac{Q(s)-Q(x)}{(s-u)^2+v^2}ds. \qquad (8.17)$$

The technical challenge is to estimate the integral in the last right-hand side. We break it into several pieces. If we assume $x \geq 1, 0 \leq u \leq 2x$ and $Q(x) \geq 0$, we see that

$$I_1 := \frac{|v|}{\pi} \int_{3u/2}^{\infty} \frac{Q(s) - Q(x)}{(s-u)^2 + v^2} ds \leq 9 \frac{\varepsilon}{\pi} \int_{3u/2}^{\infty} \frac{Q(s)}{s^2} ds,$$

and

$$I_2 := \frac{|v|}{\pi} \int_{-\infty}^{0} \frac{Q(s) - Q(x)}{(s-u)^2 + v^2} ds \leq 9 \frac{\varepsilon}{\pi} \int_{-\infty}^{0} \frac{Q(s)}{s^2} ds.$$

(Recall that $|v| \leq |t - x| \leq \varepsilon$.) In $\left[0, \frac{u}{2}\right]$, we see that the integrand is non-positive, as $s \leq u/2 \leq x$, so

$$I_3 := \frac{|v|}{\pi} \int_{0}^{u/2} \frac{Q(s) - Q(x)}{(s-u)^2 + v^2} ds \leq 0.$$

Finally, we handle the difficult central integral

$$\begin{aligned} I_4 &:= \frac{|v|}{\pi} \int_{u/2}^{3u/2} \frac{Q(s) - Q(x)}{(s-u)^2 + v^2} ds \\ &\leq \frac{|v|}{\pi} \int_{u/2}^{3u/2} \frac{Q(s) - Q(u)}{(s-u)^2 + v^2} ds + |Q(u) - Q(x)| \\ &=: I_{41} + I_{42}. \end{aligned}$$

Here

$$I_{41} = \frac{|v|}{\pi} \int_{u/2}^{3u/2} \frac{Q(s) - Q(u)}{(s-u)^2 + v^2} ds = \frac{|v|}{\pi} \int_{u/2}^{u} \frac{Q(r) - 2Q(u) + Q(2u - r)}{(r-u)^2 + v^2} dr.$$

For some ξ between r and u,

$$Q(r) - 2Q(u) + Q(2u - r) = Q''(\xi)(r - u)^2$$

and one can show that [69, p. 53]

$$Q''(\xi) \leq C \frac{Q(r)}{r^2},$$

so

$$I_{41} \leq C \frac{\varepsilon}{\pi} \int_{u/2}^{2u} \frac{Q(r)}{r^2} dr.$$

Also,

$$I_{42} = |Q(u) - Q(x)| = Q'(\xi) |u - x| \leq C Q'(x) \varepsilon \leq C \varepsilon \int_{x}^{2x} \frac{Q(s)}{s^2} ds.$$

Combining the above estimates, we have shown that for $x \geq 1$,

$$\log \left| P'W \right| (x) \leq \log \frac{1}{\varepsilon} + \log M + C \varepsilon \int_{1}^{\infty} \frac{Q(y)}{y^2 + 1} dy.$$

We simply choose $\varepsilon = 1$. Similar estimates hold over $(-\infty, -1]$, and the range $[-1, 1]$ is easy to handle. This leads to the Markov inequality

$$\|P'W\|_{L_\infty(\mathbb{R})} \leq C \|PW\|_{L_\infty(\mathbb{R})},$$

valid for all $n \geq 1$ and polynomials of degree $\leq n$.

When dealing with weights $W = \exp(-Q)$, where Q grows faster than $|x|^\alpha$, some $\alpha > 1$, one typically chooses

$$\varepsilon = \frac{a_n}{n},$$

uses restricted range inequalities, and needs more work to estimate the various integrals.

8.5 Levin–Lubinsky's Method

Like the previous method, we use Cauchy's integral formula for derivatives, and then go back to the real line. However, instead of using (8.16), we use the potential theoretic functions associated with exponential weights. These give analogues of the Bernstein–Walsh inequality for growth of polynomials in the complex plane. For $p < \infty$, going back from the plane to the real line is quite complicated, and requires Carleson measures. We shall outline the method for $1 \leq p < \infty$.

Fix $x \geq 0$, $\varepsilon > 0$, and define an entire function F_x by

$$F_x(z) := \exp(-Q(x) - Q'(x)(z - x)).$$

Observe that

$$F_x^{(j)}(x) = W^{(j)}(x) \text{ for } j = 0, 1.$$

We have

$$\begin{aligned} \left|(PW)'(x)\right| &= \left|(PF_x)'(x)\right| = \left|\frac{1}{2\pi i}\int_{|t-x|=\varepsilon} \frac{(PF_x)(t)}{(t-x)^2} dt\right| \\ &\leq \left(\frac{1}{2\pi}\int_{|t-x|=\varepsilon} |PF_x|(t)^p\,|dt|\right)^{1/p} \left(\frac{1}{2\pi}\int_{|t-x|=\varepsilon} |t-x|^{-2q}|dt|\right)^{1/q} \\ &= \frac{1}{\varepsilon}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |PF_x|^p(x + \varepsilon e^{i\theta})d\theta\right)^{1/p}. \end{aligned} \tag{8.18}$$

In the second last line, we used Hölder's inequality, and $q = \frac{p}{p-1}$ there. We shall choose

$$\varepsilon = \varepsilon\,(n, x) = \varphi_n(x),$$

where $\varphi_n(x)$ is given by (7.8). This guarantees [85, Lemma 10.6, p. 301] that

$$|F_x(z)| \leq CW\,(|z|) \qquad \text{for} \qquad |z - x| \leq \varphi_n(x). \tag{8.19}$$

The proof of this involves a Taylor series expansion of $Q(\operatorname{Re} z)$ about $Q(x)$. Now comes the potential theory bit. There is [85, Lemma 10.7, p. 303] a function G_n analytic in $\mathbb{C}\backslash\,[-a_n, a_n]$, with boundary values $G_n(x)$ on $[-a_n, a_n]$ from the upper half-plane satisfying

$$|G_n(x)| = W(x), \qquad x \in [-a_n, a_n].$$

Moreover, $z^n G_n(z)$ has a finite limit at ∞, and uniformly for $x \in [-a_{2n}, a_{2n}]$,

$$W(|z|) \leq C\,|G_n(z)|\,, \qquad |z-x| \leq \varphi_n(x). \tag{8.20}$$

One representation for G_n is

$$G_n(z) = \exp\left(-\int_{-a_n}^{a_n} \log(z-t)d\mu_n(t) - c_n\right),$$

where μ_n is the equilibrium measure of mass n for the weight W, and c_n is an equilibrium constant. Recall that we used something similar in deriving restricted range inequalities, in Section 6. Then we obtain from (8.18) to (8.20),

$$\varphi_n^p(x)\left|(PW)'(x)\right|^p \leq C\int_{-\pi}^{\pi} |PG_n|^p(x+\varphi_n(x)e^{i\theta})d\theta.$$

Integrating gives

$$\begin{aligned}\int_{-a_{2n}}^{a_{2n}} \varphi_n^p|(PW)'|^p &\leq C\int_{-a_{2n}}^{a_{2n}}\left[\int_{-\pi}^{\pi}|PG_n|^p(x+\varphi_n(x)e^{i\theta})d\theta\right]dx \\ &= C\int |PG_n|^p\,d\left[\nu_n^+ + \nu_n^-\right], \qquad (8.21)\end{aligned}$$

where ν_n^+ is a measure on the upper half-plane, and ν_n^- is a measure on the lower half-plane. For Borel measurable sets S, with characteristic function χ_S, we have

$$\begin{aligned}\nu_n^+(S) &= \int_{-a_{2n}}^{a_{2n}}\left[\int_0^{\pi}\chi_S(x+\varphi_n(x)e^{i\theta})d\theta\right]dx; \\ \nu_n^-(S) &= \int_{-a_{2n}}^{a_{2n}}\left[\int_{-\pi}^{0}\chi_S(x+\varphi_n(x)e^{i\theta})d\theta\right]dx.\end{aligned}$$

Next, we use a famous inequality of Carleson. We say a measure ν on the upper half-plane is a **Carleson measure** if there exists $A > 0$ such that for all squares K in the upper half-plane, with base on the real axis, and side $h > 0$,

$$\nu(K) \leq Ah.$$

The smallest such number A is called $N(\nu)$, the **Carleson norm** of ν. Let H^p denote the Hardy space in the upper half-plane, consisting of all functions analytic there, whose boundary values on the real axis lie in $L_p(\mathbb{R})$. For $f \in H^p$,

$$\int |f|^p d\nu \leq CN(\nu)\int_{-\infty}^{\infty}|f|^p.$$

(For $p < 1$, there is an analogous statement.)

Applying this to (8.21) gives

$$\int_{-a_{2n}}^{a_{2n}}\varphi_n^p\left|(PW)'\right|^p \leq C\left(N\left[\nu_n^+\right]+N\left[\nu_n^-\right]\right)\int_{-\infty}^{\infty}|PG_n|^p.$$

Here although ν_n^- is a measure on the lower half-plane, it is obvious what is meant by its Carleson norm. Next, one shows that

$$\sup_n \left(N\left[\nu_n^+\right] + N\left[\nu_n^-\right] \right) < \infty.$$

Finally,

$$\int_{-a_n}^{a_n} |PG_n|^p = \int_{-a_n}^{a_n} |PW|^p ,$$

and one can show using Hilbert transform estimates that

$$\int_{|x| \geq a_n} |PG_n|^p \leq C \int_{-a_n}^{a_n} |PW|^p ,$$

giving

$$\int_{-a_{2n}}^{a_{2n}} \varphi_n \left| (PW)' \right|^p \leq C \int_{-a_n}^{a_n} |PW|^p .$$

The integral over $\mathbb{R} \backslash [-a_{2n}, a_{2n}]$ may be handled using restricted-range inequalities.

This method is the deepest of those we presented — and it is the only one that gives the Bernstein inequalities in Theorem 7.4 and 7.6. For $p = \infty$, the proof is easier, as one can avoid Carleson measures [82], [133].

8.6 Sieved Markov–Bernstein Inequalities

The methods of this section have been extensively developed by Paul Nevai and others [105], [108], and illustrate the power of Jensen's inequality. On finite intervals, they go back at least to Zygmund [193]. The idea is to start with the Christoffel type estimates

$$(PW)^2(\xi) \quad \leq \quad C \frac{n}{a_n} \int_{-\infty}^{\infty} (PW)^2, \tag{8.22}$$

and

$$(P'W)^2(\xi) \quad \leq \quad C \left(\frac{n}{a_n} \right)^3 \int_{-\infty}^{\infty} (PW)^2, \tag{8.23}$$

valid for Freud weights W, $n \geq 1$, polynomials P of degree $\leq n$, and $\xi \in \mathbb{R}$. We used these already in Section 8.1.

We now extend these to L_p type Christoffel function estimates. Let $0 < p < 2$. From (8.22), we derive

$$\|PW\|_{L_\infty(\mathbb{R})}^2 \leq C \frac{n}{a_n} \|PW\|_{L_\infty(\mathbb{R})}^{2-p} \int_{-\infty}^{\infty} |PW|^p ,$$

and hence

$$\|PW\|_{L_\infty(\mathbb{R})}^p \leq C \frac{n}{a_n} \int_{-\infty}^{\infty} |PW|^p . \tag{8.24}$$

Next, (8.23) followed by (8.24) give

$$\begin{aligned} \left\| P'W \right\|_{L_\infty(\mathbb{R})}^2 &\leq C \left(\frac{n}{a_n} \right)^3 \|PW\|_{L_\infty(\mathbb{R})}^{2-p} \int_{-\infty}^{\infty} |PW|^p \\ &\leq C \left(\frac{n}{a_n} \right)^3 \left(\frac{n}{a_n} \int_{-\infty}^{\infty} |PW|^p \right)^{\frac{2-p}{p}} \int_{-\infty}^{\infty} |PW|^p . \end{aligned}$$

Rearranging gives

$$\left\|P'W\right\|^p_{L_\infty(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^{p+1}\int_{-\infty}^{\infty}|PW|^p. \tag{8.25}$$

Thus far we have (8.24) and (8.25) for $0 < p \leq 2$. Our weight is $W = \exp(-Q)$. We apply these inequalities instead to the weight $\exp(-rQ)$ for fixed $r > 0$. Its Mhaskar–Rakhmanov–Saff number is a multiple of that for $W = \exp(-Q)$. Since for any fixed $s > 0$,

$$a_{sn} \sim a_n \text{ uniformly in } n,$$

we obtain

$$\|PW^r\|^p_{L_\infty(\mathbb{R})} \leq C\frac{n}{a_n}\int_{-\infty}^{\infty}|PW^r|^p, \tag{8.26}$$

and

$$\left\|P'W^r\right\|^p_{L_\infty(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^{p+1}\int_{-\infty}^{\infty}|PW^r|^p, \tag{8.27}$$

for all $0 < p \leq 2$, $r > 0$, $n \geq 1$, and P of degree $\leq n$.

Now comes the sieving idea. Let L, M be positive integers and fix ξ. We apply these inequalities to the polynomial in t,

$$P(t) = S(t)\left(K_{Mn}(\xi, t)\right)^L, \tag{8.28}$$

where S has degree $\leq n$, and

$$K_n(\xi, t) = \sum_{j=0}^{n-1} p_j(\xi)p_j(t)$$

is the nth reproducing kernel for the weight W^2. As

$$\frac{(LM+1)n}{a_{(LM+1)n}} \sim \frac{n}{a_n},$$

uniformly in n, we obtain from (8.26),

$$\left\{\left|S(\xi)K^L_{Mn}(\xi,\xi)\right| W^r(\xi)\right\}^p \leq C\frac{n}{a_n}\int_{-\infty}^{\infty}\left|(SW^r)(t)K_{Mn}(\xi,t)^L\right|^p dt.$$

Here, for Freud weights $W \in \mathcal{F}^*$ [83],

$$K_n(\xi,\xi) = 1/\lambda_n(W^2,\xi) \sim \frac{n}{a_n}W^{-2}(\xi), \qquad |\xi| \leq \frac{1}{2}a_n, \tag{8.29}$$

while

$$K_n(\xi,\xi) = 1/\lambda_n(W^2,\xi) \leq C\frac{n}{a_n}W^{-2}(\xi), \qquad \xi \in \mathbb{R}, \tag{8.30}$$

so if M is so large that

$$\frac{1}{2}a_{Mn} \geq 2a_n,$$

we obtain for all polynomials S of degree $\leq n$, and all $|\xi| \leq 2a_n$,

$$|S(\xi)|^p W^{(r-2L)p}(\xi) \leq C\left(\frac{n}{a_n}\right)^{1-Lp}\int_{-\infty}^{\infty}\left|(SW^r)(t)K_{Mn}(\xi,t)^L\right|^p dt.$$

Now assume that L is chosen so large that $Lp > 2$. By Cauchy–Schwarz, for all n and $t, \xi \in \mathbb{R}$,

$$\begin{aligned} |K_{Mn}(\xi,t)|^{Lp-2} &\leq \left(K_{Mn}(\xi,\xi)^{1/2}K_{Mn}(t,t)^{1/2}\right)^{Lp-2} \\ &\leq C\left(\frac{n}{a_n}\right)^{Lp-2}(W^{-1}(\xi)W^{-1}(t))^{Lp-2}. \end{aligned} \tag{8.31}$$

So

$$|S(\xi)|^p W^{(r-L)p-2}(\xi) \leq C\frac{a_n}{n}\int_{-\infty}^{\infty}|S(t)|^p K_{Mn}(\xi,t)^2 W^{(r-L)p+2}(t)dt. \tag{8.32}$$

We now choose $r = L + 1$, giving

$$|(SW)(\xi)|^p \leq C\frac{a_n}{n}\int_{-\infty}^{\infty}|(SW)(t)|^p\,[W(\xi)W(t)K_{Mn}(\xi,t)]^2dt, \tag{8.33}$$

valid for all $0 < p \leq 2$, $n \geq 1$, S of degree $\leq n$, and $|\xi| \leq 2a_n$. Since (by orthonormality)

$$\int_{-\infty}^{\infty}K_{Mn}^2(\xi,t)W(t)^2dt = K_{Mn}(\xi,\xi) \sim \frac{n}{a_n}W^{-2}(\xi), \tag{8.34}$$

we can also write this as:

Lemma 8.2 *Let $0 < p \leq 2$, $n \geq 1$, S of degree $\leq n$, and $|\xi| \leq 2a_n$. Then*

$$|(SW)(\xi)|^p \leq C\frac{\int_{-\infty}^{\infty}|(SW)(t)|^p K_{Mn}(\xi,t)^2W^2(t)dt}{\int_{-\infty}^{\infty}K_{Mn}(\xi,t)^2W^2(t)dt}. \tag{8.35}$$

From this follows:

Lemma 8.3 (Fundamental lemma of sieving) *Let $\psi : [0,\infty) \to [0,\infty)$ be a convex increasing function of x, with $\psi(0) = 0$. Let $p > 0$, $n \geq 1$, S of degree $\leq n$, and $|\xi| \leq 2a_n$. Then*

$$\psi(|SW|^p)(\xi) \leq \frac{\int_{-\infty}^{\infty}\psi(C|SW|^p)(t)K_{Mn}(\xi,t)^2W^2(t)dt}{\int_{-\infty}^{\infty}K_{Mn}(\xi,t)^2W^2(t)dt}. \tag{8.36}$$

If in addition, for some $A > 0$,

$$\psi(2t) \leq A\psi(t), \qquad t \in [0,\infty),$$

then

$$\psi(|SW|^p)(\xi) \leq C\frac{\int_{-\infty}^{\infty}\psi(|SW|^p)(t)K_{Mn}(\xi,t)^2W^2(t)dt}{\int_{-\infty}^{\infty}K_{Mn}(\xi,t)^2W^2(t)dt}. \tag{8.37}$$

Proof. For $p \leq 2$, this follows from (8.35) by a single application of Jensen's inequality. For general p, one instead applies Jensen's inequality with the convex function $t \mapsto \psi(t^a)$, with large enough a. $\square$

Next, we extend this to derivatives. We again use the polynomial P of (8.28). We let

$$K_n'(x,t) = \frac{\partial}{\partial x} K_n(x,t).$$

By (8.25),

$$\begin{aligned}
&\left| S'(\xi) K_{Mn}^L(\xi,\xi) + L\left(K_{Mn}(\xi,\xi)\right)^{L-1} K_{Mn}'(\xi,\xi) S(\xi) \right|^p W^{rp}(\xi) \\
&\quad \leq \; C \left(\frac{n}{a_n}\right)^{p+1} \int_{-\infty}^{\infty} \left| (SW^r)(t) K_{Mn}(\xi,t)^L \right|^p dt.
\end{aligned}$$

Here we recall (8.29) and (8.30), while Cauchy–Schwarz and (8.2) give for all $x, t \in \mathbb{R}$,

$$\left| K_{Mn}'(\xi,t) \right| \leq \left(\sum_{k=0}^{Mn-1} p_k'(\xi)^2 \right)^{1/2} \left(\sum_{k=0}^{Mn-1} p_k(t)^2 \right)^{1/2} \leq C \left(\frac{n}{a_n}\right)^2 W^{-1}(\xi) W^{-1}(t).$$

Then

$$\begin{aligned}
\left| S'(\xi) \right|^p W(\xi)^{(r-L)p-2} \;&\leq\; C \left(\frac{n}{a_n}\right)^p |S(\xi)|^p W^{(r-L)p-2}(\xi) \\
&\quad + C \left(\frac{n}{a_n}\right)^{p-1} \int_{-\infty}^{\infty} |S(t)|^p K_{Mn}(\xi,t)^2 W^{(r-L)p+2}(t) dt \\
&\leq\; C \left(\frac{n}{a_n}\right)^{p-1} \int_{-\infty}^{\infty} |S(t)|^p K_{Mn}(\xi,t)^2 W^{(r-L)p+2}(t) dt,
\end{aligned}$$

by (8.32). We now choose $r = L+1$, giving

$$|S'W|^p(\xi) \leq C \left(\frac{n}{a_n}\right)^{p-1} \int_{-\infty}^{\infty} |SW|^p(t) \left[W(\xi) W(t) K_{Mn}(\xi,t)\right]^2 dt, \tag{8.38}$$

which we can reformulate as

Lemma 8.4 *Let $0 < p \leq 2$, $n \geq 1$, S of degree $\leq n$, and $|\xi| \leq 2a_n$. Then*

$$|S'W|^p(\xi) \leq C \left(\frac{n}{a_n}\right)^p \frac{\int_{-\infty}^{\infty} |SW|^p(t) K_{Mn}(\xi,t)^2 W^2(t) dt}{\int_{-\infty}^{\infty} K_{Mn}(\xi,t)^2 W^2(t) dt}. \tag{8.39}$$

From this follows:

Lemma 8.5 *Let $\psi : [0,\infty) \to [0,\infty)$ be a convex increasing function of x, with $\psi(0) = 0$. Let $p > 0$, $n \geq 1$, S of degree $\leq n$, and $|\xi| \leq 2a_n$. Then*

$$\psi(|S'W|^p)(\xi) \leq \frac{\int_{-\infty}^{\infty} \psi(C \left| \frac{n}{a_n} SW \right|^p)(t) K_{Mn}(\xi,t)^2 W^2(t) dt}{\int_{-\infty}^{\infty} K_{Mn}(\xi,t)^2 W^2(t) dt}. \tag{8.40}$$

If in addition, for some $A > 0$,

$$\psi(2t) \leq A\psi(t), \qquad t \in [0,\infty), \tag{8.41}$$

then

$$\psi(|S'W|^p)(\xi) \leq C \frac{\int_{-\infty}^{\infty} \psi(\left| \frac{n}{a_n} SW \right|^p)(t) K_{Mn}(\xi,t)^2 W^2(t) dt}{\int_{-\infty}^{\infty} K_{Mn}(\xi,t)^2 W^2(t) dt}. \tag{8.42}$$

Proof. The proof uses Jensen's inequality as in Lemma 8.3. □

The inequalities (8.40) and (8.42) are useful for more than Markov–Bernstein inequalities. But for the moment, we deduce by integration and restricted range inequalities [108]:

Theorem 8.6 *Let $W \in \mathcal{F}^*$. Let $\psi : [0,\infty) \to [0,\infty)$ be a convex increasing function of x, with $\psi(0) = 0$. Let $p > 0$, $n \geq 1$, S of degree $\leq n$. Then*

$$\int_{-\infty}^{\infty} \psi(|S'W|^p)(\xi)d\xi \leq \int_{-\infty}^{\infty} \psi(C\frac{n}{a_n}|SW|^p)(t)dt. \tag{8.43}$$

If in addition, for some $A > 0$, (8.41) holds, then

$$\int_{-\infty}^{\infty} \psi(|S'W|^p)(\xi)d\xi \leq C\int_{-\infty}^{\infty} \psi(\frac{n}{a_n}|SW|^p)(t)dt. \tag{8.44}$$

9 Nikolskii Inequalities

We already proved an inequality of this type in the last section:

$$\|PW\|_{L_\infty(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^{1/p} \|PW\|_{L_p(\mathbb{R})},$$

for $n \geq 1$, and polynomials P of degree $\leq n$ — recall (8.24) and (8.35). Thus we compared the weighted sup norm and weighted L_p norm of a polynomial. More generally, inequalities that compare the norms of polynomials of degree $\leq n$ in different spaces are called **Nikolskii inequalities**. They are not difficult to prove, here is a sample:

Theorem 9.1 *Let $W \in \mathcal{F}^*$. Let $0 < p, r \leq \infty$. Then for $n \geq 1$ and polynomials P of degree $\leq n$,*

$$\|PW\|_{L_p(\mathbb{R})} \leq CN_n(p,r)\|PW\|_{L_r(\mathbb{R})},$$

where

$$N_n(p,r) = \begin{cases} a_n^{\frac{1}{p}-\frac{1}{r}}, & r > p \\ \left(\frac{n}{a_n}\right)^{\frac{1}{r}-\frac{1}{p}}, & r < p \end{cases}.$$

Proof. If first $\infty > r > p$, we can use restricted-range inequalities, and then Hölder's inequality:

$$\|PW\|^p_{L_p(\mathbb{R})} \leq C\int_{-a_n}^{a_n} |PW|^p \leq C\left(\int_{-a_n}^{a_n} |PW|^{p\frac{r}{p}}\right)^{\frac{p}{r}} \left(\int_{-a_n}^{a_n} 1\right)^{1-\frac{p}{r}} \leq C\|PW\|^p_{L_r(\mathbb{R})} a_n^{1-\frac{p}{r}},$$

that is

$$\|PW\|_{L_p(\mathbb{R})} \leq Ca_n^{\frac{1}{p}-\frac{1}{r}}\|PW\|_{L_r(\mathbb{R})}.$$

If $p = \infty$, the proof is easier. Next, if $r < p < \infty$, we use

$$\|PW\|^p_{L_p(\mathbb{R})} = \int_{-\infty}^{\infty} |PW|^p \leq \|PW\|^{p-r}_{L_\infty(\mathbb{R})}\int_{-\infty}^{\infty} |PW|^r.$$

Letting

$$\Lambda_{n,p} := \sup_{\deg(P) \leq n} \left(\frac{\|PW\|_{L_\infty(\mathbb{R})}}{\|PW\|_{L_p(\mathbb{R})}} \right)^p,$$

we obtain

$$\|PW\|_{L_p(\mathbb{R})} \leq \Lambda_{n,p}^{\frac{1}{r}-\frac{1}{p}} \|PW\|_{L_r(\mathbb{R})}^p. \tag{9.1}$$

In the case of Freud weights $W \in \mathcal{F}^*$ that grow at least as fast as $|x|^\alpha$, some $\alpha > 1$, we know from (8.24) that

$$\Lambda_{n,p} \leq C \frac{n}{a_n}. \qquad \square$$

For the canonical weights W_α, we have

$$\Lambda_{n,p} \sim \begin{cases} n^{1-1/\alpha}, & \alpha > 1 \\ \log(n+1), & \alpha = 1 \\ 1, & \alpha < 1 \end{cases}.$$

The sharpness of these was proved by Nevai and Totik [143], [152]. Observe that $\Lambda_{n,p}$ grows independently of p. In general, it seems that

$$\Lambda_{n,p} \sim \sup_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x) W^2(x).$$

For general convex exponential weights, it is known [85, p. 295] that:

Theorem 9.2 *Let $W \in \mathcal{F}_{even}\left(C^2\right)$ be as in Definition 7.5. Let $0 < p, r \leq \infty$. Then for $n \geq 1$ and polynomials P of degree $\leq n$,*

$$\|PW\|_{L_p(\mathbb{R})} \leq C N_n(p,r) \|PW\|_{L_r(\mathbb{R})},$$

where

$$N_n(p,r) = \begin{cases} a_n^{\frac{1}{p}-\frac{1}{r}}, & r > p \\ \left(\frac{nT(a_n)^{1/2}}{a_n} \right)^{\frac{1}{r}-\frac{1}{p}}, & r < p \end{cases}.$$

10 Orthogonal Expansions

There is an old mathematical saying that L_1, L_2 and L_∞ were invented by the Almighty, and man invented all else. The author heard this many years ago, but was interested to see it used as the opening quote in a chapter of Simon's treatise [172]. That orthogonal expansions naturally live in L_2 is obvious. Among the many manifestations of this, is the best approximation property

$$\|(f - S_n[f]) W\|_{L_2(\mathbb{R})} = E_n[f; W]_2 = \inf_{\deg(P) \leq n} \|(f - P)W\|_{L_2(\mathbb{R})}.$$

This ensures that when the polynomials are dense (in an obvious sense)

$$\lim_{n \to \infty} \|(f - S_n[f]) W\|_{L_2(\mathbb{R})} = 0$$

for all functions f for which $fW \in L_2(\mathbb{R})$.

But it is part of the mathematician's spirit to take important tools out of their natural domain, so it is not surprising that much effort has been devoted to convergence of $\{S_n[f]\}$ in L_p, or in a uniform sense, or at a specific point, and so on. A lot of fundamental advances have ensued: for example, the boundedness of the Hilbert transform in L_p, $1 < p < \infty$, was established in order to prove that classic Fourier series converge in such L_p. The theory of A_p weights started with Muckenhoupt's efforts to prove convergence in L_p of Hermite expansions.

In this section, we shall discuss pointwise and mean convergence. We begin with the latter.

10.1 Mean Convergence

Recall the reproducing kernel and the Christoffel–Darboux formula:

$$K_n(x,t) = \sum_{j=0}^{n-1} p_j(x)p_j(t) = \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t}.$$

Define the Hilbert transform

$$H[g](x) = \int_{-\infty}^{\infty} \frac{g(t)}{x-t}dt = \lim_{\varepsilon\to 0+}\int_{(-\infty,\infty)\setminus(x-\varepsilon,x+\varepsilon)} \frac{g(t)}{x-t}dt,$$

whenever the limit exists. If $g \in L_1(\mathbb{R})$, the transform exists for a.e. x. For a function f with $fW \in L_2(\mathbb{R})$, we see that

$$\begin{aligned} S_n[f](x) &= \int_{-\infty}^{\infty} K_n(x,t)f(t)W^2(t)dt \\ &= \frac{\gamma_{n-1}}{\gamma_n}\left\{p_n(x)H\left[p_{n-1}fW^2\right](x) - p_{n-1}(x)H\left[p_nfW^2\right](x)\right\}. \end{aligned}$$

Thus if u is a given function, and $1 < p < \infty$,

$$\begin{aligned} \left\|S_n[f]Wu^2\right\|_{L_p(\mathbb{R})} \leq \frac{\gamma_{n-1}}{\gamma_n}\Big\{ & \|p_nWu\|_{L_\infty(\mathbb{R})}\left\|H\left[p_{n-1}fW^2\right]u\right\|_{L_p(\mathbb{R})} \\ & + \|p_{n-1}Wu\|_{L_\infty(\mathbb{R})}\left\|H\left[p_nfW^2\right]u\right\|_{L_p(\mathbb{R})}\Big\}. \end{aligned} \tag{10.1}$$

Next, we use the aforementioned theorem of Riesz that the Hilbert transform is a bounded operator on L_p, provided $1 < p < \infty$: For some $C \neq C(g)$,

$$\|H[g]\|_{L_p(\mathbb{R})} \leq C\|g\|_{L_p(\mathbb{R})}.$$

In his investigations of Hermite expansions, Muckenhoupt [144] considered weighted versions: for suitable functions u,

$$\|H[g]u\|_{L_p(\mathbb{R})} \leq C\|gu\|_{L_p(\mathbb{R})}.$$

Such u are severely restricted, in particular satisfying Muckenhoupt's condition; examples are

$$u(t) = (1+|t|)^a, \qquad -\frac{1}{p} < a < 1 - \frac{1}{p}. \tag{10.2}$$

The theorem fails if $p = 1$ or $p = \infty$, though in L_1, other inequalities are available. So we continue (10.1) as

$$\begin{aligned}\left\|S_n[f]\,Wu^2\right\|_{L_p(\mathbb{R})} &\leq C\frac{\gamma_{n-1}}{\gamma_n}\left\{\|p_nWu\|_{L_\infty(\mathbb{R})}\left\|p_{n-1}fW^2u\right\|_{L_p(\mathbb{R})}\right.\\ &\qquad\left.+\|p_{n-1}Wu\|_{L_\infty(\mathbb{R})}\left\|p_nfW^2u\right\|_{L_p(\mathbb{R})}\right\}. \qquad (10.3)\end{aligned}$$

For exponential weights in the real line, recall from (4.14) that

$$\frac{\gamma_{n-1}}{\gamma_n} \leq Ca_n.$$

If we have the bounds

$$\|p_nWu\|_{L_\infty(\mathbb{R})} \leq Ca_n^{-1/2}, \qquad (10.4)$$

then for some $C \neq C(n, f)$,

$$\left\|S_n[f]\,Wu^2\right\|_{L_p(\mathbb{R})} \leq C\,\|fW\|_{L_p(\mathbb{R})}.$$

Once we have such a bound, the reproducing property of S_n (that $S_n[P] = P$ when P has degree $\leq n$) and density of polynomials gives

$$\lim_{n\to\infty}\left\|(f - S_n(f))\,Wu^2\right\|_{L_p(\mathbb{R})} = 0.$$

All that is required of f is that $fW \in L_p(\mathbb{R})$.

The problem with this procedure is that the bound (10.4) is hardly ever true. For Freud weights like W_α, or the weights in the class $\mathcal{F}^*$, we typically have

$$|p_n(x)|\,W(x) \leq Ca_n^{-1/2}\left(\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}\right)^{-1/4}, \qquad x \in \mathbb{R} \qquad (10.5)$$

and the upper bound reflects the real growth of p_n near $\pm a_n$:

$$\|p_nW\|_{L_\infty(\mathbb{R})} \sim a_n^{-1/2}n^{1/6}.$$

However, at least in $[-\rho a_n, \rho a_n]$, with $\rho \in (0, 1)$ fixed, we do have

$$|p_nW| \leq Ca_n^{-1/2}. \qquad (10.6)$$

This problematic growth of the orthogonal polynomials near the (effective) endpoints of the interval of orthogonality, already occurs for the Legendre weight on $[-1, 1]$, and more generally, Jacobi weights. The same Pollard that solved Bernstein's approximation problem, came up with a fix for this. In the context of Freud weights (a similar feature occurs for Jacobi weights), the fix is based on the observation that $p_n - p_{n-2}$ has a much better bound than p_n:

$$|p_n(x) - p_{n-2}(x)|\,W(x) \leq Ca_n^{-1/2}\left(\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}\right)^{1/4}, \qquad x \in \mathbb{R}. \qquad (10.7)$$

In particular,

$$|p_n(x) - p_{n-2}(x)|\,W(x) \leq Ca_n^{-1/2}, \qquad x \in \mathbb{R}.$$

(Think of $\widetilde{p}_n(\cos\theta) = \cos n\theta$ to see from whence this comes.)

Pollard [158], [159], [160] found a clever way to rewrite the Christoffel–Darboux formula to exploit this: let us set

$$\alpha_n = \frac{\gamma_{n-1}}{\gamma_n}.$$

Then (see [90] or [141] for an accessible proof)

$$K_n(x,y) = K_{n,1}(x,y) + K_{n,2}(x,y) + K_{n,3}(x,y),$$

where

$$\begin{aligned} K_{n,1}(x,y) &= \frac{\alpha_n}{\alpha_n + \alpha_{n-1}} p_{n-1}(x) p_{n-1}(y); \\ K_{n,2}(x,y) &= \frac{\alpha_n \alpha_{n-1}}{\alpha_n + \alpha_{n-1}} p_{n-1}(y) \left[\frac{p_n(x) - p_{n-2}(x)}{x-y} \right]; \\ K_{n,3}(x,y) &= K_{n,2}(y,x). \end{aligned}$$

Note that there is no $x-y$ in the denominator in $K_{n,1}$, while in $K_{n,2}$ and $K_{n,3}$, we have the term $p_n - p_{n-2}$ to help. This clever decomposition is not enough on its own; we emphasize that in investigating mean convergence, one still has to break up integrals into several different pieces, and work over several different ranges.

For the Hermite weight, the simplest bound is due to Askey and Wainger [2]:

Theorem 10.1 *Let $W(x) = \exp(-x^2)$. Let $\frac{4}{3} < p < 4$. There exists $C \neq C(n,f)$ such that*

$$\|S_n[f]\,W\|_{L_p(\mathbb{R})} \leq C\,\|fW\|_{L_p(\mathbb{R})}.$$

This inequality is not true for $p \leq \frac{4}{3}$ or $p \geq 4$.

Muckenhoupt found the correct extension to $1 < p < \infty$. In what follows we let $u_a(x) = (1+|x|)^a$.

Theorem 10.2 *Let $W(x) = \exp(-x^2)$ and $1 < p < \infty$. Let*

$$b < 1 - \frac{1}{p}; \quad B > -\frac{1}{p}; \quad b \leq B. \tag{10.8}$$

Assume in addition that

$$-B + \max\left\{b, -\frac{1}{p}\right\} + \frac{4}{3p} - 1 \leq 0 \quad \textit{if } p < \frac{4}{3}; \tag{10.9}$$

and

$$b - \min\left\{B, 1 - \frac{1}{p}\right\} + \frac{1}{3} - \frac{4}{3p} \leq 0 \quad \textit{if } p > 4. \tag{10.10}$$

Then

$$\|S_n[f]\,Wu_b\|_{L_p(\mathbb{R})} \leq C\,\|fWu_B\|_{L_p(\mathbb{R})} \tag{10.11}$$

for some $C \neq C(n,f)$. If $b = B$ and $p = \frac{4}{3}$ or 4, then we insert a factor of $\log(|x|+2)$ in the right-hand side of (10.11). In the case of equality in (10.9) or (10.10), we need strict inequality and replace the max *or* min *by their second terms. All these inequalities are also necessary.*

For general Freud weights, Shing Wu Jha and the author [52] extended Muckenhoupt's result, also closing up slight gaps between the latter's necessary and sufficient conditions in "boundary" cases: we let

$$L_{\sigma,\tau}(n)=\begin{cases}(\log(n+1))^{|\sigma|}, & \text{if } \sigma=\tau\\ 1, & \text{otherwise.}\end{cases}$$

Theorem 10.3 *Let $W=\exp(-Q)\in\mathcal{F}^*$. Let $1<p<\infty$, and $b,B\in\mathbb{R}$.*

(a) *Then for*

$$\|S_n[f]\,Wu_b\|_{L_p(\mathbb{R})}\le C\,\|fWu_B\|_{L_p(\mathbb{R})}$$

to hold with some $C\neq C(n,f)$, it is necessary that:

(I) (10.8) *holds.*

(II) *If $p<\frac{4}{3}$, then*

$$a_n^{\max\left\{b,-\frac{1}{p}\right\}-B}\,n^{\frac{1}{6}\left(\frac{4}{p}-3\right)}=O\left(\frac{1}{L_{b,-\frac{1}{p}(n)}}\right).$$

(III) *If $p=\frac{4}{3}$ or $p=4$, then strict inequality holds in the third inequality in* (10.8).

(IV) *If $p>4$, then*

$$a_n^{b-\min\left\{B,1-\frac{1}{p}\right\}-B}\,n^{\frac{1}{6}\left(1-\frac{4}{p}\right)}=O\left(\frac{1}{L_{B,1-\frac{1}{p}(n)}}\right).$$

(b) *Assume in addition that the orthonormal polynomials for W^2 also satisfy* (10.7). *Then the conditions above are also sufficient.*

When $W=W_\alpha$, and $a_n=Cn^{1/\alpha}$, the conditions above reduce to essentially Muckenhoupt's:

$$\max\left\{b,-\frac{1}{p}\right\}-B+\frac{\alpha}{6}\left(\frac{4}{p}-3\right)\begin{cases}\le 0, & b\neq-\frac{1}{p}\\ <0, & b=-\frac{1}{p}\end{cases}.$$

$$b-\min\left\{B,1-\frac{1}{p}\right\}-B+\frac{\alpha}{6}\left(1-\frac{4}{p}\right)\begin{cases}\le 0, & B\neq 1-\frac{1}{p}\\ <0, & B=1-\frac{1}{p}\end{cases}.$$

The bound (10.7) was established for $\exp\left(-x^{2m}\right)$, $m=1,2,3,\ldots$ in [52]. For general $\alpha>1$, it follows from results in that paper and work of Kriecherbauer and McLaughlin [68]. It is an interesting unsolved problem to establish the bound (10.7) for $W\in\mathcal{F}^*$, or to come up with some method to prove it that works without much deeper tools. See [76] for some work in this direction. There is also an L_1 analogue of these [52, p. 337].

Further work on mean convergence of orthogonal expansions appears in [56], [77], [123], [130], [131]. Discrete analogues of orthogonal expansions have been considered by Mashele [112], [113].

10.2 Uniform and Pointwise Convergence

One of the simplest and yet most elegant and general theorems on pointwise convergence is an extension of the Dirichlet–Jordan criterion, due to Freud [39]. It was subsequently extended by Mhaskar [129], [131]. Its proof is quite simple, and follows readily from estimates of previous sections, so we provide it:

Theorem 10.4 (Dirichlet–Jordan Criterion) *Let $W \in \mathcal{F}^*$ and let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, with $f'W \in L_1(\mathbb{R})$. Then*

$$\lim_{n\to\infty} \|(f - S_n[f])\, W\|_{L_\infty(\mathbb{R})} = 0.$$

Proof. Note first that fW is bounded and has limit 0 at $\pm\infty$. To see this let $A > 0$, and $x \geq A$. Then

$$|fW|(x) = \left| f(A) + \int_A^x f' \right| W(x) \leq |f(A)|\, W(x) + \int_A^\infty |f'|\, W,$$

as W is decreasing. Hence

$$\limsup_{x\to\infty} |fW|(x) \leq \int_A^\infty |f'|\, W.$$

Our integrability condition on $f'W$ ensures that this last right-hand side can be made as small as we please. So, indeed, fW has limit 0 at $\pm\infty$.

Next, let $m = \lfloor n/2 \rfloor$. Then

$$\|(f - S_n[f])\, W\|_{L_\infty(\mathbb{R})} \leq \|(f - V_m[f])\, W\|_{L_\infty(\mathbb{R})} + \|(V_m[f] - S_n[f])\, W\|_{L_\infty(\mathbb{R})}. \tag{10.12}$$

Here by Nikolskii inequalities, such as in Theorem 9.1,

$$\begin{aligned} \|(V_m[f] - S_n[f])\, W\|_{L_\infty(\mathbb{R})} &\leq C\sqrt{\frac{n}{a_n}}\, \|(V_m[f] - S_n[f])\, W\|_{L_2(\mathbb{R})} \\ &\leq C\sqrt{\frac{n}{a_n}} \left\{ \|(V_m[f] - f)\, W\|_{L_2(\mathbb{R})} + \|(f - S_n[f])\, W\|_{L_2(\mathbb{R})} \right\} \\ &\leq 2C\sqrt{\frac{n}{a_n}}\, \|(V_m[f] - f)\, W\|_{L_2(\mathbb{R})}. \end{aligned} \tag{10.13}$$

Recall the L_2 best approximation property of the partial sums S_n, and that V_m has degree $\leq 2m \leq n$. Next,

$$\begin{aligned} \|(V_m[f] - f)\, W\|_{L_2(\mathbb{R})}^2 &\leq \|(V_m[f] - f)\, W\|_{L_1(\mathbb{R})}\, \|(V_m[f] - f)\, W\|_{L_\infty(\mathbb{R})} \\ &\leq C E_m[f;W]_1\, \|(V_m[f] - f)\, W\|_{L_\infty(\mathbb{R})} \\ &\leq C\frac{a_m}{m}\, \|f'W\|_{L_1(\mathbb{R})}\, \|(V_m[f] - f)\, W\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

We used the Jackson–Favard inequality (4.5) in the last line. Since $m \sim n$, so $\frac{a_m}{m} \sim \frac{a_n}{n}$. By substituting this into (10.13), we obtain

$$\|(V_m[f] - S_n[f])W\|_{L_\infty(\mathbb{R})} \leq C\left[\|f'W\|_{L_1(\mathbb{R})}\|(V_m[f]-f)W\|_{L_\infty(\mathbb{R})}\right]^{1/2},$$

so (10.12) becomes

$$\begin{aligned}\|(f - S_n[f])W\|_{L_\infty(\mathbb{R})} &\leq \|(f - V_m[f])W\|_{L_\infty(\mathbb{R})} + C\left[\|f'W\|_{L_1(\mathbb{R})}\|(V_m[f]-f)W\|_{L_\infty(\mathbb{R})}\right]^{1/2} \\ &\leq C_1\left(E_m[f;W]_\infty + E_m[f;W]_\infty^{1/2}\right),\end{aligned}$$

by Theorem 4.8 again. Since fW is continuous, and has limit 0 at infinity, we have $E_m[f;W]_\infty \to 0$ as $n \to \infty$. $\square$

This theorem is attractive because of the simplicity of the hypotheses. We don't need to assume anything about the "tail" of f. There are also results which give convergence at a point, under local assumptions. A result of Mhaskar [136, p. 224] is the archetype:

Theorem 10.5 *Let $W \in \mathcal{F}^*$ and in addition assume that Q'' is increasing on $(0,\infty)$. Let f be of bounded variation, and x be a point of continuity of f. Then*

$$|S_n[f](x) - f(x)| \leq C(x)\left[\int_{x-\sqrt{\frac{a_n}{n}}}^{x+\sqrt{\frac{a_n}{n}}} W\,|df| + \sqrt{\frac{a_n}{n}}\int_{-\infty}^{\infty} W\,|df| + \int_{|t|\geq Ca_n} W\,|df|\right].$$

Here $C(x)$ depends on x, but not on n or f.

Mashele [114] has established analogues of this for general exponential weights. Unfortunately in passing to the more general weights, some of the power of this result is lost. Ky [77] has investigated a.e. convergence of orthonormal expansions for Freud weights. Trilinear kernels and related expansions have been explored by Osilenker [154].

There is an extensive literature on expansions in Hermite polynomials, including transplantation theorems [186], [187] that specifically exploit the structure of the Hermite weight. There are also many results for Laguerre and related expansions, on the interval $[0,\infty)$, see [119].

11 Lagrange Interpolation

Lagrange interpolation at zeros of orthogonal polynomials is probably the most studied of all approximation processes associated with exponential weights. Recall that we denoted the zeros of p_n, the nth orthonormal polynomial for the weight W^2, by

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty.$$

Given a function f, we define the nth Lagrange interpolation polynomial to f at $\{x_{jn}\}_{j=1}^n$ to be the unique polynomial of degree $\leq n-1$ that satisfies

$$L_n[f](x_{jn}) = f(x_{jn}), \qquad 1 \leq j \leq n.$$

One formula for $L_n[f]$ is

$$L_n[f] = \sum_{j=1}^{n} f(x_{jn})\,\ell_{jn}$$

where $\{\ell_{jn}\}_{j=1}^{n}$ are fundamental polynomials of Lagrange interpolation, satisfying

$$\ell_{jn}(x_{kn}) = \delta_{jk}.$$

The $\{\ell_{jn}\}$ admit the representation

$$\ell_{jn}(x) = \prod_{k=1,k\neq j}^{n} \frac{x - x_{kn}}{x_{jn} - x_{kn}},$$

but because we are using zeros of orthogonal polynomials, there are others:

$$\ell_{jn}(x) = \frac{K_n(x, x_{jn})}{K_n(x_{jn}, x_{jn})}.$$

(To see this, look at the Christoffel–Darboux formula.)

Another advantage of interpolating at zeros of orthogonal polynomials is the ease of analysis, largely because we have the Gauss quadrature formula. For polynomials P of degree $\leq 2n-1$,

$$\int_{-\infty}^{\infty} PW^2 = \sum_{j=1}^{n} \lambda_{jn} P(x_{jn}),$$

where

$$\lambda_{jn} = \lambda_n(W^2, x_{jn}) = 1/K_n(x_{jn}, x_{jn})$$

are called the **Christoffel numbers**. Since $L_n[f]^2$ has degree $\leq 2n-2$, and agrees with f at $\{x_{jn}\}$, we see that

$$\int_{-\infty}^{\infty} L_n[f]^2\, W^2 = \sum_{j=1}^{n} \lambda_{jn} f^2(x_{jn}). \tag{11.1}$$

Thus boundedness and convergence of $\{L_n[f]\}$ is closely associated with convergence of Gauss quadrature. There are at least two recent monographs devoted to Lagrange interpolation [116], [179].

11.1 Mean Convergence

Just as $\{S_n[f]\}$ converge in L_2, so do the $\{L_n[f]\}$, a famous theorem of Erdős-Turán. Indeed, we may think of $L_n[f]$ as a discretization of $S_n[f]$:

$$L_n[f](x) = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn})\, K_n(x, x_{jn}) \approx \int_{-\infty}^{\infty} W^2(t) f(t) K_n(x,t)\,dt = S_n[f](x).$$

At this stage, may I share two Erdős stories with the reader. I was doing a postdoc at the Technion in Haifa in 1982. Erdős often visited the Technion, and I was one of several who went to lunch with him. Later that day, I carried his suit to the drycleaners, and the reward was some time with him. He asked me what I was doing, and I mentioned numerical quadrature. He replied that I should go and read the book of Freud on orthogonal polynomials, which I did, and it changed my career.

Here is another story, which illustrates how little we communicate across fields. In early 2006, I had the privilege of visiting Texas A&M in College Station, to give some Frontiers Lectures. I mentioned the famous theorem of Erdős–Turán on distribution of zeros of polynomials. At the end of the talk, one of the very first comments was "I thought Erdős was a graph theorist". Hmmph. He spent a large amount of his life on Lagrange interpolation, orthogonal polynomials and approximation theory!

Since Lagrange interpolation samples a function only at a discrete sets of points, and sets of measure zero make no difference to Lebesgue integrals, it is clear that we shall need to use something other than Lebesgue integrability. As Gauss quadrature sums are Riemann sums or Riemann–Stieltjes sums [181, p. 50], the Riemann integral is the appropriate tool. The following result was actually proved by Shohat [171]; Erdős–Turán considered only finite intervals. The growth of f at ∞ poses extra problems for the infinite interval.

Theorem 11.1 (Shohat's version of the Erdős–Turán Theorem) *Let f be bounded and Riemann integrable on each finite interval. Assume, moreover, that there is an even entire function G with all non-negative Maclaurin series coefficients such that GW^2 is integrable over the real line, and*

$$\lim_{|x|\to\infty} f^2(x)/G(x) = 0. \tag{11.2}$$

Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (f - L_n[f])^2 W^2 = 0. \tag{11.3}$$

Proof. Let P be a fixed polynomial. For large enough n, $L_n[P] = P$, so

$$\begin{aligned}\int_{-\infty}^{\infty} (f - L_n[f])^2 W^2 &= \int_{-\infty}^{\infty} (f - P - L_n[f-P])^2 W^2 \\ &\leq 2\int_{-\infty}^{\infty} (f-P)^2 W^2 + 2\int_{-\infty}^{\infty} (L_n[f-P])^2 W^2 \\ &= 2\int_{-\infty}^{\infty} (f-P)^2 W^2 + \sum_{j=1}^{n} \lambda_{jn} (f-P)^2(x_{jn}). \end{aligned} \tag{11.4}$$

In the second last line, we used the inequality

$$(x+y)^2 \leq 2\left(x^2+y^2\right)$$

and in the last line we used (11.1). Now if G is not a polynomial (which we may assume, just add a very slowly growing entire function if it is),

$$\lim_{|x|\to\infty} (f-P)^2(x)/G(x) = 0.$$

Let $\varepsilon > 0$. Choose $A > 0$ such that

$$|x| \geq A \Rightarrow (f-P)^2(x)/G(x) \leq \varepsilon.$$

Then

$$\sum_{j:|x_{jn}|\geq A} \lambda_{jn}(f-P)^2(x_{jn}) \leq \varepsilon \sum_{j:|x_{jn}|\geq A} \lambda_{jn} G(x_{jn}) \leq \varepsilon \sum_{j=1}^{n} \lambda_{jn} G(x_{jn}) \leq \varepsilon \int_{-\infty}^{\infty} GW^2,$$

by a special case of the Posse–Markov–Stieltjes inequalities [38, p. 92]. These inequalities estimate quadrature sums of absolutely monotone functions in terms of integrals. Next, since $(f-P)^2\chi_{[-A,A]}$ is Riemann integrable, while Gauss quadrature sums are Riemann sums,

$$\lim_{n\to\infty} \sum_{j:|x_{jn}|\leq A} \lambda_{jn}(f-P)^2(x_{jn}) = \int_{-A}^{A} (f-P)^2 W^2.$$

Substituting all these in (11.4) gives

$$\limsup_{n\to\infty} \int_{-\infty}^{\infty} (f - L_n[f])^2 W^2 \leq 4 \int_{-\infty}^{\infty} (f-P)^2 W^2 + \varepsilon \int_{-\infty}^{\infty} GW^2.$$

Here P is independent of ε, and by L_2 analogues of the Bernstein problem, may be chosen so that $\int_{-\infty}^{\infty}(f-P)^2W^2$ is as small as we please. $\square$

Note that if f is of polynomial growth at ∞, we can always choose G to satisfy (11.2). For large classes of exponential weights, the author [87], [88] constructed entire functions G satisfying the hypotheses above, and with

$$G(x)W^2(x) \sim (1+|x|)^{-a},$$

any $a > 1$. So typically, the growth restriction on f is hardly more than that required for $(fW)^2$ to be integrable.

Of course this is a really beautiful result, and its proof is elegant! The extensions to L_p require a whole lot more work. But like their cousins in orthogonal expansions, a lot of great mathematics has come out of these efforts. It was Paul Nevai [149] who obtained the first major result in L_p, for the Hermite weight.

Theorem 11.2 *Let f be continuous on the real line. Let $W(x) = \exp\left(-\frac{1}{2}x^2\right)$ be the Hermite weight, and assume that*

$$\lim_{|x|\to\infty} f(x)(1+|x|)W(x) = 0.$$

Let $\{L_n[f]\}$ denote the Lagrange interpolation polynomials to f at the zeros of Hermite polynomials. Then for $p > 1$,

$$\lim_{n\to\infty} \|(f - L_n[f])W\|_{L_p(\mathbb{R})} = 0.$$

Note that the pth power of the weight W appears in the integral. Nevai showed that the presence of such a factor is necessary — see his still relevant survey for more perspectives on this [147]. The author and D. Matjila [106] subsequently extended Nevai's result to general Freud weights:

Theorem 11.3 *Let f be continuous on the real line. Let $\beta > 1$, $p > 1$, and $W(x) = \exp\left(-\frac{1}{2}|x|^{\beta}\right)$. Let $\Delta \in \mathbb{R}$, $\alpha > 0$, and $\hat{\alpha} := \min\{1, \alpha\}$. Moreover, let*

$$\tau := \tau(p) := \frac{1}{p} - \hat{\alpha} + \frac{\beta}{6}\max\left\{0, 1 - \frac{4}{p}\right\}. \tag{11.5}$$

Let $\{L_n[f]\}$ denote the Lagrange interpolation polynomials to f at the zeros of $\left\{p_n(W^2,x)\right\}$. For

$$\lim_{n\to\infty}\left\|\left(f-L_n[f]\right)(x)W(x)\left(1+|x|\right)^{-\Delta}\right\|^p_{L_p(\mathbb{R})}=0 \tag{11.6}$$

to hold for every continuous function satisfying

$$\lim_{|x|\to\infty}f(x)\left(1+|x|\right)^{\alpha}W(x)=0 \tag{11.7}$$

it is necessary and sufficient that

$$\Delta>\tau \quad \textit{if } 1<p\leq 4;$$

$$\Delta>\tau \quad \textit{if } p>4 \textit{ and } \alpha=1;$$

$$\Delta\geq\tau \quad \textit{if } p>4 \textit{ and } \alpha\neq 1.$$

Moreover, for (11.6) *to hold for every $1<p<\infty$ and every continuous f satisfying (11.7), it is necessary and sufficient that $\Delta\geq-\hat{\alpha}+\max\left\{1,\frac{\beta}{6}\right\}$.*

This in turn, is a special case of a result for general Freud weights:

Theorem 11.4 *Let $W=\exp(-Q)\in\mathcal{F}^*$. Let $p>1$, $\Delta\in\mathbb{R}$, $\alpha>0$, and $\hat{\alpha}:=\min\{1,\alpha\}$. Let $\{L_n[f]\}$ denote the Lagrange interpolation polynomials to f at the zeros of $\left\{p_n(W^2,x)\right\}$. For* (11.6) *to hold for every continuous function satisfying* (11.7), *it is necessary and sufficient that*

$$\begin{aligned}
&\Delta>-\hat{\alpha}+\tfrac{1}{p} && \textit{if } 1<p\leq 4;\\
&a_n^{\frac{1}{p}-(\hat{\alpha}+\Delta)}n^{\frac{1}{6}\left(1-\frac{4}{p}\right)}=O\left(\tfrac{1}{\log n}\right) && \textit{if } p>4 \textit{ and } \alpha=1;\\
&a_n^{\frac{1}{p}-(\hat{\alpha}+\Delta)}n^{\frac{1}{6}\left(1-\frac{4}{p}\right)}=O(1) && \textit{if } p>4 \textit{ and } \alpha\neq 1.
\end{aligned} \tag{11.8}$$

The main ideas of proof of these results already appear in Nevai's 1980 paper, although, obviously, technical details are more complicated for general weights. In analyzing $L_n[f]$, one fixes a polynomial P and splits the function $f-P$ into two pieces, for example,

$$L_n[f-P]=L_n\left[(f-P)\chi_{\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]}\right]+L_n\left[(f-P)\chi_{\mathbb{R}\setminus\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]}\right],$$

and handles them separately, using different techniques. One of the reasons for the split is the bound

$$|p_n(x)|\,W(x)\leq Ca_n^{-1/2},\qquad |x|\leq(1-\varepsilon)\,a_n,$$

any fixed $\varepsilon>0$, which does not extend all the way to $\pm a_n$; recall the discussion at (10.5) and (10.6). Indeed, it is the fact that $p_n(x)W(x)$ behaves roughly like $\left(1-\frac{|x|}{a_n}\right)^{-1/4}$ near $\pm a_n$ that accounts for the change in phenomena at $p=4$, as well as for the complicated conditions.

The weighted L_p norms of the two "pieces" of f are themselves split into a number of pieces, with different techniques used to estimate over different ranges. Tools include estimates of quadrature sums in terms of integrals (see Section 12), bounds on Hilbert transforms, and estimates of special quadrature sums. Here we shall establish just one of the main estimates, using ideas that go back to Richard Askey [1].

Lemma 11.5 *Let*

$$\psi_n = (f - P)\chi_{\left[-\frac{1}{4}a_n, \frac{1}{4}a_n\right]} \tag{11.9}$$

and

$$\Lambda = \left\| (f - P)(x)W(x)\left(1 + |x|\right)^{\alpha} \right\|_{L_\infty(\mathbb{R})}. \tag{11.10}$$

Then for $n \geq 1$,

$$\left\| L_n\left[\psi_n\right](x)W(x)\left(1 + |x|\right)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} \leq C\Lambda, \tag{11.11}$$

where C *is independent of* f, n, P.

Proof. We let

$$G(x) = W(x)^{-1}(1 + |x|)^{-\alpha}$$

and first show that

$$\begin{aligned} &\left\| L_n\left[\psi_n\right](x)W(x)\left(1 + |x|\right)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} \\ &\quad \leq \; C\Lambda \sup_{\|h\|_{L_\infty(\mathbb{R})} \leq 1} \left\| S_n\left[hG\right](x)W(x)\left(1 + |x|\right)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]}. \end{aligned} \tag{11.12}$$

Then we show that

$$\sup_{\|h\|_{L_\infty(\mathbb{R})} \leq 1} \left\| S_n\left[hG\right](x)\, W(x)\left(1 + |x|\right)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} \leq C, \tag{11.13}$$

giving the result.

Proof of (11.12). We use duality, but in a very precise form. Let

$$g_n(x) := \operatorname{sign}\left\{L_n\left[\psi_n\right](x)\right\} \left|L_n\left[\psi_n\right](x)\right|^{p-1} W^{p-2}(x)\left(1 + |x|\right)^{-\Delta p} \chi_{\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]}(x).$$

Then

$$\begin{aligned} \left\| L_n\left[\psi_n\right](x)W(x)\left(1 + |x|\right)^{-\Delta} \right\|^p_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} &= \int_{-\infty}^{\infty} L_n\left[\psi_n\right] g_n(x)W^2(x)dx \\ &= \int_{-\infty}^{\infty} L_n\left[\psi_n\right] S_n\left[g_n\right](x)W^2(x)dx, \end{aligned} \tag{11.14}$$

by orthogonality of $g - S_n[g]$ to polynomials of degree $\leq n-1$. Using the Gauss quadrature formula, and the interpolatory properties of ψ_n, we continue this as

$$= \sum_{k=1}^{n} \lambda_{kn}\psi_n(x_{kn})S_n\left[g\right](x_{kn}) \leq \Lambda \sum_{|x_{kn}| \leq \frac{1}{4}a_n} \lambda_{kn} \left|S_n\left[g\right](x_{kn})\right| W^{-1}(x_{kn})\left(1 + |x_{kn}|\right)^{-\alpha}.$$

We estimate the quadrature sum by an integral. These types of inequalities will be discussed in Section 12. For the moment, we note that λ_{kn} behaves roughly like $\frac{a_n}{n}W^2\left(x_{kn}\right)$ and continue this as

$$\begin{aligned} \leq C\Lambda \int_{-\infty}^{\infty} \left|S_n\left[g_n\right](x)\right| W(x)\left(1 + |x|\right)^{-\alpha} dx &= C\Lambda \int_{-\infty}^{\infty} S_n\left[g_n\right](x)h_n(x)G(x)W^2(x)dx \\ &= C\Lambda \int_{-\infty}^{\infty} g_n(x)S_n\left[h_nG\right](x)W^2(x)dx, \end{aligned}$$

where $h_n(x) := \operatorname{sign}(S_n[g_n](x))$, and we have used the self-adjointness of S_n. Since g_n vanishes outside $\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]$, and letting $q = \frac{p}{p-1}$, we continue this as

$$\begin{aligned}
&= C\Lambda \int_{-\frac{1}{2}a_n}^{\frac{1}{2}a_n} g_n(x) S_n[h_n G](x) W^2(x)dx \\
&\leq C\Lambda \left\| g_n(x)W(x)(1+|x|)^{\Delta} \right\|_{L_q\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]} \left\| S_n[h_n G](x)W(x)(1+|x|)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]} \\
&= C\Lambda \left\| L_n[\psi_n](x)W(x)(1+|x|)^{-\Delta} \right\|^{p-1}_{L_p\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]} \left\| S_n[h_n G](x)W(x)(1+|x|)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]},
\end{aligned}$$

by the form of g_n. Cancelling the $(p-1)$th power of

$$\left\| L_n[\psi_n](x)W(x)(1+|x|)^{-\Delta} \right\|$$

in the extreme left of (11.14), we obtain (11.12).

Proof of (11.13). We use the Christoffel–Darboux formula much as we did in the section on mean convergence of orthogonal expansions:

$$S_n[hG](x) = \frac{\gamma_{n-1}}{\gamma_n}\left\{p_n(x)H\left[p_{n-1}hGW^2\right](x) - p_{n-1}(x)H\left[p_n hGW^2\right](x)\right\}.$$

Recalling the bound (10.6) on p_n and the bound

$$\frac{\gamma_{n-1}}{\gamma_n} \leq Ca_n,$$

we obtain

$$|S_n[hG]W|(x) \leq Ca_n^{1/2}\sum_{j=n-1}^{n}\left|H\left[p_j hGW^2\right](x)\right|.$$

We next let

$$\chi_n^* = \chi_{\left[-\frac{3}{4}a_n,\frac{3}{4}a_n\right]},$$

and split

$$\begin{aligned}
|S_n[hG]W|(x) &\leq Ca_n^{1/2}\sum_{j=n-1}^{n}\left|H\left[p_j hG\chi_n^* W^2\right](x)\right| \\
&\quad + Ca_n^{1/2}\sum_{j=n-1}^{n}\left|H\left[p_j hG(1-\chi_n^*)W^2\right](x)\right|. \qquad (11.15)
\end{aligned}$$

We next use weighted bounds for Hilbert transforms, of the type established by Muckenhoupt [144]. Let

$$\hat{\Delta} = \min\left\{\Delta, \frac{1}{p} - \delta\right\},$$

for some small $\delta > 0$. Then by Muckenhoupt's bounds for weighted Hilbert transforms,

$$\begin{aligned}
& a_n^{1/2} \sum_{j=n-1}^{n} \left\| H\left[p_j h G \chi_n^* W^2\right](x) (1+|x|)^{-\hat{\Delta}} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} \\
\leq \; & C a_n^{1/2} \sum_{j=n-1}^{n} \left\| \left(p_j h G \chi_n^* W^2\right)(x) (1+|x|)^{-\hat{\Delta}} \right\|_{L_p(\mathbb{R})} \leq C \left\| (hGW)(x) (1+|x|)^{-\hat{\Delta}} \right\|_{L_p\left[-\frac{3}{4}a_n, \frac{3}{4}a_n\right]} \\
\leq \; & C \left\| (1+|x|)^{-\alpha-\hat{\Delta}} \right\|_{L_p\left[-\frac{3}{4}a_n, \frac{3}{4}a_n\right]} \leq C, \qquad (11.16)
\end{aligned}$$

as $\left(\alpha + \hat{\Delta}\right) p > 1$. (Recall our hypotheses in Theorem 11.4 — they imply $\Delta + \alpha > \frac{1}{p}$.) Next, for $|x| \leq \frac{1}{2} a_n$, and $j = n-1, n$,

$$\begin{aligned}
\left| H\left[p_j h G \left(1 - \chi_n^*\right) W^2\right](x) \right| &= \left| \int_{\left\{t : |t| \geq \frac{3}{4}a_n\right\}} \frac{\left(p_j h G W^2\right)(t)}{x - t} dt \right| \leq C \int_{\left\{t : |t| \geq \frac{3}{4}a_n\right\}} |p_j W|(t) t^{-1-\alpha} dt \\
&\leq C \left[\int_{-\infty}^{\infty} (p_j W)^2\right]^{1/2} \left[\int_{\frac{3}{4}a_n}^{\infty} t^{-2-2\alpha} dt\right]^{1/2} \leq C a_n^{-\frac{1}{2}-\alpha}.
\end{aligned}$$

So

$$\begin{aligned}
& a_n^{1/2} \sum_{j=n-1}^{n} \left\| H\left[p_j h G \left(1 - \chi_n^*\right) W^2\right](x) (1+|x|)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} \leq C a_n^{-\alpha} \left\| (1+|x|)^{-\Delta} \right\|_{L_p\left[-\frac{1}{2}a_n, \frac{1}{2}a_n\right]} \\
& \leq C a_n^{-\alpha} \begin{cases} 1, & \Delta p > 1, \\ (\log n)^{\frac{1}{p}}, & \Delta p = 1, \\ a_n^{\frac{1}{p}-\Delta}, & \Delta p < 1. \end{cases} = o(1), \qquad (11.17)
\end{aligned}$$

since

$$-\alpha + \frac{1}{p} - \Delta < 0.$$

Combining (11.15), (11.16), and (11.17), yields (11.13) and hence the result. □

Of course, this is just part of the estimation, but is the most difficult part. For the full details, see [106].

Mean convergence of Lagrange interpolation associated with Erdős weights has been investigated by S. Damelin and the author [25], [26]. The author has investigated Lagrange interpolation associated with exponential weights on $[-1, 1]$ [94], while D. Kubayi and the author investigated mean convergence associated with general exponential weights [75]. The convergence in L_p, $p < 1$, has been investigated by Matjila [125].

One obvious question is whether other interpolation sets lead to cleaner, or simpler, results. It was J. Szabados [176], [192] who came up with this idea, the "method of additional points", in the context of Lebesgue functions, which we shall discuss shortly. Szabados' idea was to add two extra points of interpolation, near $\pm a_n$, to damp the growth of $p_n(x)$ (recall, roughly $\left(1 - \frac{|x|}{a_n}\right)^{-1/4}$) near $\pm a_n$. Suitable points are those where $p_n W$ attains its sup norm on the real line:

$$|p_n W|(\xi_n) = \|p_n W\|_{L_\infty(\mathbb{R})}.$$

One can show that

$$|1 - \xi_n / a_n| \leq C n^{-2/3}$$

and that

$$|x_{1n} - \xi_n| / a_n \sim n^{-2/3}.$$

We let $L_n^* [f]$ denote the polynomial of degree $\leq n+1$ that interpolates to f at the $n+2$ points

$$\{-\xi_n, x_{nn}, x_{n-1,n}, \ldots, x_{2n}, x_{1n}, \xi_n\}.$$

Inspired by Szabados' work on Lebesgue functions, G. Mastroianni and the author [103] proved:

Theorem 11.6 *Let $W \in \mathcal{F}^*$. Let $p > 1$, $\Delta \in \mathbb{R}$, $\alpha > 0$, and $\hat{\alpha} := \min\{1, \alpha\}$. For*

$$\lim_{n \to \infty} \left\| (f - L_n^* [f]) (x) W(x) (1 + |x|)^{\Delta} \right\|_{L_p(\mathbb{R})}^p = 0 \tag{11.18}$$

to hold for every continuous function satisfying

$$\lim_{|x| \to \infty} f(x) (1 + |x|)^{\alpha} W(x) = 0 \tag{11.19}$$

it is necessary and sufficient that

$$\Delta > \frac{1}{p} - \hat{\alpha}.$$

What a difference the extra two points make! Let us explain a little of the mechanics behind this. $L_n^* [f]$ admits the representation

$$L_n^* [f] (x) = f(\xi_n) \ell_{0n}^*(x) + f(-\xi_n) \ell_{n+1,n}^*(x) + \sum_{j=1}^{n} f(x_{jn}) \ell_{j,n}^*(x),$$

where for $1 \leq j \leq n$,

$$\ell_{j,n}^*(x) = \ell_{jn}(x) \frac{\xi_n^2 - x^2}{\xi_n^2 - x_{jn}^2},$$

and

$$\ell_{jn}(x) = \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})}.$$

The factor $\xi_n^2 - x^2$ damps the growth of $p_n(x)$ near $\pm a_n$. Of course, there is an extra factor of $\xi_n^2 - x_{jn}^2$ in the denominator, but this is large for most x_{jn}, so actually helps.

We already noted that convergence of $\{L_n [f]\}$ associated with Freud weights in L_p, $p < 1$, has been investigated by Matjila [125]. If one sacrifices precision in this case, one can achieve very general results, using distribution functions, rearrangements of functions, and a classic lemma of Loomis [3]. The latter asserts that for any $\{c_j\}_{j=1}^n$,

$$\operatorname{meas} \left\{ x : \left| \sum_{j=1}^{n} \frac{c_j}{x - x_j} \right| > \lambda \right\} \leq \frac{8}{\lambda} \sum_{j=1}^{n} |c_j|.$$

(It is often stated in the case where all $c_j > 0$ and without absolute values.) If one writes

$$L_n[f](x) = p_n(x) \sum_{j=1}^{n} \frac{f(x_{jn})}{p_n'(x_{jn})(x - x_{jn})} =: p_n(x) g_n(x),$$

we see that for any positive function ϕ and real numbers b, c,

$$\|L_n[f] W\phi^b\|_{L_p(\mathbb{R})} \leq \|p_n W\phi^{b+c}\|_{L_{2p}(\mathbb{R})} \|g_n\phi^{-c}\|_{L_{2p}(\mathbb{R})}. \tag{11.20}$$

Next, Loomis' Lemma shows that

$$\begin{aligned} m_{g_n}(\lambda) := \text{meas}\{x : |g_n(x)| > \lambda\} &\leq \frac{8\|fW\phi^c\|_{L_\infty(\mathbb{R})}}{\lambda} \sum_{j=1}^{n} \frac{1}{|p_n' W\phi^c|(x_{jn})} \\ &=: \frac{8\|fW\phi^c\|_{L_\infty(\mathbb{R})}}{\lambda} \Omega_n. \end{aligned}$$

For the moment, let h^* denote the decreasing rearrangement of a function h. In particular, the decreasing rearrangement of g_n is

$$g_n^*(t) = \sup\{\lambda : m_{g_n}(\lambda) > t\} \leq \sup\left\{\lambda : \frac{8\|fW\phi^c\|_{L_\infty(\mathbb{R})}}{\lambda}\Omega_n > t\right\} = \frac{8\|fW\phi^c\|_{L_\infty(\mathbb{R})}}{t}\Omega_n.$$

Inequalities for decreasing rearrangements give [3]

$$\begin{aligned} \left\|g_n\phi^{-c}\right\|_{L_{2p}(\mathbb{R})}^{2p} &= \int_{-\infty}^{\infty} \left|g_n\phi^{-c}\right|^{2p} \leq \int_0^\infty \left(|g_n|^{2p}\right)^* \left(\phi^{-2pc}\right)^* = \int_0^\infty (g_n^*)^{2p} \left(\left(\phi^{-1}\right)^*\right)^{2pc} \\ &\leq \left(8\|fW\phi^c\|_{L_\infty(\mathbb{R})}\Omega_n\right)^{2p} \int_0^\infty t^{-2p}\left((\phi^{-1})^*(t)\right)^{2pc} dt. \end{aligned} \tag{11.21}$$

Now if $p < \frac{1}{2}$, this integrand poses no problem at 0. If for example,

$$\phi(t) = 1 + |t|,$$

then

$$(\phi^{-1})^*(t) = \left(1 + \frac{t}{2}\right)^{-1}, \qquad t \geq 0,$$

and the integral will converge provided $2p + 2pc > 1$. Combining this, (11.20) and (11.21), we obtain

$$\|L_n[f] W\phi^b\|_{L_p(\mathbb{R})} \leq C\|fW\phi^c\|_{L_\infty(\mathbb{R})}\Omega_n\|p_n W\phi^{b+c}\|_{L_{2p}(\mathbb{R})}$$

with C independent of n, f. Recalling the definition of Ω_n above, we have the main part of [97, Theorem 3, p. 155]:

Theorem 11.7 *Let W be even and positive in $[0,\infty)$ and $\phi(t) = 1 + |t|$. The following are equivalent:*

(I) *There exist $b, c \in \mathbb{R}$ and $C, p > 0$ such that for every continuous f*

$$\|L_n[f] W\phi^b\|_{L_p(\mathbb{R})} \leq C\|fW\phi^c\|_{L_\infty(\mathbb{R})}.$$

(II) *There exist $\beta, \gamma \in \mathbb{R}$ and $r > 0$ such that*

$$\sup_{n\geq 1} \|p_n W \phi^\beta\|_{L_r(\mathbb{R})} \sum_{j=1}^{n} \frac{1}{|p_n' W \phi^\gamma|(x_{jn})} < \infty.$$

From this one may easily obtain necessary and sufficient conditions for mean convergence of $\{L_n[f]\}$ in L_p, some $p < \frac{1}{2}$, with appropriate weights. The striking feature of these results is the simplicity of the proofs, and their generality. However, one sacrifices a great deal of precision, for even the specific L_p is not fixed in advance. For further work on mean convergence of Lagrange interpolation associated with exponential weights, see [19], [21], [24], [53], [117], [118], [123], [153], [168], [169].

11.2 Lebesgue Functions and Pointwise Convergence

Since $L_n[P] = P$ for any polynomial P of degree $\leq n-1$, the convergence of $\{L_n\}$ in almost any norm is equivalent to the norms of these operators being bounded independent of n. The **Lebesgue function** is the weighted norm of $L_n[f]$ at a given point. When working on infinite intervals, it makes more sense to investigate weighted version of these. The idea is

$$\begin{aligned} |L_n[f](x)|\, W(x) &\leq \sum_{j=1}^{n} |f(x_{jn})|\, |\ell_{jn}(x)|\, W(x) \\ &\leq \|fW\|_{L_\infty(\mathbb{R})} W(x) \sum_{j=1}^{n} |\ell_{jn}(x)|\, W^{-1}(x_{jn}) =: \|fW\|_{L_\infty(\mathbb{R})} \Lambda_n(x). \end{aligned}$$

It is easy to see by suitable choice of f that

$$\Lambda_n(x) = \sup\left\{ |L_n[f](x) W(x)| : \|fW\|_{L_\infty(\mathbb{R})} \leq 1 \right\},$$

so the Lebesgue function $\Lambda_n(x)$ is indeed a norm. The **Lebesgue constant** of L_n is

$$\Lambda_n := \|\Lambda_n\|_{L_\infty(\mathbb{R})}.$$

Observe that for any function f, there is the pointwise error estimate

$$\begin{aligned} W(x)\,|f - L_n[f]|(x) &\leq (1 + \Lambda_n(x)) \inf_{\deg(P)\leq n-1} \|(f-P)W\|_{L_\infty(\mathbb{R})} \\ &= (1 + \Lambda_n(x))\, E_n[f;W]_\infty \end{aligned}$$

and hence the global estimate

$$\|W(f - L_n[f])\|_{L_\infty(\mathbb{R})} \leq (1 + \Lambda_n)\, E_n[f;W]_\infty.$$

Amongst the contributors to estimation of $\Lambda_n(x)$ on infinite intervals are Damelin, Freud, Horváth, Kubayi, Matjila, Nevai, Sklyarov, Szabados, Szabó, Szili and Vértesi. The earliest results were due to Freud and Sklyarov.

The following result for general Freud weights is due to Szabados [176]. Indeed, it was in this context that Szabados introduced his "method of additional points". Matjila [124], [126] independently obtained the upper bound implicit in (a).

Theorem 11.8 *Let $W = \exp(-Q) \in \mathcal{F}^*$.*

(a) *Let Λ_n denote the Lebesgue constant for the Lagrange interpolation polynomials $L_n[\cdot]$ at the zeros of $p_n\left(W^2, \cdot\right)$. Then*

$$\Lambda_n \sim n^{1/6}, \qquad n \geq 1.$$

(b) *Let Λ_n denote the Lebesgue constant for the Lagrange interpolation polynomials $L_n^*[\cdot]$ at the zeros of $p_n\left(W^2, \cdot\right)$, together with the points $\pm\xi_n$ where $|p_n W|$ attains its maximum. Then*

$$\Lambda_n^* \sim \log(n+1), \qquad n \geq 1.$$

The $n^{1/6}$ arises from the factor $\left(1 - \frac{|x|}{a_n}\right)^{-1/4}$ that gives the growth of p_n near $\pm a_n$, together with the fact that

$$1 - \frac{x_{1n}}{a_n} \sim n^{-2/3}.$$

It is remarkable that Szabados' addition of the two points $\pm\xi_n$ gives the optimal factor of $\log n$. Yes, that is the same lower bound of $\log n$ for the sup norm of projection operators onto the space of polynomials of degree $\leq n$, in the context of finite intervals [179]. In the context of exponential weights, this optimality was proved by Szabados [176]. There are extensions of this result to Erdős weights, and exponential weights on $(-1, 1)$ [12], [13], [71], [73], [74]. Typically there one obtains the same result for Λ_n^*, while

$$\Lambda_n \sim (nT(a_n))^{1/6}, \qquad n \geq 1.$$

Some inkling of the proofs comes from the representation for the fundamental polynomials

$$\ell_{jn}(x) = \lambda_{jn} K_n(x, x_{jn}),$$

when we interpolate at the zeros of p_n. Then

$$\begin{aligned}
\Lambda_n(x) &= W(x) \sum_{j=1}^{n} |\ell_{jn}(x)| W^{-1}(x_{jn}) = W(x) \sum_{j=1}^{n} \lambda_{jn} W^{-1}(x_{jn}) |K_n(x, x_{jn})| \\
&\approx W(x) \int_{-\infty}^{\infty} W(t) |K_n(x,t)| \, dt.
\end{aligned}$$

Recall that $\lambda_{jn} \sim \frac{a_n}{n} W^2(x_{jn})$ at least for x_{jn} well inside the interval $[-a_n, a_n]$.

For general sets of interpolation nodes, lower bounds for Lebesgue functions have been obtained by Vértesi and his coworkers [189], [190]. These show that most of the time the Lebesgue function is bounded below by $\log n$ [190, p. 359]:

Theorem 11.9 *Let $W \in \mathcal{F}^*$. For $n \geq 1$, let $\Lambda_n(x)$ denote the Lebesgue function corresponding to n distinct points (not necessarily the zeros of p_n). Let $\varepsilon > 0$. Then there is a set $\mathcal{H}_n$ of linear Lebesgue measure $\leq 2\varepsilon a_n$, such that*

$$\Lambda_n(x) \geq C\varepsilon \log n, \qquad x \in [-a_n, a_n] \backslash \mathcal{H}_n.$$

The constant is independent of n, x, ε, and the array of interpolation points.

Szabados analyzed the influence of the distribution of the interpolation points, and in particular the largest interpolation point, on the size of the fundamental polynomials and Lebesgue constant [177]. Kubayi [72] and Damelin [17] explored Szabados' result in the context of more general weights. Summability of weighted Lagrange interpolation has been investigated by Szili and Vértesi [183], [184], [185].

When one increases the degree of the polynomial interpolating at n points from $n-1$ to $\lfloor n(1+\varepsilon)\rfloor$, some fixed $\varepsilon > 0$, then one can avoid the factor of $\log n$ above. For finite intervals, this was an old result of Erdős. For Freud weights, it is due to Vértesi [191], and for exponential weights on $[-1,1]$, due to Szili and Vértesi [182].

Theorem 11.10 *Let $W \in \mathcal{F}^*$. Let us be given an array of interpolation points $\{x_{jn}\}_{n\geq 1, 1\leq j\leq n}$ with associated fundamental polynomials of interpolation $\{\ell_{jn}\}_{n\geq 1, 1\leq j\leq n}$. Assume that these admit the bound*

$$\sup_{x\in\mathbb{R}} |\ell_{jn}(x)|\, W^{-1}(x_{jn})\, W(x) \leq A,$$

for $n \geq 1$ and $1 \leq j \leq n$. Let $\varepsilon > 0$ and let f be a continuous function such that $fW^{1+\varepsilon}$ has limit 0 at $\pm\infty$. Then there exist polynomials $\{P_n\}$ such that

(I) *P_n has degree $N_n \leq n\left(1+\varepsilon + C\varepsilon n^{-2/3}\right)$;*

(II) *$P_n(x_{jn}) = f(x_{jn})$, $1 \leq j \leq n$;*

(III) $\|W^{1+\varepsilon}(f-P_n)\|_{L_\infty(\mathbb{R})} \leq CE_{N_n}\left[f; W^{1+\varepsilon}\right]_\infty$.

Note that the fundamental polynomials associated with the orthogonal polynomials $p_n(W^2,x)$ satisfy the above bound [83].

Despite the many investigations of Lagrange interpolation, there are relatively few results where local assumptions are imposed that guarantee pointwise convergence of $\{L_n[f]\}$. It seems that the following interesting result of Nevai has never been extended beyond the case of the Hermite weight [146]:

Theorem 11.11 *Let $W(x) = \exp(-x^2)$. Let f be Riemann integrable in each finite interval, and assume that for some $c < \frac{1}{2}$,*

$$\|fW^c\|_{L_\infty(\mathbb{R})} < \infty.$$

Assume that on $[a,b]$, f satisfies the one-sided Dini condition

$$f(x+t) - f(x) \geq -v(t)\,|\log t|^{-1}, \qquad a < x < x+t < b,$$

where $v(t)$ decreases to 0 as t decreases to 0. Then

$$\lim_{n\to\infty} L_n[f](x) = f(x) \tag{11.22}$$

at each point of continuity of f in (a,b). The convergence is uniform in every closed subinterval of (a,b) where f is continuous.

Note that any increasing function f satisfies this Dini condition, so (11.22) holds also when f is of bounded variation in $[a,b]$. (For then f is the difference of monotone increasing functions.) Finally, we note that convergence of Lagrange interpolation to entire functions of suitably restricted growth leads to a geometric rate of convergence, see [136].

12 Marcinkiewicz–Zygmund Inequalities

The classical Marcinkiewicz–Zygmund inequality has the form

$$C_1 \leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |R(\theta)|^p \, d\theta \right)^{1/p} \Big/ \left(\frac{1}{2n+1} \sum_{j=0}^{2n} \left| R\left(\frac{j\pi}{2n+1} \right) \right|^p \right) \leq C_2,$$

where $n \geq 1$, R is a trigonometric polynomial of degree $\leq n$, and C_1 and C_2 are independent of n and R [193, Vol. 2, p. 28]. These types of inequalities are useful, for example, in discretisation problems, quadrature theory, and Lagrange interpolation.

In the context of exponential weights, these have chiefly been used for mean convergence of Lagrange interpolation. In fact, the proof of the Erdős–Turán theorem above was based on a converse quadrature sum, namely,

$$\int_{-\infty}^{\infty} L_n [f]^2 \, W^2 = \sum_{j=1}^{n} \lambda_{jn} f^2(x_{jn}).$$

This of course is an immediate consequence of the Gauss quadrature formula. In our outline of the proof of Lemma 11.5 above we used a forward quadrature sum. Let us repeat this, in simplified form: assume that we have an inequality

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|(x_{jn}) \leq C \int_{-\infty}^{\infty} |PW| \, . \tag{12.1}$$

If $q = \frac{p}{p-1}$, duality gives

$$\| L_n [f] \, W \|_{L_p(\mathbb{R})} = \sup_g \int_{-\infty}^{\infty} L_n [f] \, g W^2,$$

where the sup is taken over all g with $\|gW\|_{L_q(\mathbb{R})} \leq 1$. Using orthogonality of $g - S_n [g]$ to polynomials of degree $\leq n - 1$, and then the Gauss quadrature formula gives

$$\begin{aligned}
&= \sup_g \int_{-\infty}^{\infty} L_n [f] \, S_n [g] \, W^2 = \sup_g \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}) S_n [g] (x_{jn}) \\
&\leq \|fW\|_{L_\infty(\mathbb{R})} \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \left| S_n [g] (x_{jn}) W(x_{jn}) \right| \\
&\leq C \|fW\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} |S_n [g]| \, W.
\end{aligned}$$

In the last line, we used the quadrature sum estimate (12.1). Setting $\sigma_n := \operatorname{sign}(S_n [g])$, and then using self-adjointness of S_n, we continue the above as

$$\begin{aligned}
= C \|fW\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} \sigma_n W^{-1} S_n [g] \, W^2 &= C \|fW\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} S_n \left[\sigma_n W^{-1} \right] g W^2 \\
&\leq C \|fW\|_{L_\infty(\mathbb{R})} \left\| S_n \left[\sigma_n W^{-1} \right] W \right\|_{L_p(\mathbb{R})} .
\end{aligned}$$

If S_n is bounded in a suitable sense, we obtain

$$\|L_n[f]W\|_{L_p(\mathbb{R})} \leq C\|fW\|_{L_\infty(\mathbb{R})}$$

with C independent of n. We emphasize that this is a simplified version of what one needs. Typically there are extra factors inside the quadrature sum. In subsequent sections, we shall present several methods to establish such quadrature sums.

12.1 Nevai's Method for Forward Quadrature Sums

We let $u \in [x_{jn}, x_{j-1,n}]$, $p \geq 1$, and start with the following consequence of the fundamental theorem of calculus:

$$|PW|^p(x_{jn}) \leq |PW|^p(u) + p\int_u^{x_{j-1,n}} |PW|^{p-1}(s)\left|(PW)'(s)\right| ds.$$

We may assume that u is the point in $[x_{jn}, x_{j-1,n}]$ where $|PW|^p$ attains its minimum. We now need estimates for Christoffel functions, and spacing of zeros of orthogonal polynomials. In fact the former imply the latter via Markov–Stieltjes inequalities. We use

$$\lambda_{jn}W^{-2}(x_{jn}) = \lambda_n(W^2, x_{jn})W^{-2}(x_{jn}) \sim \varphi_n(x_{jn}),$$

where φ_n is defined by (7.8), and

$$x_{jn} - x_{j-1,n} \geq C\lambda_{jn}W^{-2}(x_{jn}).$$

Then

$$\begin{aligned}\lambda_{jn}W^{-2}(x_{jn})|PW|^p(x_{jn}) \quad &\leq \quad C\int_{x_{jn}}^{x_{j-1,n}} |PW|^p(u)du \\ &\quad + C\varphi_n(x_{jn})\int_{x_{jn}}^{x_{j-1,n}} |PW|^{p-1}(s)|(PW)'(s)|ds.\end{aligned}$$

Summing over j, and using the fact that φ_n does not change much in $[x_{jn}, x_{j-1,n}]$, one obtains

$$\begin{aligned}\sum_{j=1}^{n} \lambda_{jn}W^{-2}(x_{jn})|PW|^p(x_{jn}) &\leq C\int_{-\infty}^{\infty} |PW|^p(u)du \\ &\quad + C\int_{-\infty}^{\infty} |PW|^{p-1}(s)|(PW)'(s)|\varphi_n(s)ds.\end{aligned} \tag{12.2}$$

We apply Hölder's inequality, and then the Bernstein inequality Theorem 7.6 to the second term:

$$\int_{-\infty}^{\infty} |PW|^{p-1}(s)|(PW)'(s)|\varphi_n(s)ds \leq \|PW\|_{L_p(\mathbb{R})}^{p-1}\|(PW)'\varphi_n\|_{L_p(\mathbb{R})} \leq C\|PW\|_{L_p(\mathbb{R})}^{p-1}\|PW\|_{L_p(\mathbb{R})}.$$

Together with (12.2), this gives

$$\sum_{j=1}^{n} \lambda_{jn}W^{-2}(x_{jn})|PW|^p(x_{jn}) \leq C\|PW\|_{L_p(\mathbb{R})}^p. \tag{12.3}$$

Here is a sample of what can be proved using this method [107, Thm. 2, p. 287]:

Theorem 12.1 *Let $W \in \mathcal{F}^*$, $1 < p < \infty$, $-\infty < b \leq 2$ and $a \in \mathbb{R}$. Then*

$$\sum_{j=1}^{n} \lambda_{jn} W^{-b}(x_{jn}) \left(1 + |x_{jn}|\right)^a |PW|^p(x_{jn}) \leq C \int_{-\infty}^{\infty} |PW|^p(t) \left(1 + |t|\right)^a W^{2-b}(t) dt.$$

This method has also been used by H. König in a vector valued and Banach space setting, with W as the Hermite weight [65], [66]. It was used first for Hermite weights by Nevai [149]. Of course, it can be used for points other than zeros of orthogonal polynomials — all we need are suitable estimates on the spacing between successive zeros.

12.2 The Large Sieve Method

As far as the author is aware, this also was developed by P. Nevai, based on the large sieve of number theory, and used by him and his coworkers for exponential weights [105]. We already presented the idea in the context of Markov–Bernstein inequalities in Section 8.6. We start with Lemma 8.3. For polynomials S of degree $\leq n$,

$$|SW|^p(\xi) \leq C \frac{\int_{-\infty}^{\infty} |SW|^p(t) K_{Mn}(\xi, t)^2 W^2(t) dt}{\int_{-\infty}^{\infty} K_{Mn}(\xi, t)^2 W^2(t) dt},$$

which is equivalent to

$$|SW|^p(\xi) \leq C \lambda_{Mn}(\xi) \int_{-\infty}^{\infty} |SW|^p(t) K_{Mn}(\xi, t)^2 W^2(t) dt.$$

Here $p > 0$, S is a polynomial of degree $\leq n$, and $|\xi| \leq 2a_n$, while M is a fixed large enough positive integer, independent of n, P, ξ. We now simply take linear combinations of this, and use the bound

$$\lambda_{Mn}(W^2, \xi) \leq C \frac{a_n}{n} W^2(\xi), \qquad |\xi| \leq 2a_n.$$

For example, if we use the zeros of p_n, we obtain

$$\sum_{j=1}^{n} |SW|^p(x_{jn}) \leq C \int_{-\infty}^{\infty} |SW|^p(t) \Sigma_n(t) dt,$$

where

$$\Sigma_n(t) = \frac{a_n}{n} W^2(t) \sum_{j=1}^{n} K_{Mn}^2 \left(x_{jn}, t\right) W^2(x_{jn}).$$

Here, by the Christoffel–Darboux formula, and our bounds for p_n, and recalling our bound (8.31) for $K_{Mn}\left(\xi, \xi\right)$, we see that for $|\xi|, |t| \leq 2a_n$,

$$W(\xi) W(t) \left|K_{Mn}(\xi, t)\right| \leq \min\left\{\frac{n}{a_n}, \frac{1}{|\xi - t|}\right\}.$$

Moreover for Freud weights, uniformly in j, n

$$x_{jn} - x_{j+1,n} \geq C \frac{a_n}{n}.$$

Hence

$$\Sigma_n(t) \le C \sum_{j=1}^{n} (x_{jn} - x_{j+1,n}) \min\left\{\frac{a_n}{n}, |x_{jn} - t|\right\}^{-2} \le C \int_{-\infty}^{\infty} \min\left\{\frac{a_n}{n}, |u-t|\right\}^{-2} du \le C_1 \frac{n}{a_n}.$$

Thus we arrive at the estimate

$$\sum_{j=1}^{n} \frac{a_n}{n} |SW|^p (x_{jn}) \le C \int_{-\infty}^{\infty} |SW|^p(t) dt,$$

valid for $n \ge 1$ and polynomials P of degree $\le n$. We note that this does not imply the estimate

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |SW|^p(x_{jn}) \le C \int_{-\infty}^{\infty} |SW|^p(t) dt$$

since

$$\lambda_{jn} W^{-2}(x_{jn}) \sim \frac{a_n}{n} \left(\left| 1 - \frac{|x_{jn}|}{a_n} \right| + n^{-2/3} \right)^{-1/2} \gg \frac{a_n}{n},$$

when x_{jn} is close to $\pm a_n$. Thus this method is in some respects not as powerful as Nevai's method. However, it can easily yield estimates like

$$\sum_{j=1}^{n} \frac{a_n}{n} \psi(|SW|^p(x_{jn})) \le C \int_{-\infty}^{\infty} \psi(|SW|^p(t)) dt$$

where ψ is an increasing convex function that is positive in $(0, \infty)$ and has $\psi(0) = 0$, and works for any set of suitably spaced points.

12.3 The Duality Method

The idea here is to use duality of L_p spaces to derive a forward quadrature sum estimate from a converse one. It was probably first used by König [65], [66]. Let us illustrate this in the context of quadrature sums at zeros of orthogonal polynomials. Let n be fixed and μ_n be a pure jump measure having mass $\lambda_{jn} W^{-2}(x_{jn})$ at x_{jn}. Let, as usual, $q = \frac{p}{p-1}$. If P is a polynomial of degree $\le n$,

$$\left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|^p(x_{jn}) \right)^{1/p} = \left(\int |PW|^p d\mu_n \right)^{1/p} = \sup_g \int PW^2 g \, d\mu_n,$$

where the sup is taken over all g with

$$\int |gW|^q \, d\mu_n \le 1.$$

Since g needs to be defined only at $\{x_{jn}\}_{j=1}^{n}$, we can assume that g is a polynomial of degree $\le n-1$. So

$$\int PW^2 g \, d\mu_n = \sum_{j=1}^{n} \lambda_{jn} (Pg)(x_{jn}) = \int_{-\infty}^{\infty} PgW^2 \le \|PW\|_{L_p(\mathbb{R})} \|gW\|_{L_q(\mathbb{R})}.$$

Now we assume a suitable converse quadrature sum inequality: for $n \geq 1$ and all S of degree $\leq n$,

$$\|SW\|_{L_q(\mathbb{R})} \leq C \left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |SW|^q (x_{jn}) \right)^{1/q}. \tag{12.4}$$

Applying this to g, gives

$$\|gW\|_{L_q(\mathbb{R})} \leq C \left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |gW|^q (x_{jn}) \right)^{1/q} = C \left(\int |gW|^q d\mu_n \right)^{1/q} \leq C.$$

Thus we obtain

$$\left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |SW|^p (x_{jn}) \right)^{1/p} \leq C \|PW\|_{L_p(\mathbb{R})}.$$

In König's work, he first proved a converse inequality like (12.4) for $1 < q < 4$, and then deduced a forward estimate for $\frac{4}{3} < p < \infty$. Note that one can also use Carleson measures to derive forward quadrature estimates, just as we derived Markov–Bernstein inequalities in Section 8.5. For this approach on finite intervals, see the survey [95].

12.4 The Duality Method for Converse Quadrature Sum estimates

This was perhaps the earliest method used for converse quadrature sum estimates. For trigonometric polynomials, it appears in Zygmund's treatise [193, Ch. X, pp. 28–29]. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. Let P be a polynomial of degree $\leq n-1$. By duality,

$$\|PW\|_{L_p(\mathbb{R})} = \sup_g \int gPW^2,$$

where the sup is taken over all g with $\|gW\|_{L_q(\mathbb{R})} \leq 1$. Let $S_n[g]$ denote the nth partial sum of the orthonormal expansion of g with respect to $\{p_j\}_{j=0}^{\infty}$. By orthogonality, we continue this as

$$\|PW\|_{L_p(\mathbb{R})} = \sup_g \int S_n[g] PW^2.$$

Using the Gauss quadrature formula, and then Hölder's inequality, we continue this as

$$\begin{aligned}
&= \sup_g \sum_{j=1}^{n} \lambda_{jn} S_n[g](x_{jn}) P(x_{jn}) \\
&\leq \sup_g \left\{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |S_n[g] W|^q (x_{jn}) \right\}^{1/q} \left\{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|^p (x_{jn}) \right\}^{1/p} \\
&=: \sup_g T_1 \times T_2.
\end{aligned}$$

If we have a suitable forward quadrature sum estimate, and mean boundedness of S_n in a weighted L_q norm, we can bound T_1 as follows:

$$T_1 \leq C \left\| S_n [g] W \right\|_{L_q(\mathbb{R})} \leq C \left\| gW \right\|_{L_q(\mathbb{R})} \leq C.$$

Then for $n \geq 1$ and all polynomials P of degree $\leq n$,

$$\|PW\|_{L_p(\mathbb{R})} \leq C \left\{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|^p (x_{jn}) \right\}^{1/p}.$$

This method does yield estimates for full Gauss quadrature sums. However, it requires mean boundedness of orthogonal expansions, such as that in Theorem 10.3, which is difficult to prove. It also requires a forward quadrature sum estimate, which is not all that difficult to prove. It works best for $p \geq 4$. For $p < 4$, the next method gives better results. Here is a sample of what one can achieve [91, Thm. 1.2, p. 531]. We note that because we are considering a special case of results there, the hypotheses there are simplified below:

Theorem 12.2 *Let $W \in \mathcal{F}^*$, and assume that the orthonormal polynomials $\{p_n\}$ for W^2 satisfy the bound* (10.7)*. Let $p > 4$ and $r, R \in \mathbb{R}$ satisfy*

$$R > -\frac{1}{p} \tag{12.5}$$

and

$$a_n^{r - \min\left\{R, 1-\frac{1}{p}\right\}} n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} = \begin{cases} O(1), & R \neq 1 - \frac{1}{p}, \\ O\left((\log n)^{-R}\right), & R = 1 - \frac{1}{p} \end{cases}. \tag{12.6}$$

Then for $n \geq 1$ and polynomials P of degree $\leq n-1$,

$$\left\| (PW)(x) \left(1 + |x|\right)^r \right\|_{L_p(\mathbb{R})} \leq C \left\{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|^p (x_{jn}) \left(1 + |x_{jn}|\right)^{Rp} \right\}^{1/p}. \tag{12.7}$$

For $p = 4$, this persists provided (12.5) *holds and*

$$r - \min\left\{R, 1 - \frac{1}{p}\right\} < 0.$$

Weights that satisfy the bound (10.7) include $\exp(-|x|^\alpha)$, $\alpha > 1$. In [91], we showed that slightly weaker forms of these conditions on r, R are necessary.

12.5 König's Method

König's method [65], [66] is based on a clever estimate for Hilbert transforms of characteristic functions of intervals. It is technically the most difficult of all the methods we present (for forward or converse estimates), but is very powerful, and relatively direct. It does not depend on mean boundedness of orthogonal expansions. For $n \geq 1$ and $1 \leq j \leq n$, let

$$I_{jn} := [x_{jn}, x_{j-1,n}); \qquad |I_{jn}| = x_{j-1,n} - x_{j,n}; \qquad \chi_{jn} := \chi_{I_{jn}}.$$

Let P be a polynomial of degree $\leq n-1$, and

$$y_{jn} := a_n^{-1/2} \frac{P(x_{jn})}{p_n'(x_{jn})}.$$

Then

$$\begin{aligned} P(x) &= L_n\left[P\right](x) = a_n^{1/2} p_n(x) \sum_{j=1}^{n} \frac{y_{jn}}{x - x_{jn}} \\ &= a_n^{1/2} p_n(x) \sum_{j=1}^{n} y_{jn} \left\{ \frac{1}{x - x_{jn}} - \frac{1}{|I_{jn}|} H\left[\chi_{jn}\right](x) \right\} + a_n^{1/2} p_n(x) H\left[\sum_{j=1}^{n} \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \right](x) \\ &=: J_1(x) + J_2(x). \end{aligned}$$

Here (as above),

$$H\left[f\right](x) = \lim_{\varepsilon \to 0+} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{t - x} dt$$

is the Hilbert transform of f. The term J_2 is easier, so let us deal with it first. We use the bound

$$|p_n W|(x) \leq C a_n^{-1/2} \left| 1 - \frac{|x|}{a_n} \right|^{-1/4}, \qquad x \in \mathbb{R},$$

valid when $W \in \mathcal{F}^*$, and the Hilbert transform bound [144]

$$\|H\left[g\right](x) \left| 1 - \frac{|x|}{a_n} \right|^{-1/4} \|_{L_p(|x| \leq 2a_n)} \leq C \|g(x) \left| 1 - \frac{|x|}{a_n} \right|^{-1/4} \|_{L_p(|x| \leq 2a_n)},$$

valid for $1 < p < 4$. Here $C \neq C(n, g)$. Then

$$\begin{aligned} \|J_2 W\|_{L_p[-2a_n, 2a_n]} &= a_n^{1/2} \|p_n W H \left[\sum_{j=1}^{n} \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \right] \|_{L_p[-2a_n, 2a_n]} \\ &\leq C \| \left| 1 - \frac{|x|}{a_n} \right|^{-1/4} \sum_{j=1}^{n} \frac{y_{jn}}{|I_{jn}|} \chi_{jn}(x) \|_{L_p[-2a_n, 2a_n]} \\ &= C \left\{ \sum_{j=1}^{n} \left(\frac{y_{jn}}{|I_{jn}|} \right)^p \int_{I_{jn}} \left| 1 - \frac{|x|}{a_n} \right|^{-p/4} dx \right\}^{1/p} \\ &\leq C \left\{ \sum_{j=1}^{n} \left(\frac{y_{jn}}{|I_{jn}|} \right)^p |I_{jn}| \, \psi_n^{-p/4}(x_{jn}) \right\}^{1/p}. \end{aligned}$$

Here

$$\psi_n(x) = \max\left\{ n^{-2/3}, 1 - \frac{|x|}{a_n} \right\}.$$

It can be shown that uniformly in j,n,

$$\frac{y_{jn}}{|I_{jn}|}\sim |PW|(x_{jn})\psi_n^{1/4}(x_{jn});\quad a_n^{1/2}\left|p_n'W\right|(x_{jn})\sim |I_{jn}|^{-1}\psi_n^{-1/4}(x_{jn});\quad |I_{jn}|\sim\lambda_{jn}W^{-2}(x_{jn}).$$

Hence

$$\|J_2W\|_{L_p[-2a_n,2a_n]}\leq C\left\{\sum_{j=1}^{n}\lambda_{jn}W^{-2}(x_{jn})|PW|^p(x_{jn})\right\}^{1/p}.$$

The estimation of J_1 is difficult. One shows that for $|x-x_{jn}|>2\,|I_{jn}|$,

$$\begin{aligned}\left|\frac{1}{x-x_{jn}}-\frac{1}{|I_{jn}|}H\left[\chi_{jn}\right](x)\right|a_n^{1/2}|p_nW|(x) &\leq C\psi_n^{-1/4}(x)\frac{|I_{jn}|}{|x-x_{jn}|}\left\{\frac{1}{|x-x_{jn}|}+\frac{1}{1+|x_{jn}|}\right\}\\ &=: Cf_{jn}(x),\end{aligned}$$

and for $|x-x_{jn}|\leq 2\,|I_{jn}|$,

$$\left|\frac{1}{x-x_{jn}}-\frac{1}{|I_{jn}|}H\left[\chi_{jn}\right](x)\right|a_n^{1/2}|p_nW|(x)\leq C\frac{\psi_n^{-1/4}(x)}{|I_{jn}|}=:Cf_{jn}(x),$$

so that

$$|J_1(x)|\leq C\sum_{j=1}^{n}|y_{jn}|\,f_{jn}(x).$$

As each f_{jn} does not change much in I_{kn}, one obtains

$$\|J_1W\|_{L_p[x_{nn},x_{1n}]}\leq C\left\{\sum_{k=2}^{n}|I_{kn}|\left[\sum_{j=1}^{n}|y_{jn}|\,f_{jn}(x_{kn})\right]^p\right\}^{1/p}.$$

One now splits this sum into three pieces. The main part is estimated by showing that certain n by n matrices are uniformly bounded in norm. These in turn depend on clever consequences of Hölder's inequality. After a lot of technical work, one obtains

$$\|J_1W\|_{L_p(\mathbb{R})}\leq C\left\{\sum_{j=1}^{n}\lambda_{jn}W^{-2}(x_{jn})|PW|^p(x_{jn})\right\}^{1/p}.$$

Here is a sample of what can be proved [91, Thm. 1.2, p. 531]:

Theorem 12.3 *Let $W\in\mathcal{F}^*$ and $1<p<4$. Let*

$$r<1-\frac{1}{p};\qquad r\leq R;\qquad R>-\frac{1}{p}. \tag{12.8}$$

Then for $n\geq 1$ and polynomials P of degree $\leq n$,

$$\|(PW)(x)\,(1+|x|)^r\|_{L_p(\mathbb{R})}\leq C\left\{\sum_{j=1}^{n}\lambda_{jn}W^{-2}(x_{jn})|PW|^p(x_{jn})(1+|x_{jn}|)^{Rp}\right\}^{1/p}. \tag{12.9}$$

Here $C\neq C\,(n,P)$.

There it was also shown that the first two conditions on r, R are necessary. For $p \geq 4$, the author and Mastroianni used König's method to prove [104, Thm. 1.2, p. 149].

Theorem 12.4 *Let $W \in \mathcal{F}^*$ and $p \geq 4$. Let (12.8) hold, and in addition, assume that for some $\delta > 0$,*

$$n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} a_n^{r-\min\left\{R,1-\frac{1}{p}\right\}} = O(n^{-\delta}). \tag{12.10}$$

Then (12.9) *holds for $n \geq 1$ and polynomials P of degree $\leq n$.*

For further work on quadrature sum estimates in the context of exponential weights, see [18], [23], [96]. In particular, in the latter paper, an attempt was made to provide a very general framework for König's method.

13 Hermite and Hermite–Féjer Interpolation

The Hermite interpolation polynomial to a function f at the zeros $\{x_{jn}\}_{j=1}^n$ of $p_n(x)$ is a polynomial $H_n[f]$ of degree $\leq 2n-1$ such that for $1 \leq j \leq n$,

$$\begin{aligned} H_n[f](x_{jn}) &= f(x_{jn}), \\ H_n'[f](x_{jn}) &= f'(x_{jn}). \end{aligned}$$

Of course, we have to assume f' is defined at the zeros $\{x_{jn}\}$. To avoid this restriction, the Hermite–Féjer interpolant $H_n^*[f]$ is often used instead. It is a polynomial of degree $\leq 2n-1$ such that for $1 \leq j \leq n$,

$$\begin{aligned} H_n^*[f](x_{jn}) &= f(x_{jn}), \\ H_n^{*\prime}[f](x_{jn}) &= 0. \end{aligned}$$

Hermite–Féjer polynomials first came to prominence when Féjer used them (with interpolation at zeros of Chebyshev polynomials) to provide another proof of Weierstrass' approximation theorem. A close relative of the Hermite–Féjer interpolation polynomial is the Grünwald operator

$$Y_n[f](x) = \sum_{j=1}^{n} f(x_{jn}) \ell_{jn}^2(x),$$

where $\{\ell_{jn}\}$ are the fundamental polynomials of Lagrange interpolation taken at the zeros of $p_n(x) = p_n\left(W^2, x\right)$. Y_n has the advantage of being a positive operator, which makes convergence more general, though possibly at a slower rate than interpolatory operators.

Convergence of Hermite and Hermite–Féjer polynomials associated with exponential weights has been investigated in [20], [22], [54], [55], [58], [59], [60], [62], [63], [64], [89], [109], [110], [142], [180]. In some cases, processes involving higher order derivatives have been studied, and the derivatives of the interpolating polynomials have also been investigated. Here we note just one result of Szabó [180]:

Theorem 13.1 *Let $W = \exp(-Q) \in \mathcal{F}^*$, with A as in* (4.18). *Let*

$$W_2(x) = W^2(x)\left(1 + \left|Q'(x)\right|\right)^{\frac{1}{3(1-1/A)}} \log\left(2 + \left|Q'(x)\right|\right).$$

Then for every continuous $f : \mathbb{R} \to \mathbb{R}$ *with*

$$\lim_{|x|\to\infty} fW_2(x) = 0,$$

we have

$$\lim_{n\to\infty} \left\| W^2 \left(f - H_n^* \right) \right\|_{L_\infty(R)} = 0.$$

Note the square of the weight in the condition on f and in the approximating norm. Szabó showed that the factor

$$\left(1 + \left|Q'(x)\right|\right)^{\frac{1}{3(1-1/A)}} \log\left(2 + \left|Q'(x)\right|\right)$$

is essential and best possible.

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15 Summary of Notation

In this section, we repeat the main notation introduced earlier in this paper, and repeat some definitions. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t and polynomials P of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively show that C depends on α, or does not depend on α.

We use the notation $\sim$ in the following sense: given sequences of real numbers $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if for some positive constants C_1, C_2 independent of n, we have

$$C_1 \leq c_n/d_n \leq C_2.$$

If x is a real number, then $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

Throughout, $W = \exp(-Q)$ is an even exponential weight. Associated with W are the Mhaskar–Rakhmanov–Saff numbers $a_n = a_n(Q)$, the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt.$$

If $xQ'(x)$ is strictly increasing, with limits 0 and ∞ at 0 and ∞ respectively, then a_n is uniquely defined. Corresponding to the weight W, we define orthonormal polynomials

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m W^2 = \delta_{mn}.$$

Note that the weight is W^2, not W. We denote the zeros of p_n by

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

The nth Christoffel function for W^2 is

$$\lambda_n(W^2, x) = \inf_{\deg(P) \leq n-1} \frac{\int_{-\infty}^{\infty} (PW)^2}{P^2(x)}. \tag{15.1}$$

It also satisfies

$$\lambda_n(W^2, x) = 1 \Big/ \sum_{j=0}^{n-1} p_j^2(x).$$

The reproducing kernel of order m is

$$K_m(x,t) := \sum_{j=0}^{m-1} p_j(x) p_j(t) = \frac{\gamma_{m-1}}{\gamma_m} \frac{p_m(x)p_{m-1}(t) - p_{m-1}(x)p_m(t)}{x-t}.$$

Our four main classes of weights follow:

Definition 15.1 (Freud Weights $\mathcal{F}$): *Let $W = \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, Q' exists and is positive in $(0, \infty)$. Moreover, assume that $xQ'(x)$ is strictly increasing, with right limit 0 at 0, and for some $\lambda, A, B > 1, C > 0$,*

$$A \leq \frac{Q'(\lambda x)}{Q'(x)} \leq B, \qquad x \geq C.$$

Then we write $W \in \mathcal{F}$.

Definition 15.2 (Freud Weights $\mathcal{F}^*$): *Let $W = \exp(-Q)$, where Q'' exists and is positive in $(0, \infty)$, while Q' is positive there, with limit 0 at 0, and for some $A, B > 1$,*

$$A - 1 \leq \frac{xQ''(x)}{Q'(x)} \leq B - 1, \qquad x \in (0, \infty).$$

Then we write $W \in \mathcal{F}^*$.

The canonical examples of W in $\mathcal{F}^*$ are

$$W_\alpha(x) = \exp(-|x|^\alpha), \qquad \alpha > 0,$$

while $\mathcal{F}^* \subset \mathcal{F}$.

Definition 15.3 (Erdős Weights $\mathcal{E}$): *Let $W = \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, Q' exists and is positive in $(0, \infty)$. Assume that $xQ'(x)$ is strictly increasing, with right limit 0 at 0, and the function*

$$T(x) := \frac{xQ'(x)}{Q(x)} \tag{15.2}$$

is quasi-increasing in the sense that

$$0 \leq x < y \quad \Longrightarrow \quad T(x) \leq CT(y),$$

while

$$\lim_{x\to\infty} T(x) = \infty. \tag{15.3}$$

Assume, moreover, that for some C_1, C_2 *and* $C_3 > 0$,

$$\frac{yQ'(y)}{xQ'(x)} \le C_1 \left(\frac{Q(y)}{Q(x)} \right)^{C_2}, \qquad y \ge x \ge C_3.$$

Then we write $W \in \mathcal{E}$.

Definition 15.4 (General Exponential Weights $\mathcal{F}_{even}(C^2)$): *Let* $I = (-d, d)$ *where* $0 < d \le \infty$. *Let* $Q : I \to [0, \infty)$ *satisfy the following properties:*

(a) Q' *is continuous and positive in* I *and* $Q(0) = 0$;

(b) Q'' *exists and is positive in* $(0, d)$;

(c)

$$\lim_{t\to d-} Q(t) = \infty;$$

(d) *The function*

$$T(t) = \frac{tQ'(t)}{Q(t)}$$

is quasi-increasing in $(0, d)$, *in the sense that*

$$0 \le x < y < d \quad \Longrightarrow \quad T(x) \le CT(y).$$

Moreover, assume for some $\Lambda > 1$,

$$T(t) \ge \Lambda > 1, \qquad t \in (0, d).$$

(e) *There exists* $C_1 > 0$ *such that*

$$\frac{|Q''(x)|}{Q'(x)} \le C_1 \frac{Q'(x)}{Q(x)}, \qquad x \in (0, d)$$

Then we write $W \in \mathcal{F}_{even}(C^2)$. *If also, there exists* $c \in (0, d)$ *such that*

$$\frac{|Q''(x)|}{Q'(x)} \le C_2 \frac{Q'(x)}{Q(x)}, \qquad x \in (c, d),$$

then we write $W \in \mathcal{F}_{even}(C^2+)$.

For $k \ge 1$, the kth iterated exponential is

$$\exp_k = \exp(\exp(\cdots \exp()))$$

Define the symmetric differences

$$\Delta_h f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2}); \qquad \Delta_h^k f(x) = \Delta_h \left(\Delta_h^{k-1} f(x) \right), \quad k \ge 1,$$

so that

$$\Delta_h^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i f(x + k\frac{h}{2} - ih).$$

Given $r \geq 1$ and a Freud weight $W = \exp(-Q)$ with Mhaskar–Rakhmanov–Saff numbers $\{a_n\}$, we define the rth order modulus of continuity as follows: First define the decreasing function of t,

$$\sigma(t) := \inf \left\{ a_n : \frac{a_n}{n} \leq t \right\}, \qquad t > 0.$$

The rth order modulus of continuity for W is

$$\omega_{r,p}(f, W, t) = \sup_{0<h\leq t} \|W\left(\Delta_h^r f\right)\|_{L_p[-\sigma(h),\sigma(h)]} + \inf_{\deg(P)\leq r-1} \|(f-P)W\|_{L_p(\mathbb{R}\setminus[-\sigma(t),\sigma(t)])}.$$

The rth order K-functional associated with W is

$$K_{r,p}\left(f, W, t^r\right) := \inf_g \left\{ \|(f-g)W\|_{L_p(\mathbb{R})} + t^r \|g^{(r)} W\|_{L_p(\mathbb{R})} \right\}, \qquad t \geq 0.$$

For Erdős weights $W \in \mathcal{E}$, the modulus of continuity involves the function

$$\Phi_t(x) := \sqrt{1 - \frac{|x|}{\sigma(t)}} + T(\sigma(t))^{-1/2},$$

where $\sigma(t)$ is as above. The rth order modulus is then

$$\begin{aligned} \omega_{r,p}(f, W, t) &= \sup_{0<h\leq t} \|W(x)\left(\Delta_{h\Phi_t(x)}^r f(x)\right)\|_{L_p[-\sigma(2t),\sigma(2t)]} \\ &\quad + \inf_{\deg(P)\leq r-1} \|(f-P)W\|_{L_p(\mathbb{R}\setminus[-\sigma(4t),\sigma(4t)])}. \end{aligned}$$

The error in weighted approximation of f by polynomials of degree $\leq n$ in the L_p norm is

$$E_n\left[f; W\right]_p := \inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})}.$$

If f is such that $fW^2 \in L_1(\mathbb{R})$, we define its orthonormal expansion

$$f \sim \sum_{j=0}^{\infty} c_j p_j,$$

where

$$c_j := \int_{-\infty}^{\infty} f p_j W^2, \qquad j \geq 0.$$

The nth partial sum of the orthonormal expansion is

$$S_n\left[f\right] = \sum_{j=0}^{n-1} c_j p_j,$$

while the nth de la Vallée Poussin operator is

$$V_n[f] = \frac{1}{n}\sum_{j=n+1}^{2n} S_j[f].$$

Given a function f, we define the nth Lagrange interpolation polynomial to f at $\{x_{jn}\}_{j=1}^{n}$ to be the unique polynomial of degree $\leq n-1$ that satisfies

$$L_n[f](x_{jn}) = f(x_{jn}), \qquad 1 \leq j \leq n.$$

One formula for $L_n[f]$ is

$$L_n[f] = \sum_{j=1}^{n} f(x_{jn})\ell_{jn}$$

where $\{\ell_{jn}\}_{j=1}^{n}$ are fundamental polynomials of Lagrange interpolation, satisfying

$$\ell_{jn}(x_{kn}) = \delta_{jk}.$$

The $\{\ell_{jn}\}$ admit the representation

$$\ell_{jn}(x) = \prod_{k=1,k\neq j}^{n} \frac{x - x_{kn}}{x_{jn} - x_{kn}}.$$

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School of Mathematics
Georgia Institute of Technology
Atlanta
GA 30332-0160
lubinsky@math.gatech.edu