Approximation in $\mathbb{C}^N$

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Abstract. This is a survey article on selected topics in approximation theory. The topics either use techniques from the theory of several complex variables or arise in the study of the subject. The survey is aimed at readers having an acquaintance with standard results in classical approximation theory and complex analysis but no apriori knowledge of several complex variables is assumed.

MSC: 32-02, 41-02

1 Introduction and motivation

Let $\mathbb{C}^N = \{(z_1, \ldots, z_N) : z_j \in \mathbb{C}\}$ where $z_j = x_j + iy_j$ and identify $\mathbb{R}^N = \{(x_1, \ldots, x_N) : x_j \in \mathbb{R}\}$. A complex-valued function f defined on an open subset of $\mathbb{C}^N$ is **holomorphic** if it is separately holomorphic in the appropriate planar region as a function of one complex variable when each of the remaining $N-1$ variables are fixed. This deceptively simple-minded criterion is equivalent to any other standard definition; e.g., f is locally representable by a convergent power series in the complex coordinates; or f is of class C^1 and satisfies the Cauchy-Riemann system

$$\frac{\partial f}{\partial \bar{z}_j} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\Big) = 0, \qquad j = 1, \ldots, N.$$

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In particular, holomorphic functions are smooth, indeed, real-analytic; whereas the separately holomorphic criterion makes no apriori assumption on continuity (Hartogs separate analyticity theorem, circa 1906; cf., [Sh] section 6). We make no assumptions nor demands on the reader's knowledge of several complex variables (SCV) but we do require basic knowledge of classical one complex variable (CCV) theory. An acquaintance with potential theory in CCV, i.e., the study of subharmonic functions, would be helpful in motivating analogies with pluripotential theory, the study of plurisubharmonic functions in SCV, but it is not essential. Sections 2 and 3 provide some background on the important notions of polynomial hulls and plurisubharmonic functions in SCV. Section 4 recalls some classical approximation theory results from CCV. In addition, two short appendices are included (sections 13 and 14) for those interested in a brief discussion of a few specialized topics in SCV: pluripolar sets, extremal plurisubharmonic functions, and the complex Monge-Ampère operator. We highly recommend the texts by

(1) Ransford [Ra] on potential theory in the complex plane;
(2) Klimek [K] on pluripotential theory; and
(3) Shabat [Sha] on several complex variables.

Hörmander's SCV text [Hö] is a classic. Range's book [Ran] is an excellent source for integral formulas in SCV; these will occur at several places in our discussion (cf., sections 3, 7 and 10). Many of the approximation topics we mention are described in the monograph of Alexander and Wermer [AW].

Zeros of holomorphic functions locally look like zero sets of holomorphic polynomials (Weierstrass Preparation Theorem; e.g., [Sha] section 23). In particular, in $\mathbb{C}^N$ for $N > 1$ these sets are never isolated. Consider, for example, $f(z_1, \ldots, z_N) = z_1$: the zero set is a copy of $\mathbb{C}^{N-1} \subset \mathbb{C}^N$. This means that, apriori, Runge-type pole-pushing arguments do not exist in SCV. Henceforth the term "polynomial" will refer to a **holomorphic polynomial**, i.e., a polynomial in $z_1, \ldots, z_N$, unless otherwise noted. We use the notation $\mathcal{P}_d = \mathcal{P}_d(\mathbb{C}^N)$ for the polynomials of degree at most d.

Continuing on this theme, rational functions, i.e., ratios of polynomials, behave quite differently in SCV than in CCV. Consider, in $\mathbb{C}^2$, the function $r(z_1, z_2) := z_1/z_2$. The "zero-set" of f contains the punctured plane $\{z_1 = 0\} \setminus (0,0)$ and the "pole-set" contains the punctured plane $\{z_2 = 0\} \setminus (0,0)$, but the point $(0,0)$ itself forms the "indeterminacy locus": f is not only undefined at this point, but, as is easily seen by simply considering complex lines $z_2 = tz_1$ through $(0,0)$, f attains all complex values in any arbitrarily small neighborhood of this point.

It is still the case that polynomials are the nicest examples of holomorphic functions and rational functions are the nicest examples of meromorphic functions (which we won't define) in SCV. Thus one wants to utilize these classes in approximation problems. Many standard tools from CCV either don't exist in SCV or are often more complicated.

In this introductory section, we first recall some classical approximation-theoretic results in the plane with an eye towards generalization, if possible, to $\mathbb{C}^N$, $N > 1$. Let K be a compact subset of $\mathbb{C}^N$, and let $C(K)$ denote the uniform algebra of continuous, complex-valued functions endowed with the supremum (uniform) norm on K. Let $P(K)$ be the uniform algebra (subalgebra of $C(K)$) consisting of uniform limits of polynomials restricted to K. Finally, let $R(K)$ be the uniform closure in $C(K)$ of rational functions $r = p/q$ where $q(z) \neq 0$ for $z \in K$.

As a sample, a question which has a complete and common answer in CCV and SCV, to be given in sections 3 and 4, is: *For which compact sets $K \subset \mathbb{C}^N$ is it true that for* **any** *function f that is holomorphic in a neighborhood of K there exists a sequence $\{p_n\}$ of polynomials which converges uniformly to f on K; i.e., $f|_K \in P(K)$? Moreover, for such compacta, estimate $d_n(f,K) := \inf\{\|f - p\|_K : \deg p \leq n\}$ in terms of the "size" of the neighborhood in which f is holomorphic.*

For example, if $N = 1$ and $K = \bar{\Delta} := \{z : |z| \leq 1\}$ is the closed unit disk, writing $f(z) = \sum_{k=0}^{\infty} a_k z^k$ as a Taylor series about the origin, the Taylor polynomials $p_n(z) = \sum_{k=0}^{n} a_k z^k$ converge uniformly to f on K. More precisely, if f is holomorphic in the disk $\Delta(0, R) := \{z : |z| < R\}$ of radius $R > 1$, the Cauchy estimates give

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{k+1}} dz \right| \leq \frac{\sup_{|z|\leq\rho} |f(z)|}{\rho^k} \tag{1}$$

for any $1 < \rho < R$ yielding

$$d_n(f, \bar{\Delta}) \leq \|f - p_n\|_{\bar{\Delta}} \leq \frac{1}{(1 - 1/\rho)} \frac{\sup_{|z|\leq\rho} |f(z)|}{\rho^{n+1}} \tag{2}$$

so that $\limsup_{n\to\infty} d_n(f, \bar{\Delta})^{1/n} \leq 1/R$. On the other hand, taking $K = T := \partial\Delta := \{z : |z| = 1\}$ the unit circle, the function $f(z) = 1/z$ is holomorphic in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ but if $p(z)$ is a polynomial with $|f(z) - p(z)| < \epsilon < 1$ on T, then, multiplying by z, we have $|1 - zp(z)| < \epsilon < 1$ on T and hence, by the maximum modulus principle, on $\bar{\Delta}$. This gives a contradiction at $z = 0$.

The difference in these sets is explained, and a continuation of our review of classical complex approximation theory proceeds, if we recall a version of the Runge theorem for $N = 1$:

Theorem (Ru). *Let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \setminus K$ connected. Then for any function f holomorphic on a neighborhood of K, there exists a sequence $\{p_n\}$ of holomorphic polynomials which converges uniformly to f on K.*

The condition "$\mathbb{C} \setminus K$ connected" is equivalent, when $N = 1$, to $K = \hat{K}$ where

$$\hat{K} := \{z \in \mathbb{C}^N : |p(z)| \leq \|p\|_K \text{ for all holomorphic polynomials } p\}$$

is the **polynomial hull** of K. Clearly a uniform limit on K of a sequence of polynomials yields a holomorphic function on the interior K^o of K; this observation motivates one of the conditions in Lavrentiev's result:

Theorem (La). *Let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \setminus K$ connected. Then $P(K) = C(K)$ if and only if $K^o = \emptyset$.*

In any number of (complex) dimensions, the maximal ideal space of the uniform algebra $C(K)$ is K and that of $P(K)$ is $\hat{K}$. Thus a necessary condition that $P(K) = C(K)$ is that $K = \hat{K}$. Lavrentiev's theorem shows that in the complex plane, removing the only other obvious obstruction yields a necessary and sufficient condition for the density of the polynomials in the space of continuous functions. A nice exposition of these results (and more) in a succinct, clear manner is given in Alexander-Wermer [AW], section 2. The techniques utilized are elementary functional analysis (Hahn-Banach), classical potential theory (logarithmic potentials) and classical complex analysis (Cauchy transforms).

If we allow K to have interior, then we may ask if functions in $C(K)$ which are holomorphic on K^o are uniformly approximable on K by polynomials. This is the content of Mergelyan's theorem:

Theorem (Me). *Let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \setminus K$ connected. Then for any function $f \in C(K)$ which is holomorphic on K^o, there exists a sequence $\{p_n\}$ of polynomials which converges uniformly to f on K.*

What happens in $\mathbb{C}^N$ for $N > 1$? The complex structure plays a major role. As an elementary, but illustrative, example, consider two disks K_1 and K_2 in $\mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ defined as follows:

$$K_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \text{ and}$$

$$K_2 := \{(z_1, 0) : |z_1| \leq 1\}.$$

Both of these sets are "polynomially convex" in $\mathbb{C}^2$; i.e., $\hat{K}_1 = K_1$ and $\hat{K}_2 = K_2$; thus each set satisfies the obvious necessary condition for holomorphic polynomials to be dense in the space of continuous functions on the set. However, K_2 lies in the complex z_1-plane and $P(K_2)$ can be identified with $P(K)$ where K is the closed unit disk in *one* complex variable; the observation made regarding Lavrentiev's theorem shows that $P(K_2) \neq C(K_2)$.

To understand K_1 and to motivate an attempt to generalize Lavrentiev's theorem in SCV, we first recall the classical theorem of Stone-Weierstrass:

Theorem (SW). *Let $\mathcal{U}$ be a subalgebra of $C(K)$ containing the constant functions and separating points of K. If $f \in \mathcal{U}$ implies that $\bar{f} \in \mathcal{U}$, then $\mathcal{U} = C(K)$.*

As an immediate corollary, we have the *real* Stone-Weierstrass theorem (which includes the classical Weierstrass theorem for a real interval):

Theorem (RSW). *Let K be a compact subset of $\mathbb{R}^N \subset \mathbb{C}^N$. Then $P(K) = C(K)$.*

Thus by (RSW), $P(K_1) = C(K_1)$. The difference here is that *the real submanifold* $\mathbb{R}^2 = \mathbb{R}^2 + i0$ *of* $\mathbb{C}^2$ *is totally real; i.e.,* $\mathbb{R}^2$ *contains no complex tangents.* We will generalize this example in Theorem (HW) of section 8. The extremely difficult question of determining when $P(K) = C(K)$ will be partially analyzed in the next section.

Recall that $R(K)$ is the uniform subalgebra of $C(K)$ generated by rational functions which are holomorphic on K. The Hartogs-Rosenthal theorem gives a sufficient condition for $R(K) = C(K)$ if $K \subset \mathbb{C}$.

Theorem (HR1). *Let K be a compact subset of $\mathbb{C}$ with two-dimensional Lebesgue measure zero. Then $R(K) = C(K)$.*

A similar result holds in $\mathbb{C}^N$, $N > 1$. For $\alpha > 0$, we let h_α denote α-Hausdorff measure.

Theorem (HRN). *Let K be a compact subset of $\mathbb{C}^N$ with $h_2(K) = 0$. Then $R(K) = C(K)$.*

This follows since the conjugates $\bar{z}_j$ of the coordinate functions belong to $R(K)$, by Theorem (HR1); from this it follows trivially that $R(K)$ is closed under complex conjugation. Then Theorem (SW) implies the conclusion.

We turn to a $\mathbb{C}^N$-version of Theorem (Ru). Note that if we take the "boundary circles" of our sets K_1 and K_2, i.e., take

$$X_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \text{ and}$$

$$X_2 := \{(z_1, 0) : |z_1| = 1\},$$

then a higher-dimensional version of Theorem (Ru) is valid for X_1 but not for X_2, i.e., if f is holomorphic on a neighborhood of X_1 (in $\mathbb{C}^2$!), then there exists a sequence $\{p_n\}$ of holomorphic polynomials which converges uniformly to f on X_1 (e.g., $f|_{X_1} \in C(X_1)$ and $P(X_1) = C(X_1)$ follows from Theorem (RSW)); the analogous statement is not true for X_2 (why?). Here the difference can simply be explained by the fact that X_1 is polynomially convex while X_2 is not (indeed, $\hat{X}_2 = K_2$). This is the content of the Oka-Weil theorem:

Theorem (OW). *Let $K \subset \mathbb{C}^N$ be compact with $\hat{K} = K$. Then for any function f holomorphic on a neighborhood of K, there exists a sequence $\{p_n\}$ of polynomials which converges uniformly to f on K.*

This result was first proved by André Weil in 1935 by using a multivariate generalization of the Cauchy integral formula for certain polynomial polyhedra. We sketch his argument in section 3. In 1936 Kyoshi Oka gave a different proof that made use of his celebrated "lifting principle" (cf. [AW] Chapter 7).

As a motivational example for what to expect, let $K = \{(z_1, \ldots, z_N) : |z_j| \leq 1,\ j = 1, \ldots, N\}$ be the closed unit polydisk. If f is holomorphic in a larger polydisk $D_R := \{(z_1, \ldots, z_N) : |z_j| < R,\ j = 1, \ldots, N\},\ R > 1$, then iterating the one-variable Cauchy integral formula, for $\rho < R$ we obtain the formula

$$f(z) = (\frac{1}{2\pi i})^N \int_{|\zeta_1|=\rho} \cdots \int_{|\zeta_N|=\rho} \frac{f(\zeta_1, \ldots, \zeta_N)}{(\zeta_1 - z_1)\cdots(\zeta_N - z_N)} d\zeta_1 \cdots d\zeta_N \tag{3}$$

valid for $z \in D_\rho$. We can write a Taylor series expansion $f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multiindex with $|\alpha| := \sum_{j=1}^N \alpha_j$ and $z^\alpha := z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ and

$$a_\alpha = (\frac{1}{2\pi i})^N \int_{|z_1|=\rho} \cdots \int_{|z_N|=\rho} \frac{f(z_1, \ldots, z_N)}{z_1^{\alpha_1+1} \cdots z_N^{\alpha_N+1}} dz_1 \cdots dz_N.$$

The same estimates as in (1) and (2) show that not only is $f|_K \in P(K)$ but we obtain, using the Taylor polynomials $p_n(z) = \sum_{|\alpha|=0}^{n} a_\alpha z^\alpha$, the quantitative estimate $\limsup_{n\to\infty} d_n(f, K)^{1/n} \leq 1/R$. Note that in the Cauchy integral formula (3), the integration takes place over the N-dimensional torus $\{(z_1, \ldots, z_N) : |z_j| = \rho,\ j = 1, \ldots, N\}$, which is a proper subset of the $(2N-1)$-dimensional topological boundary ∂D_ρ if $N > 1$.

2 Polynomial hulls and polynomial convexity

The condition that $K = \hat{K}$ occurs in Theorems (Ru), (La), (Me) and (OW); indeed, this condition is implicit in Theorem (RSW): any compact subset of $\mathbb{R}^N$ is polynomially convex (exercise!). If $K \subset \mathbb{C}$, $\hat{K}$ is the union of K with the bounded components of $\mathbb{C} \setminus K$. For $K \subset \mathbb{C}^N$ if $N > 1$, $\hat{K}$ contains the union of K with the bounded components of $\mathbb{C}^N \setminus K$ but it can be much, much more. An elementary example is the connected and simply connected set $K := K_1 \cup K_2$ which is the union of two bidisks

$$K_1 := \{(z_1, z_2) : |z_1| \leq 1,\ |z_2| \leq r < 1\}$$

and

$$K_2 := \{(z_1, z_2) : |z_2| \leq 1,\ |z_1| \leq r < 1\}.$$

We'll see in the next section that

$$\hat{K} = \{(z_1, z_2) : |z_1| \leq 1,\ |z_2| \leq 1,\ |z_1 z_2| \leq r\}$$

(draw a picture in $|z_1|, |z_2|$-space). Note it is clear that $\hat{K}$ is contained in the right-hand-side by considering the polynomial $p(z_1, z_2) = z_1 z_2$. In general, the polynomial hull of a compact set is difficult to describe.

It follows readily from the maximum modulus principle that if we have a bounded holomorphic mapping $f = (f_1, \ldots, f_N) : \Delta \to \mathbb{C}^N$; i.e., each $f_j : \Delta \to \mathbb{C}$ is a bounded holomorphic function, with (componentwise) radial limit values $f^*(e^{i\theta}) := \big(f_1^*(e^{i\theta}), \ldots, f_N^*(e^{i\theta})\big) \in K$ for almost all θ, then $f(\Delta) \subset \hat{K}$. In general, we will say that a set $S \subset \mathbb{C}^N$ has **analytic structure** if it contains a nonconstant analytic disk $f(\Delta)$. Thus one way to obtain (lots of) points in $\hat{K}$ is the existence of analytic structure in $\hat{K}$. Moreover, existence of analytic structure in a compact set S precludes the possibility of $C(S) = P(S)$ for the set.

In 1963, Stolzenberg [Sto] gave an example of a compact set K in the topological boundary

$$\partial(\Delta \times \Delta) = \{(z_1, z_2) : |z_1| = 1,\ |z_2| \leq 1,\ \text{or } |z_2| = 1,\ |z_1| \leq 1\}$$

of the bidisk $\Delta \times \Delta \subset \mathbb{C}^2$ such that the origin $(0,0) \in \hat{K}$ but the projections $\pi_{z_1}(\hat{K}), \pi_{z_2}(\hat{K})$ of $\hat{K}$ in each coordinate plane contain no nonempty open set; thus $\hat{K}$ contains no analytic structure. From the lack of analytic structure in $\hat{K}$ one may be tempted to conjecture that $P(\hat{K}) = C(\hat{K})$. However, there clearly exist $f \in C(\hat{K})$ with $|f(0,0)| > \|f\|_K$ for this set K; e.g., $f(z_1, z_2) = 1 - \max[|z_1|, |z_2|]$. For any $p \in P(\hat{K})$, we obviously have $\|p\|_{\hat{K}} = \|p\|_K$. Thus $f \notin P(\hat{K})$.

How can one tell if $\hat{K} \setminus K$ contains analytic structure? Note that an analytic disk has locally finite Hausdorff two-measure.

Theorem (Alexander-Sibony). *Let K be a compact subset of $\mathbb{C}^N$ and let $q \in \hat{K} \setminus K$. If there exists a neighborhood U of q with $h_2(\hat{K} \cap U) < +\infty$, then $\hat{K} \cap U$ is a one-dimensional analytic subvariety of U.*

This means that $\hat{K} \cap U$ is essentially a one-dimensional complex manifold (modulo some singular points) and hence looks locally like a nonconstant analytic disk $f(\Delta)$. In particular, if $\hat{K} \setminus K \neq \emptyset$ and $\hat{K} \setminus K$ contains no analytic structure, then $h_2(\hat{K} \setminus K) = +\infty$. A nice discussion of the Alexander-Sibony result can be found in section 21 of [AW]. In [DL] the authors constructed examples of compact sets $K \subset \mathbb{C}^N$ whose polynomial hull $\hat{K}$ contains no analytic structure but such that $\hat{K} \setminus K$ has positive $2N$-Hausdorff measure.

Recall that a compact subset K of $\mathbb{R}^N$ is automatically polynomially convex and, moreover, $P(K) = C(K)$ for such sets. From Theorem (HRN) and Theorem (OW), we also get the following result.

Corollary. *Let $K = \hat{K} \subset \mathbb{C}^N$ with $h_2(K) = 0$. Then $P(K) = C(K)$.*

Question: For an arbitrary compact set $K \subset \mathbb{C}^N$, find a "nice" condition (C) on K so that if $K = \hat{K}$, then K satisfies (C) if and only if $P(K) = C(K)$; i.e., find a $\mathbb{C}^N$-version of Theorem (La).

We return to this matter in section 8. As a final note to reinforce the delicate nature of polynomial hulls in $\mathbb{C}^N$ for $N > 1$, we mention the curious results of E. Kallin [Ka]. The union of any *two* disjoint convex compact sets in $\mathbb{C}^N$ is polynomially convex. The union of any *three* disjoint closed Euclidean *balls* is polynomially convex. On the other hand, there exist *three* disjoint *convex* sets whose union is *not* polynomially convex. Moreover, it is unknown whether the union of *four or more* disjoint closed *balls* is polynomially convex, unless, e.g., the centers of the balls lie on the real subpace $\mathbb{R}^N$ of $\mathbb{C}^N$ [Kh].

3 Plurisubharmonic functions and the Oka-Weil theorem

We outline the basic notions and sketch a proof of the Oka-Weil theorem, Theorem (OW). First of all, in the complex plane, any domain $D \subset \mathbb{C}$ is a **domain of holomorphy**; i.e., there exists f

holomorphic in D – we write $f \in \mathcal{O}(D)$ – which does not extend holomorphically across any boundary point of D. This follows from the classical Weierstrass theorem which allows the construction of a nontrivial holomorphic function with prescribed discrete zero set in D. However, in $\mathbb{C}^N$, $N > 1$, there exist domains D with the property that *every* $f \in \mathcal{O}(D)$ extends holomorphically to a larger domain $\tilde{D}$ (independent of f). Products of planar domains, e.g., polydisks, are obviously domains of holomorphy. A simple example of a domain $D \subset \mathbb{C}^2$ which is not a domain of holomorphy is

$$D = \{(z_1, z_2) : |z_1| < 2,\ |z_2| < 2\} \setminus \{(z_1, z_2) : 1 \leq |z_1| < 2,\ |z_2| \leq 1\}.$$

It is straightforward to see (use Laurent series!) that any $f \in \mathcal{O}(D)$ extends holomorphically to the bidisk $\{(z_1, z_2) : |z_1| < 2,\ |z_2| < 2\}$. Another example is the interior of the set $K = K_1 \cup K_2$ from the previous section: any f holomorphic on

$$D := \{(z_1, z_2) : |z_1| < 1,\ |z_2| < r\} \cup \{(z_1, z_2) : |z_1| < r,\ |z_2| < 1\}$$

extends holomorphically to the domain

$$\tilde{D} := \{(z_1, z_2) : |z_1| < 1,\ |z_2| < 1,\ |z_1 z_2| < r\}.$$

Exercise: *For each $f \in \mathcal{O}(D)$, if $\tilde{f}$ denotes the holomorphic extension of f to $\tilde{D}$, then $f(D) = \tilde{f}(\tilde{D})$.*

As a sample of holomorphic extendability situations in SCV, consider the following three results on bounded domains $D \subset \mathbb{C}^N$. The first two can be reduced to classical one-variable arguments if one utilizes the Cauchy integral formula for polydisks (3).

1. (**Morera type**): *Let $N \geq 1$ and let S be a smooth, real hypersurface in D; i.e., (locally) $S = \{\rho = 0\}$ where ρ is a smooth, real-valued function on a neighborhood of S and $d\rho \neq 0$ on S. If $f \in \mathcal{O}(D \setminus S) \cap C(D)$, then $f \in \mathcal{O}(D)$.*
2. (**Riemann removable singularity type**): *Let $N \geq 1$ and let A be a complex analytic hypersurface in D; i.e., (locally) $A = \{g = 0\}$ where g is holomorphic on a neighborhood of A. If $f \in \mathcal{O}(D \setminus A)$ is locally bounded on A (i.e., for each $z \in A$ there is a neighborhood U of z with f bounded on $(D \setminus A) \cap U$), then f has a holomorphic extension $F \in \mathcal{O}(D)$.*
3. (**Hartogs type**) *Let $N > 1$ and let A be a complex analytic subvariety of (complex) codimension two in D; i.e., A is (locally) the common zero set of two holomorphic functions. If $f \in \mathcal{O}(D \setminus A)$, then f has a holomorphic extension $F \in \mathcal{O}(D)$.*

We refer the reader to section 32 of [Sha]. As an example of 3., any function holomorphic in a punctured ball

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1 - a_1|^2 + |z_2 - a_2|^2 < R^2\}$$

in $\mathbb{C}^2$ extends holomorphically across the puncture $a = (a_1, a_2)$. Note that no boundedness assumptions on f are required. Indeed, a theorem of Hartogs states that *if K is a compact subset of a domain $D \subset \mathbb{C}^N$ $(N > 1)$ such that $D \setminus K$ is connected, then every $f \in \mathcal{O}(D \setminus K)$ extends holomorphically to D*. Thus $D \setminus K$ is not a domain of holomorphy.

A real-valued function $u : D \to [-\infty, +\infty)$ defined on a domain $D \subset \mathbb{C}^N$ is called **plurisubharmonic** (**psh**) if u is uppersemicontinuous (usc) on D and $u|_{D \cap L}$ is subharmonic on (components of) $D \cap L$ for any complex affine line $L = L_{z_0,a} := \{z_0 + ta : t \in \mathbb{C}\}$ ($z_0, a \in \mathbb{C}^N$ fixed). The canonical examples of such functions are those of the form $u = \log |f|$ where $f \in \mathcal{O}(D)$. The class of psh functions on a domain D, denoted $PSH(D)$, forms a convex cone; i.e., if $u, v \in PSH(D)$ and $\alpha, \beta \geq 0$, then $\alpha u + \beta v \in PSH(D)$. The limit function $u(z) := \lim_{n\to\infty} u_n(z)$ of a decreasing sequence $\{u_n\} \subset PSH(D)$ is psh in D (we may have $u \equiv -\infty$); while for any family $\{v_\alpha\} \subset PSH(D)$

(resp., sequence $\{v_n\} \subset PSH(D)$) which is uniformly bounded above on any compact subset of D, the functions

$$v(z) := \sup_{\alpha} v_\alpha(z) \text{ and } w(z) := \limsup_{n\to\infty} v_n(z)$$

are "nearly" psh: the usc regularizations

$$v^*(z) := \limsup_{\zeta\to z} v(\zeta) \text{ and } w^*(z) := \limsup_{\zeta\to z} w(\zeta)$$

are psh in D. Finally, if ϕ is a real-valued, convex increasing function of a real variable, and u is psh in D, then so is $\phi \circ u$.

If $u \in C^2(D)$, then u is psh if and only if for each $z \in D$ and vector $a \in \mathbb{C}^N$, the Laplacian of $t \mapsto u(z + ta)$ is nonnegative at $t = 0$; i.e., *the complex Hessian* $[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)]$ *of* u *is positive semidefinite on* D:

$$\sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) a_j \bar{a}_k \geq 0.$$

In particular, the trace of the complex Hessian is nonnegative so that u is $\mathbb{R}^{2N}-$subharmonic. Indeed, $u : D \to [-\infty, +\infty)$ *is psh if and only if* $u \circ A$ *is* $\mathbb{R}^{2N}-$*subharmonic in* $A^{-1}(D)$ *for every complex linear isomorphism* A; moreover, the notion of a psh function makes sense on any complex manifold. If $u \in C^2(D)$, we call u **strictly** psh if the complex Hessian of u is positive definite; i.e., we have strict inequality in the above displayed equation provided $a \neq (0, \ldots, 0)$. The function $u(z) = |z|^2 := |z_1|^2 + \cdots + |z_N|^2$ is strictly psh on $\mathbb{C}^N$.

A domain $D \subset \mathbb{C}^N$ is said to be (globally) **pseudoconvex** if D admits a **psh exhaustion function**: there exists u psh in D with the property that the sublevel sets $D_c := \{z \in D : u(z) < c\}$ are compactly contained in D for all real c. Any planar domain $D \subset \mathbb{C}$ is pseudoconvex: if $D = \mathbb{C}$, take $u(z) = |z|$; if $D = \mathbb{C} \setminus \{z_0\}$, take $u(z) = 1/|z - z_0|$; otherwise, take

$$u(z) = |z| + \sup_{z_0 \in \partial D} \frac{1}{|z - z_0|} = |z| + \operatorname{dist}(z, \partial D)^{-1}.$$

It turns out that *a domain* $D \subset \mathbb{C}^N$ *is pseudoconvex if and only if the function*

$$u(z) = -\log \operatorname{dist}(z, \partial D)$$

is psh in D. Note that, in this case,

$$\exp(u(z)) = \operatorname{dist}(z, \partial D)^{-1}$$

is psh since $x \to e^x$ is convex and increasing; if, e.g., D is bounded, both $u(z)$ and $\exp(u(z))$ are continuous psh exhaustion functions for D. A slightly more restrictive notion is that of **hyperconvexity**: D is hyperconvex if it admits a negative psh exhaustion function u; i.e., $u < 0$ in D and the sets D_c are compactly contained in D for all $c < 0$. Such a domain is pseudoconvex, for the function $-1/u := \phi \circ u$, where $\phi(x) = -1/x$ is convex and increasing for $x < 0$, is then a psh exhaustion function for D. In particular, a fact we will need below is that *if* D *is pseudoconvex and* $u \in PSH(D)$, *the components of the sublevel sets* $\{z \in D : u(z) < c\}$ *are pseudoconvex domains.*

The condition that a domain $D \subset \mathbb{R}^N$ with smooth (say C^2) boundary be convex can be described analytically by the existence of a smooth defining function r:

$$D = \{x \in \mathbb{R}^N : r(x) < 0\}; \quad \partial D = \{x \in \mathbb{R}^N : r(x) = 0\}; \quad dr \neq 0 \text{ on } \partial D,$$

such that the real Hessian of r is positive semidefinite on the real tangent space $T_p(\partial D)$ to ∂D; i.e., for $p \in \partial D$,

$$\sum_{j,k=1}^{N} \frac{\partial^2 r}{\partial x_j \partial x_k}(p) a_j a_k \geq 0 \quad \text{if} \quad \sum_{j=1}^{N} \frac{\partial r}{\partial x_j}(p) a_j = 0.$$

Strict convexity means that the real Hessian is positive definite on the real tangent space to ∂D. The complex analogue of convexity is **Levi-pseudoconvexity**: a smoothly bounded domain $D \subset \mathbb{C}^N$ is Levi-pseudoconvex if it admits a defining function r whose *complex* Hessian is positive semidefinite on the *complex* tangent space $T_p^{\mathbb{C}}(\partial D)$ to ∂D; i.e., for $p \in \partial D$,

$$\sum_{j,k=1}^{N} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) a_j \bar{a}_k \geq 0 \quad \text{if} \quad \sum_{j=1}^{N} \frac{\partial r}{\partial z_j}(p) a_j = 0.$$

We say that ∂D is **Levi-pseudoconvex at** $p \in \partial D$ if this holds. Note that $T_p^{\mathbb{C}}(\partial D)$ is an $(N-1)$-*complex*-dimensional linear subspace of the $(2N-1)$-*real*-dimensional space $T_p(\partial D)$. **Strict Levi-pseudoconvexity** means that the complex Hessian is positive definite on the complex tangent space to ∂D. It turns out that *if D is strictly Levi-pseudoconvex, one can make a holomorphic change of coordinates so that D is (locally) strictly convex*. Precisely, if $p \in \partial D$ is a strictly Levi-pseudoconvex boundary point, then there is a neighborhood $U \subset \mathbb{C}^N$ of p and a biholomorphic map $\phi : U \to \phi(U) \subset \mathbb{C}^N$ such that $\phi(U \cap \partial D)$ is strictly convex (in the $\mathbb{R}^{2N}$-sense) at $\phi(p)$. It is *not* the case that if D is merely Levi-pseudoconvex at p, then one can make a holomorphic change of coordinates so that in the new coordinates D is convex.

There is a relationship between pseudoconvexity and Levi-pseudoconvex: if D is pseudoconvex, then there exists an increasing sequence of bounded, strictly Levi-pseudoconvex domains $D_n \subset D$ with smooth boundary which are relatively compact in D such that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

This follows since once we have a psh exhaustion function u for D, we can modify it to get a smooth, strictly psh exhaustion function $\tilde{u}$; then we can take

$$D_n := \{z \in D : \tilde{u}(z) < m_n\}$$

for an appropriate sequence $\{m_n\}$ with $m_n \uparrow \infty$.

We already observed that in $\mathbb{C}$, every domain is both a domain of holomorphy and a pseudoconvex domain. In $\mathbb{C}^N$ for $N > 1$ this no longer holds, but these two notions are equivalent: a domain $D \subset \mathbb{C}^N$ is a domain of holomorphy if and only if it is pseudoconvex. The "if" direction is deep and is one version of the so-called **Levi problem**. Thus from now on we will simply use the terminology "pseudoconvex domain." Products of pseudoconvex domains are pseudoconvex domains. Euclidean balls and, more generally, convex domains (in the $\mathbb{R}^{2N}$-sense) are pseudoconvex; but there are many non-convex pseudoconvex domains.

The statement that canonical examples of psh functions are those of the form $u = \log|f|$ where $f \in \mathcal{O}(D)$ can now be made precise:

Theorem (B). *A psh function u on a pseudoconvex domain can be written in the form*

$$u(z) = \left[\limsup_{j\to\infty} a_j \log|f_j(z)|\right]^*$$

where $a_j \geq 0$ and $f_j \in \mathcal{O}(D)$.

Here $u^*(z)$ is the usc regularization of the function u. The result is false if D is not pseudoconvex [Le].

Sketch of proof. The domain

$$\tilde{D} := \{(z,w) \in D \times \mathbb{C} \subset \mathbb{C}^{N+1} : z \in D,\ |w| < e^{-u(z)}\}$$

is pseudoconvex, for the function $\tilde{u}(z,w) := u(z) + \log|w|$ is psh in $D \times \mathbb{C}$ and

$$\tilde{D} = \{(z,w) \in D \times \mathbb{C} : \tilde{u}(z,w) < 0\}.$$

Since $\tilde{D}$ is a pseudoconvex domain and hence a domain of holomorphy, there exists $F \in \mathcal{O}(\tilde{D})$ which is not holomorphically extendible across any boundary point of $\tilde{D}$; expanding F in a **Hartogs series**, i.e., a series of the form

$$F(z,w) = \sum_{j=0}^{\infty} f_j(z) w^j$$

where $f_j \in \mathcal{O}(D)$, the radius of convergence in w (as a function of z) is given by the usual formula

$$R(z) = \left[\limsup_{j\to\infty} |f_j(z)|^{1/j}\right]^{-1}.$$

Since F is not holomorphically extendible across any boundary point of $\tilde{D}$, $R(z) = e^{-u(z)}$ almost everywhere; i.e., $u(z) = -\log R(z)$ a.e.; taking usc regularizations (making the right-hand-side psh) the result follows. □

This result is due to Bremermann but the proof given is due to Sibony. An important remark, which we will use later, is that a slight refinement of this argument yields a local uniform approximability result (cf. [JP], Proposition 4.4.13): *if u is psh and continuous in a bounded domain D, then for any compact set $K \subset D$ and $\epsilon > 0$ there exist finitely many $f_j \in \mathcal{O}(D)$, $j = 1,\ldots,m$ and positive constants $a_1,\ldots,a_m$ such that*

$$\left|u(z) - \max_{j=1,\ldots,m} a_j \log|f_j(z)|\right| < \epsilon \quad \text{for} \quad z \in K. \tag{4}$$

Given a polynomially convex compact set $K \subset \mathbb{C}^N$; i.e., $K = \hat{K}$, and given a bounded open neighborhood U of K, by compactness of K and the definition of $\hat{K}$ we can find finitely many polynomials $q_1,\ldots,q_m$ such that

$$K \subset \Pi := \{z \in U : |q_j(z)| < 1,\ j = 1,\ldots,m\} \subset U. \tag{5}$$

Note that necessarily $m \geq N$. We call Π a **polynomial polyhedron**. By slightly modifying the polynomials q_j, we may assume that Π is a **Weil polyhedron** which simply means that the "faces"

$$\sigma_j := \{z \in U : |q_j(z)| = 1,\ |q_k(z)| \leq 1,\ k \neq j\}$$

are real $(2N-1)$-dimensional manifolds and the intersection of any s distinct faces for $2 \leq s \leq N$ has dimension at most $2N-s$. The set of N dimensional "edges" $\sigma_{i_1,\ldots,i_N} := \sigma_{i_1} \cap \cdots \cap \sigma_{i_N}$ form the "skeleton" of Π. For the unit polydisk $P := \{(z_1,\ldots,z_N) : |z_j| < 1,\ j = 1,\ldots,N\}$, the skeleton, or distinguished boundary, is the N-torus $T^N := \{(z_1,\ldots,z_N) : |z_j| = 1,\ j = 1,\ldots,N\}$. We have the following generalization of the Cauchy integral formula for polydisks (cf. [Sha] section 30).

Theorem (Weil integral formula). *Let Π be a Weil polyhedron. Then for any $f \in \mathcal{O}(\Pi)\cap C(\bar{\Pi})$,*

$$f(z) = (\frac{1}{2\pi i})^N \sum_{i_1,\ldots,i_N}{}' \int_{\sigma_{i_1,\ldots,i_N}} \frac{f(\zeta)\det\{[P_m^{i_n}(\zeta,z)]_{m,n=1,\ldots,N}\}}{\prod_{\nu=1}^N (q_{i_\nu}(\zeta)-q_{i_\nu}(z))} d\zeta_1\cdots d\zeta_N$$

for $z \in \Pi$.

Here, $\sum'_{i_1,\ldots,i_N}$ refers to a summation over increasing multiindices: $i_1 < \cdots < i_N$. The functions P_s^t are polynomials satisfying

$$q_i(\zeta) - q_i(z) = \sum_{j=1}^N (\zeta_j - z_j) P_j^i(\zeta, z).$$

This formula is a special case of a more general result known as Hefer's formula; here, we are merely rearranging the Taylor expansion of q_i at the point z. As an example, for the polydisk P, one can take $m = N$ and $q_j(z) = z_j$ in which case $P_s^t = \delta_{st}$.

Returning to the setting of the Oka-Weil theorem, Theorem (OW), given a function f holomorphic on a neighborhood U of $K = \hat{K}$, we construct a Weil polyhedron satisfying (5). We obtain a Taylor-like expansion

$$f(z) = \sum_{|\kappa|=0}^\infty \sum_I{}' A_\kappa^I(z) q_I(z)^\kappa$$

where $\kappa = (k_1,\ldots,k_N)$, $I = (i_1,\ldots,i_N)$, $q_I(z)^\kappa = q_{i_1}(z)^{k_1}\cdots q_{i_N}(z)^{k_N}$, and

$$A_\kappa^I(z) = (\frac{1}{2\pi i})^N \int_{\sigma_{i_1,\ldots,i_N}} \frac{f(\zeta)}{q_{i_1}(\zeta)^{k_1+1}\cdots q_{i_N}(\zeta)^{k_N+1}} \times$$

$$\det\{P_m^{i_n}(\zeta,z)_{m,n=1,\ldots,N}\} d\zeta_1\cdots d\zeta_N$$

are polynomials. Using truncations of this expansion, as with the polydisk at the end of section 1, one concludes that $f|_K \in P(K)$, completing the outline of the proof of Theorem (OW). An alternate proof of Theorem (OW), using Lagrange interpolation at generalized Fekete points (see section 9) has been given by Siciak [Si].

From (4), (5) and Theorem (OW), it follows that if K is a compact set in $\mathbb{C}^N$ and D is an open neighborhood of the polynomial hull $\hat{K}$, then $\hat{K}$ can just as well be constructed as a "hull" with respect to holomorphic or continuous psh functions; i.e., $\hat{K}$ coincides with both

$$\hat{K}_{\mathcal{O}(D)} := \{z : |f(z)| \le \|f\|_K \text{ for all } f \in \mathcal{O}(D)\}$$

and

$$\hat{K}_{PSH(D)} := \{z : u(z) \le \sup_{\zeta\in K} u(\zeta) \text{ for all } u \in PSH(D)\cap C(D)\}.$$

The reader may now verify the claim in the previous section about the polynomial hull of $K_1 \cup K_2$ utilizing the above observation with D being a dilation of the domain

$$\tilde{D} := \{(z_1,z_2) : |z_1| < 1,\ |z_2| < 1,\ |z_1 z_2| < r\}$$

and the exercise at the beginning of this section.

4 Quantitative approximation theorems in $\mathbb{C}$

Before jumping to a quantitative Runge-type theorem in $\mathbb{C}^N$, we recall the example of the closed unit disk $\bar{\Delta}$ to review the one-variable story. In the introduction we proved one direction of the following.

Theorem. *Let f be continuous on $\bar{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $R > 1$. Then*

$$\limsup_{n\to\infty} d_n(f, \bar{\Delta})^{1/n} \leq 1/R \tag{6}$$

if and only if f is the restriction to $\bar{\Delta}$ of a function holomorphic in $\Delta(0, R) = \{z \in \mathbb{C} : |z| < R\}$.

Proof. For the only if direction we note that for any nonconstant polynomial p, the function

$$u(z) := \frac{1}{\deg p} \log \frac{|p(z)|}{\|p\|_{\bar{\Delta}}} - \log |z|$$

is subharmonic on $\mathbb{C} \setminus \bar{\Delta}$, bounded at ∞, and nonpositive on $T = \partial\Delta$. By the maximum principle $u \leq 0$ on $\mathbb{C} \cup \{\infty\} \setminus \bar{\Delta}$ which gives

$$|p(z)| \leq \|p\|_{\bar{\Delta}} |z|^{\deg p}, \ |z| \geq 1,$$

hence

$$|p(z)| \leq \|p\|_{\bar{\Delta}} \max(|z|, 1)^{\deg p} = \|p\|_{\bar{\Delta}} (e^{\log^+ |z|})^{\deg p}, \quad z \in \mathbb{C}.$$

This is the **Bernstein-Walsh inequality**. In particular,

$$|p(z)| \leq \|p\|_{\bar{\Delta}} \, \rho^{\deg p}, \qquad |z| \leq \rho. \tag{7}$$

Let f be a continuous function on $\bar{\Delta}$ such that (6) holds and choose a polynomial p_n of degree at most n satisfying $d_n = \|f - p_n\|_{\bar{\Delta}}$. We claim that the series $p_0 + \sum_1^\infty (p_n - p_{n-1})$ converges uniformly on compact subsets of $\{z : |z| < R\}$ to a holomorphic function F which agrees with f on $\bar{\Delta}$. For if $1 < R' < R$, by hypothesis the polynomials p_n satisfy

$$\|f - p_n\|_{\bar{\Delta}} \leq \frac{M}{R'^n}, \qquad n = 0, 1, 2, \ldots, \tag{8}$$

for some $M > 0$. Then for $1 < \rho < R'$, simply apply (7) to $p_n - p_{n-1}$:

$$\begin{aligned} \sup_{|z|\leq\rho} |p_n(z) - p_{n-1}(z)| &\leq \rho^n \|p_n - p_{n-1}\|_{\bar{\Delta}} \\ &\leq \rho^n (\|p_n - f\|_{\bar{\Delta}} + \|f - p_{n-1}\|_{\bar{\Delta}}) \leq \rho^n \frac{M(1 + R')}{R'^n}. \end{aligned}$$

From (8), $F = f$ on $\bar{\Delta}$. $\square$

We generalize this. Let K be a compact subset of $\mathbb{C}$ with $\mathbb{C} \setminus K$ connected. Recall this is equivalent to the condition that $K = \hat{K}$; i.e., K is polynomially convex. We say that K is **regular** if there is a continuous function $g_K : \mathbb{C} \to [0, +\infty)$ which is identically equal to zero on K, harmonic on $\mathbb{C} \setminus K$, and has a logarithmic singularity at infinity in the sense that $g_K(z) - \log |z|$ is harmonic at infinity (this is equivalent to $\mathbb{C} \setminus K$ being a regular domain for the Dirichlet problem). We call g_K the classical **Green function** for K. For $K = \bar{\Delta}$, we have $g_{\bar{\Delta}}(z) = \log^+ |z|$. In the general case, if p is any nonconstant polynomial, then the function

$$V := \frac{1}{\deg p} \log \frac{|p|}{\|p\|_K} - g_K$$

is subharmonic on $\mathbb{C} \setminus K$, bounded at ∞, and continuously assumes nonpositive values on ∂K. By the maximum principle we have $V \leq 0$ on $\mathbb{C} \cup \{\infty\} \setminus K$, yielding the Bernstein-Walsh property

$$|p(z)| \leq \|p\|_K (e^{g_K(z)})^{\deg p}.$$

In particular, if $R > 1$ and

$$D_R := \{z : g_K(z) < \log R\}, \tag{9}$$

then

$$|p(z)| \leq \|p\|_K R^{\deg p}, \qquad z \in D_R. \tag{10}$$

Then a similar argument proves one-half of the following univariate Bernstein-Walsh theorem.

Theorem (BW1). *Let K be a regular compact subset of the plane with Green function g_K. Let $R > 1$, and define D_R by (9). Let f be continuous on K. Then $\limsup_{n\to\infty} d_n(f, K)^{1/n} \leq 1/R$ if and only if f is the restriction to K of a function holomorphic in D_R.*

Proof. We prove the other half using duality. We suppose f is holomorphic on D_R; and we will rewrite the numbers d_n in such a way that we can estimate them. Let $1 < r < \rho < R$. To get a global C^∞ extension F of f that agrees with f on a neighborhood of K, we let ϕ be a smooth cut-off function which is identically equal to 1 on $\bar{D}_\rho$ and has compact support in D_R. We then set $F = \phi f$ in D_R and let F be identically 0 outside of D_R.

For n fixed, by the Hahn-Banach theorem there exists a complex measure $\mu = \mu_n$ supported in K with total variation $|\mu|(K) = 1$, such that μ annihilates the vector space $\mathcal{P}_n$ of holomorphic polynomials of degree at most n (that is, $\int_K p_n \, d\mu = 0$ for all $p_n \in \mathcal{P}_n$) and

$$d_n = \int_K f \, d\mu.$$

Since $F = f$ on K, we can write

$$d_n = \int_K F \, d\mu = (\mu * \check{F})(0). \tag{11}$$

where $\check{F}(z) := F(-z)$. Now form the convolution

$$\mu * \check{F} = (\mu * \check{F}) * \delta = (\mu * \check{F}) * \frac{\partial}{\partial \bar{z}} E = \frac{\partial}{\partial \bar{z}} \check{F} * (\mu * E), \tag{12}$$

where δ is the point mass at 0 and $E(z) := 1/(\pi z)$ is the Cauchy kernel. Associativity of the triple convolution holds since each term has compact support. We write

$$\hat{\mu}(z) := (\mu * E)(z) = \frac{1}{\pi}\int_K \frac{d\mu(\zeta)}{z-\zeta},$$

the Cauchy transform of μ. Note that $\mu * E$ is holomorphic outside K. From (11) and (12) we then obtain (note that $\partial F/\partial\bar{z} = 0$ on D_ρ)

$$d_n = \int_{D_R\setminus D_\rho} (\mu * E)(z)\frac{\partial}{\partial\bar{z}}\check{F}(z)\, dA(z), \tag{13}$$

where A is Lebesgue measure in $\mathbb{C}$.

In order to utilize formula (13) for d_n, we need estimates for $\hat{\mu}$. We first note that since $|\mu|(K) = 1$, we have

$$|(\mu * E)(z)| \leq M, \quad z \in \partial D_r, \tag{14}$$

for some constant $M > 0$ depending only on the distance from K to ∂D_r. In addition we have the growth estimate

$$|(\mu * E)(z)| = O(1/|z|^{n+1}) \quad \text{as } |z| \to \infty; \tag{15}$$

this follows from noting that for z sufficiently large we have $1/(z-\zeta) = \sum_k \zeta^k/z^{k+1}$ uniformly for $\zeta \in K$, and then using the fact that μ satisfies $\int_K \zeta^k\, d\mu(\zeta) = 0$ for $0 \leq k \leq n$. We now consider the function

$$u(z) := g_K(z) + \frac{1}{n}\log\left(\frac{|\mu * E(z)|}{M}\right).$$

Using (14) and (15), we see that $u(z)$ is subharmonic in $\mathbb{C}\setminus\bar{D}_r$, bounded at ∞, and continuously assumes values which are at most $\log r$ on ∂D_r. By the maximum principle we have $u(z) \leq \log r$ on $\mathbb{C}\setminus D_r$; that is,

$$|(\mu * E)(z)| \leq M[e^{\log r - g_K(z)}]^n, \quad z \in \mathbb{C}\setminus D_r. \tag{16}$$

From (13) and (16) we conclude that $\limsup_{n\to\infty} d_n(f,K)^{1/n} \leq r/\rho$. Now let $r \downarrow 1$ and $\rho \uparrow R$. $\square$

Note that for each non-constant polynomial p with $\|p\|_K \leq 1$, we have $\frac{1}{\deg p}\log|p(z)| \leq g_K(z)$ so that

$$\max\left\{0, \sup_p\left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\} \leq g_K(z). \tag{17}$$

It turns out that equality holds in (17). We use this as a starting point in jumping to several complex variables in the next section.

5 The Bernstein-Walsh theorem in $\mathbb{C}^N$, $N > 1$

For a compact set $K \subset \mathbb{C}^N$, we may define

$$V_K(z) := \max\left\{0, \sup_p\left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\}$$

where the supremum is taken over all non-constant polynomials p with $\|p\|_K \leq 1$. This is a generalization of the one-variable Green function g_K. Note that from the definition of the polynomial hull $\hat{K}$, we have

$$V_K = V_{\hat{K}}.$$

The function V_K is lower semicontinuous, but it need not be upper semicontinuous. The upper semicontinuous regularization

$$V_K^*(z) = \limsup_{\zeta \to z} V_K(\zeta)$$

of V_K is either identically $+\infty$ or else V_K^* is plurisubharmonic. The first case occurs if the set K is too "small"; precisely if K is **pluripolar**: this means that there exists a psh function u defined in a neighborhood of K with $K \subset \{z : u(z) = -\infty\}$ (see section 13 for more on pluripolar sets). We say that K is **L-regular** if $V_K = V_K^*$, that is, if V_K is continuous. For example, if $\mathbb{C}^N \setminus K$ is regular with respect to $\mathbb{R}^{2N}$-potential theory, then K is L-regular. A simple example is a closed Euclidean ball $K = \{z \in \mathbb{C}^N : |z - a| \leq R\}$; in this case, $V_K(z) = V_K^*(z) = \max[0, \log|z - a|/R]$. For a product $K = K_1 \times \cdots \times K_N$ of planar compact sets $K_j \subset \mathbb{C}$, $V_K(z_1, \ldots, z_N) = \max_{j=1,\ldots,N} g_{K_j}(z_j)$. In particular, for a polydisk

$$P := \{(z_1, \ldots, z_N) : |z_j - a_j| \leq r_j,\ j = 1, \ldots, N\},$$

$V_K(z_1, \ldots, z_N) = \max_{j=1,\ldots,N}[0, \log|z_j - a_j|/r_j]$. Any compact set K can be approximated from above by the decreasing sequence of L-regular sets $K_n := \{z : \mathrm{dist}(z, K) \leq 1/n\}$. The reason for the "$L$" is that the class of plurisubharmonic functions u in $\mathbb{C}^N$ **of logarithmic growth**, i.e., such that $u(z) \leq \log|z| + C,\ |z| \to \infty$, is called the class $L = L(\mathbb{C}^N)$. The functions $\frac{1}{\deg p} \log|p(z)|$ for a polynomial p clearly belong to L; historically, for any Borel set E, the function

$$V_E(z) := \sup\{u(z) : u \in L,\ u \leq 0 \text{ on } E\}$$

was called the **L-extremal function of** E and it was proved that for compact sets K, this upper envelope coincides with that in the beginning of this section. We sketch a proof of this. An important feature of the proof is the correspondence between psh functions in $L(\mathbb{C}^N)$ and "homogeneous" psh functions in $\mathbb{C}^{N+1}$. We remind the reader of the standard correspondence between polynomials p_d of degree d in N variables and homogeneous polynomials H_d of degree d in $N + 1$ variables via

$$p_d(z_1, \ldots, z_N) \mapsto H_d(w_0, \ldots, w_N) := w_0^d p_d(w_1/w_0, \ldots, w_N/w_0).$$

Clearly $V_K(z) \leq V(z) := \sup\{u(z) : u \in L,\ u \leq 0 \text{ on } K\}$ and to prove the reverse inequality, by approximating K from above, if necessary, we may assume K is L-regular. We consider $h(z, w)$ defined for $(z, w) \in \mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$ as follows:

$$h(z, w) := \begin{cases} |w| \exp V(z/w), & w \neq 0; \\ \limsup_{(z', w') \to (z, 0)} h(z', w'), & w = 0. \end{cases}$$

This is a nonnegative *homogeneous* psh function in $\mathbb{C}^{N+1}$; i.e., $h(tz, tw) = |t| h(z, w)$ for $t \in \mathbb{C}$. We say that the function $\log h$ is **logarithmically homogeneous**: $\log h(tz, tw) = \log|t| + \log h(z, w)$. Fix a point $(z_0, w_0) \neq (0, 0)$ with $z_0/w_0 \notin K$ and fix $0 < \epsilon < 1$. Using the fact that the polynomial hull coincides with the hull with respect to continuous psh functions (see the end of section 3), it follows that the compact set

$$E := \{(z, w) \in \mathbb{C}^{N+1} : h(z, w) \leq (1 - \epsilon) h(z_0, w_0)\}$$

is polynomially convex. Moreover, E is **circled**: $(z, w) \in E$ implies $(e^{it} z, e^{it} w) \in E$ for all real t.

Exercise. *Given a compact, circled set* $E \subset \mathbb{C}^N$ *and a polynomial* $p_d = h_d + h_{d-1} + \cdots + h_0$ *of degree* d *written as a sum of homogeneous polynomials, we have* $\|h_j\|_E \leq \|p_d\|_E$, $j = 0, \ldots, d$. Hint: Fix a point $b \in E$ at which $|h_j(b)| = \|h_j\|_E$ and use Cauchy's estimates on $\lambda \mapsto p_d(\lambda b) = \sum_{j=0}^{d} \lambda^j h_j(b)$.

From the exercise, the polynomial hull of our circled set E is the same as the hull obtained using only homogeneous polynomials. Since $E = \hat{E}$ and $(z_0, w_0) \notin E$, we can find a homogeneous polynomial h_s of degree s with $|h_s(z_0, w_0)| > \|h_s\|_E$. Define

$$p_s(z,w) := \frac{h_s(z,w)}{\|h_s\|_E} \cdot [(1-\epsilon)h(z_0,w_0)]^s.$$

Then $|p_s(z,w)|^{1/s} \leq |h(z,w)|$ for $(z,w) \in \partial E$ and by homogeneity of $|p_s|^{1/s}$ and h we have $|p_s|^{1/s} \leq h$ in all of $\mathbb{C}^{N+1}$. At (z_0, w_0), we have

$$|p_s(z_0,w_0)|^{1/s} > (1-\epsilon)h(z_0,w_0);$$

since $\epsilon > 0$ was arbitrary, as was the point (z_0, w_0) (provided $z_0/w_0 \notin K$), we get that

$$h(z,w) = \sup_s \{|p_s(z,w)|^{1/s} : p_s \text{ homogeneous of degree } s,\ |p_s|^{1/s} \leq |h|\}.$$

At $w = 1$, we obtain

$$\exp V(z) = h(z,1) = \sup_s \{|Q_s(z)|^{1/s} : Q_s \text{ of degree } s,\ |Q_s|^{1/s} \leq \exp V\}$$

which proves the result (note $V \leq 0$ on K).

If the compact set $K \subset \mathbb{C}^N$ is L-regular, then for each $R > 1$ we define the set

$$D_R := \{z : V_K(z) < \log R\}; \tag{18}$$

this is an open neighborhood of $\hat{K}$ and we clearly have the Bernstein-Walsh inequality

$$|p(z)| \leq \|p\|_K R^{\deg p} = \|p\|_{\hat{K}} R^{\deg p}, \qquad z \in D_R \tag{19}$$

for every polynomial p in $\mathbb{C}^N$. Theorem (BW1) goes over exactly to several complex variables:

Theorem (BWN). *Let* K *be an* L*-regular compact set in* $\mathbb{C}^N$. *Let* $R > 1$, *and let* D_R *be defined by* (18). *Let* f *be continuous on* K. *Then*

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} \leq 1/R$$

if and only if f *is the restriction to* K *of a function holomorphic in* D_R.

Sketch of proof. The "only if" direction follows since K satisfies the Bernstein-Walsh inequality (19). For the converse, we may assume that $K = \hat{K}$. Fix $f \in \mathcal{O}(D_R)$ and $\epsilon > 0$. Since ∂D_R is compact and $V_K(z) = \max\left\{0, \sup_p \left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\}$, we can find finitely many polynomials $p_1, \ldots, p_m$ of degree d, say, with $\|p_j\|_K \leq 1$ such that

$$\max_j\{\frac{1}{d}\log|p_j(z)|\} > \log R - \epsilon \quad \text{on} \quad \partial D_R.$$

By slightly modifying the p_j's, if necessary, we may assume that

$$\Pi := \{z : |q_j(z)| < 1,\ j = 1, \ldots, m\}$$

is a Weil polyhedron where $q_j(z) := p_j(z)/(e^{-\epsilon d}R^d)$; and clearly $K \subset \Pi$ and Π approximates D_R; i.e., $D_{R-\delta} \subset \Pi \subset D_R$ for small δ. Now use the Weil integral formula from section 3 to expand $f(z)$ and truncate the series to get good polynomial approximators. □

This sketch follows the outline of the proof given by Siciak [Si]. Zaharjuta [Z2] gave the first proof of Theorem (BWN).

There is an interesting related result, due to Tom Bloom, which applies pluripotential theory to multivariate approximation theory. To motivate this, we return to the one-variable situation and let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \setminus K$ connected and g_K continuous. Let $W(K)$ denote the closure (in the uniform norm on K) of the functions holomorphic on a neighborhood of K. Given $f \in W(K)$, let $B_d(z) = b_d z^d + \cdots$ be the best approximant to f (in sup-norm on K) from $\mathcal{P}_d(\mathbb{C})$. Wojcik [W] showed that f has a holomorphic extension to D_R for some $R > 1$ if and only if

$$\limsup_{d\to\infty} |b_d|^{1/d} \leq \frac{1}{R\mathrm{cap}(K)}.$$

Here, $\mathrm{cap}(K) := \lim_{|z|\to\infty} |z| \exp(-g_K(z))$ is the **logarithmic capacity of** K. Equivalently,

$$\mathrm{cap}(K) = \lim_{d\to\infty} \inf\{\|p_d\|_K^{1/d} : p_d(z) = z^d + \cdots\},$$

the **Chebyshev constant** of K.

Now in several complex variables, what should replace the "leading coefficient" b_d of a univariate polynomial? Moreover, what replaces the asymptotics of the Green function g_K? To address the first question, recall that any polynomial P_d of degree d may be written as the sum $P_d = H_d + H_{d-1} + \cdots + H_0$ where H_j is a homogeneous polynomial of degree j; i.e., $H_j(tz) = t^j H_j(z)$ for $t \in \mathbb{C},\ z \in \mathbb{C}^N$. Going backwards, *given* a homogeneous polynomial H_d of degree d and a compact set $K \subset \mathbb{C}^N$, we define the **Chebyshev polynomial of H_d relative to** K, denoted $\mathrm{Tch}_K H_d$, to be a polynomial of the form $H_d + R_{d-1}$ with $\|H_d + R_{d-1}\|_K$ minimal among all $R_{d-1} \in \mathcal{P}_{d-1}(\mathbb{C}^N)$. Such a polynomial need not be unique if $N \geq 2$ but the number $\|\mathrm{Tch}_K H_d\|_K$ is well-defined. Note if $N = 1$ and $H_d(z) = z^d$, $\mathrm{Tch}_K H_d$ is just the classical Chebyshev polynomial for K of degree d.

For the second question, we make some preliminary definitions. Given a psh function $u \in L(\mathbb{C}^N)$ we define the **Robin function** of u to be

$$\rho_u(z) := \limsup_{|\lambda|\to\infty} \left[u(\lambda z) - \log|\lambda|\right].$$

Note that for $\lambda \in \mathbb{C}$, $\rho_u(\lambda z) = \log|\lambda| + \rho_u(z)$; i.e., ρ_u is logarithmically homogeneous. It is known ([Bl6], Proposition 2.1) that for $u \in L(\mathbb{C}^N)$, the Robin function $\rho_u(z)$ is plurisubharmonic in $\mathbb{C}^N$; indeed, either $\rho_u \in L(\mathbb{C}^N)$ or $\rho_u \equiv -\infty$. As an example, if p is a polynomial of degree d so that $u(z) := \frac{1}{d}\log|p(z)| \in L(\mathbb{C}^N)$, then $\rho_u(z) = \frac{1}{d}\log|\hat{p}(z)|$ where $\hat{p}$ is the top degree (d) homogeneous part of p. For a compact set K, we denote by ρ_K the Robin function of V_K^*; i.e., $\rho_K := \rho_{V_K^*}$.

We can now state the beautiful result of Bloom:

Theorem ([Bl6]). *Let K be an L-regular, polynomially convex compact set in $\mathbb{C}^N$. Let $f \in W(K)$ and let $B_d := H_d +$ lower degree terms, $d = 1, 2, \ldots$, be a sequence of best approximating polynomials to f on K. For $R > 1$, the following are equivalent:*

1. *f extends holomorphically to D_R;*
2. *$\limsup_{d\to\infty} \|f - B_d\|_K^{1/d} \leq 1/R$;*
3. *$\limsup_{d\to\infty} \|\mathrm{Tch}_K H_d\|_K^{1/d} \leq 1/R$;*
4. *$\limsup_{d\to\infty} \frac{1}{d}\log|H_d(z)| \leq \rho_K(z) - \log R$ for $z \in \mathbb{C}^N \setminus \{0\}$.*

Of course, the equivalence of 1. and 2. is the Bernstein-Walsh theorem. The deep part of this result is the implication 4. implies 3.; this follows from

Theorem ([Bl6]). *Let K be an L-regular, polynomially convex compact set in $\mathbb{C}^N$. Let $\{H_d\}$ be a sequence of homogeneous polynomials with $\deg H_d = d$ and assume that $\limsup_{d\to\infty} \frac{1}{d}\log|H_d(z)| \le \rho_K(z)$ for $z \in \mathbb{C}^N \setminus \{0\}$. Then*

$$\limsup_{d\to\infty} \|\mathrm{Tch}_K H_d\|_K^{1/d} \le 1.$$

To prove this theorem, Bloom constructs polynomials $W_j,\ j = 1,\dots,s$ with $\|W_j\|_K \le 1$ such that a Weil polyhedron $\{z \in \mathbb{C}^N : |\hat{W}_j(z)| < R_j,\ j = 1,\dots,s\}$ utilizing the top degree homogeneous polynomials $\hat{W}_j$ of W_j contains K. Each H_d may be expanded in a series involving the $\hat{W}_j$'s:

$$H_d(z) = \sum_{|M|=0}^{\infty} \sum_I{}' A_M^I(z)[\hat{W}_I(z)]^M.$$

Replacing each $\hat{W}_I$ by W_I creates polynomials

$$P_d(z) := \sum_{|M|=0}^{\infty} \sum_I{}' A_M^I(z)[W_I(z)]^M$$

of degree d which are competitors for $\mathrm{Tch}_K H_d$ and whose sup norms on K can be estimated.

6 Quantitative Runge-type results in multivariate approximation

The duality proof presented of the one-variable Walsh theorem, Theorem (BW1) of section 4, may be extended to yield a quantitative Runge theorem for harmonic functions in $\mathbb{R}^N$, where $N \ge 2$. To state this we let $\mathcal{H}_n^N$ be the vector space of all harmonic, real-valued polynomials of N variables of degree at most n. If f is a continuous real-valued function on a compact set $K \subset \mathbb{R}^N$, we now define

$$d_n(f,K) := \inf\{\|f - h_n\|_K : h_n \in \mathcal{H}_n^N\}.$$

Theorem ([A],[BL2]). *Let K be a compact subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus K$ is connected.*

(a) *Let Ω be an open neighborhood of K. Then there is a constant $\rho \in (0,1)$, depending only on K and Ω, with the following property: if f is harmonic on Ω, then $\limsup_{n\to\infty} d_n(f,K)^{1/n} \le \rho$.*

(b) *Suppose that when we regard $K \subset \mathbb{R}^N = \mathbb{R}^N + i0 \subset \mathbb{C}^N$, the set K is L-regular. If f is a real-valued continuous function on K such that $\limsup_{n\to\infty} d_n(f,K)^{1/n} < 1$, then f extends to a harmonic function on an open neighborhood of K.*

Sketch of proof of (a). We follow the outline of the duality proof of the Walsh theorem, replacing the Cauchy kernel by the fundamental solution $E(x-y) = c_N|x-y|^{2-N}$ for the Laplace operator Δ (here c_N is a constant depending only on the dimension). Analogous to (13), we can write $d_n = \int_L (\mu * E)(z)\Delta F(z) dA(z)$ where $\mu = \mu_n$ is a complex measure supported on K with $|\mu|(K) = 1$ which annihilates $\mathcal{H}_n^N$, L is a compact subset of $\Omega \setminus K$, and F is a C^∞ function. The details of this proof may be found in [BL2], but the main difference here is the problem of estimating the **Newtonian potential**

$$(\mu * E)(x) = \int_K c_N|x-y|^{2-N}\, d\mu(y)$$

on compact subsets of $\Omega \setminus K$ under the assumption that we have the decay estimate

$$|(\mu * E)(x)| = O(|x|^{-n}) \quad \text{as} \quad |x| \to \infty \tag{20}$$

analogous to (15). We utilize the Kelvin transform $T(x) := x/|x|^2$, under which a harmonic function $h(x)$ on a domain $G \subset \mathbb{R}^N \setminus \{0\}$ is transformed into a harmonic function $\widetilde{h}(x) := |x|^{2-N} h(x/|x|^2)$ on $T(G)$. The condition (20) of rapid decay at infinity is transformed into a condition of flatness near the origin, and the problem of estimating $\mu * E$ under the hypothesis (20) is reduced to proving the following **Schwarz lemma** for harmonic functions [BL2].

Lemma (HSL). *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $a \in \Omega$. If K is a compact subset of Ω, then there exist constants $C > 1$ and $\rho \in (0,1)$, depending only on K and Ω, with the following property: if f is a harmonic function on Ω satisfying $|f| \leq 1$ in Ω, and if $D^\alpha f(a) = 0$ whenever $|\alpha| < n$, then $\|f\|_K \leq C\rho^n$.*

Here $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$. The proof of Lemma (HSL) is based on techniques from the theory of functions of several complex variables. In the preceding section we introduced the function V_K in $\mathbb{C}^N$ as a substitute for the ordinary Green function with pole at infinity in $\mathbb{C}^1$. To prove Lemma (HSL) we introduce in $\mathbb{C}^N$ a substitute for the Green function with a finite pole in $\mathbb{C}^1$. Following Klimek [K], section 6.1, we define for each domain $\widetilde{\Omega} \subset \mathbb{C}^N$ and each point $a \in \widetilde{\Omega}$ the **pluricomplex Green function**

$$G_{\widetilde{\Omega}}(z; a) := \sup_u u(z),$$

where the supremum is taken over all nonpositive plurisubharmonic functions u on $\widetilde{\Omega}$ such that $u(z) - \log|z - a|$ has an upper bound in some neighborhood of a. It is known that if, e.g., $\widetilde{\Omega}$ is bounded, or, more generally, hyperconvex (see section 3), then $G_{\widetilde{\Omega}}(\cdot; a)$ is a nonconstant, negative plurisubharmonic function in $\widetilde{\Omega}$ (see [K], [BL2]). This fact leads to the following Schwarz lemma for holomorphic functions of several variables (see [Bi], [BL2]).

Lemma (SL). *Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{C}^N$ and let $a \in \widetilde{\Omega}$. Let $\tilde{K}$ be a compact subset of $\widetilde{\Omega}$. If f is a holomorphic function on $\widetilde{\Omega}$ satisfying $|f| \leq 1$ in $\widetilde{\Omega}$, and if $\partial^\alpha f(a) = 0$ whenever $|\alpha| < n$, then $\|f\|_{\tilde{K}} \leq \rho^n$, where*

$$\rho := \sup_{\tilde{K}} \exp(G_{\widetilde{\Omega}}(\cdot; a)) < 1.$$

Here $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_N^{\alpha_N}}$. The inequality $\rho < 1$ is clear from the fact that $G_{\widetilde{\Omega}}(\cdot; a)$ is a negative function on $\widetilde{\Omega}$ which is *subharmonic* as a function of $2N$ real variables. The rest of the lemma follows from the fact that the function $u(z) := \frac{1}{n} \log|f(z)|$ is one of the competitors in the definition of $G_{\widetilde{\Omega}}(\cdot; a)$.

A harmonic function is real analytic and thus about each point x_0 in our domain $\Omega \subset \mathbb{R}^N$ we can get a power series expansion $\sum a_\alpha (x - x_0)^\alpha$ of f which converges *as a holomorphic function* $\sum a_\alpha (z - x_0)^\alpha$ in a neighborhood of this point in $\mathbb{C}^N$. We prove Lemma (HSL) by covering the compact set K by finitely many real balls B and the union of the complex balls $\widetilde{B}$ gives a neighborhood $\widetilde{\Omega}$ of K in $\mathbb{C}^N$ to which we can apply Lemma (SL). $\square$

In [BL3], the authors prove Bernstein theorems for solutions of more general elliptic partial differential equations. Let $p(x) := \sum_{|\alpha|=m} a_\alpha x^\alpha$ be a non-constant homogeneous polynomial in $\mathbb{R}^N$, with complex coefficients, which is never equal to zero on $\mathbb{R}^N \setminus \{0\}$; here $N \geq 2$. Then the partial differential operator $p(D) := p(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is elliptic. We let $\mathcal{L}_n$ be the vector space of polynomials q of degree at most n in N variables which are solutions of the equation $p(D)q = 0$. If f is a continuous function on a compact set $K \subset \mathbb{R}^N$, define

$$d_n(f,K) = \inf\{\|f - p_n\|_K : p_n \in \mathcal{L}_n\}.$$

Theorem ([BL3]). *Let K be a compact subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus K$ is connected.*

(a) *Let Ω be an open neighborhood of K. Then there is a constant $\rho \in (0,1)$, depending only on $p(D)$, K, and Ω, with the following property: if f is a solution of $p(D)f = 0$ on Ω, then $\limsup_{n\to\infty} d_n(f,K)^{1/n} \leq \rho$.*

(b) *Suppose that when we regard $K \subset \mathbb{R}^N = \mathbb{R}^N + i0 \subset \mathbb{C}^N$, the set K is L-regular. If f is a real-valued continuous function on K such that $\limsup_{n\to\infty} d_n(f,K)^{1/n} < 1$, then f extends to a solution F of $p(D)F = 0$ on an open neighborhood of K.*

A duality proof, utilizing a fundamental solution $E(x-y)$ for the operator $p(D)$, and yet another tool from pluripotential theory, the **relative extremal function**

$$\omega^*(z,F,\Omega) := [\sup\{u(z) : u \text{ psh in } \Omega,\ u \leq 0,\ u|_F \leq -1\}]^*$$

of a set $F \subset \Omega$ relative to Ω, may be found in [BL3]. See section 13 for more on $\omega^*(z,F,\Omega)$. The need for this function arises as we don't have a Kelvin transform in this general setting; but we do get a "transfer of smallness" result analogous to Lemma (SL):

Lemma. *Let Ω be a bounded domain in $\mathbb{C}^N$. Let $F \subset \Omega$ be nonpluripolar and let $K \subset \Omega$ be compact. Then there is a constant $a \in (0,1]$ such that for any holomorphic function g on Ω with $|g| \leq M$ on Ω and $|g| \leq m < M$ on F, we have*

$$|g| \leq m^a M^{1-a} \text{ on } K.$$

The proof is trivial: simply observe that $\log|g|$ is psh and by the definition of $\omega(z,F,\Omega)$, it follows that

$$u(z) := \frac{\log(|g(z)|/M)}{\log(M/m)} \leq \omega(z,F,\Omega).$$

Then the constant a can be chosen to be $a(\Omega,F,K) := -\sup_K \omega(z,F,\Omega)$. The nonpluripolarity of F insures that $a > 0$ (see Proposition (ω) in section 13).

Finally, we mention that Jackson-type approximation theorems for solutions to $p(D)f = 0$ on Ω which are continuous on $\bar{\Omega}$ can be found in [BBL1] and [BBL2].

7 Mergelyan property and solving $\bar{\partial}$

Recall the Cauchy-Green formula: Let Ω be a bounded domain in $\mathbb{C}$ with C^1-boundary and let $f \in C^1(\bar{\Omega})$. Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_\Omega \frac{\partial f}{\partial \bar{z}}(\zeta)\cdot(\frac{1}{\zeta - z})dA(\zeta) \tag{21}$$

where dA denotes Lebesgue measure. In particular,

1. if $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}d\zeta \text{ (Cauchy integral formula);} \tag{22}$$

2. if $f \in C_0^1(\Omega)$,

$$f(z) = -\frac{1}{\pi}\int_{\Omega}\frac{\partial f}{\partial \bar{z}}(\zeta)\cdot(\frac{1}{\zeta - z})dA(\zeta).$$

As an immediate corollary, if $g \in C_0(\Omega)$, then

$$G(z) := -\frac{1}{\pi}\int_{\Omega} g(\zeta)\cdot(\frac{1}{\zeta - z})dA(\zeta)$$

solves the inhomogeneous Cauchy-Riemann equation $\partial G/\partial \bar{z} = g$ in Ω. Moreover, we see that

$$\sup_{\Omega}|G| \leq [\sup_{\Omega}|g|]\sup_{z\in\Omega}[\frac{1}{\pi}\int_{\text{supp}g}|\frac{1}{\zeta - z}|dA(\zeta)] \tag{23}$$

so that $\sup_{\Omega}|G| \leq C\sup_{\Omega}|g|$. More generally, if μ is a measure with compact support in Ω, the Cauchy transform of μ,

$$\hat{\mu}(z) := -\frac{1}{\pi}\int_{\Omega}\frac{1}{\zeta - z}d\mu(\zeta),$$

satisfies $\partial\hat{\mu}/\partial\bar{z} = \mu$ in the sense of distributions on Ω.

Now suppose $K = \bar{\Omega}$ where Ω is a simply connected domain with boundary of class C^1. An elementary proof of Mergelyan's theorem (Theorem (Me) from section 1) for such K goes as follows:

1. *Smooth approximation*. Cover ∂K by finitely many open sets $U_1,\ldots,U_n$ such that for each $j = 1,\ldots,n$ there is a vector t_j transverse to $\partial\Omega$ at each point of $\partial\Omega \cap U_j$ pointing outward (into $\mathbb{C}\setminus\bar{\Omega}$). Take a partition of unity $\phi,\phi_1,\ldots,\phi_n$ for a neighborhood of $\bar{\Omega}$ subordinate to the cover consisting of $\Omega, U_1,\ldots,U_n$. Given $f \in \mathcal{O}(\Omega)\cap C(\bar{\Omega})$, for sufficiently large j,

$$g_j(z) := \phi(z)f(z) + \sum_{k=1}^{n}\phi_k(z)f(z - t_k/j)$$

is defined and C^∞ on a neighborhood Ω_j of $\bar{\Omega}$. Since $f \in C(\bar{\Omega})$, $g_j \to f$ uniformly on $\bar{\Omega}$.

2. *Holomorphic correction*. We have that

$$\frac{\partial g_j}{\partial\bar{z}}(z) = f(z)\cdot\frac{\partial\phi}{\partial\bar{z}}(z) + \sum_{k=1}^{n}f(z - t_k/j)\cdot\frac{\partial\phi_k}{\partial\bar{z}}(z)$$

is uniformly small on Ω_j since $\phi + \sum_k\phi_k = 1$ there; say $\sup_{\Omega_j}|\frac{\partial g_j}{\partial\bar{z}}| \leq \delta_j$ where $\delta_j \to 0$. From the previous discussion, utilizing (23), for each j we can find $G_j \in C^\infty(\Omega_j)$ with

$$\frac{\partial G_j}{\partial\bar{z}} = \frac{\partial g_j}{\partial\bar{z}}$$

in Ω_j and $\sup_{\Omega_j}|G_j| \leq C_j\delta_j \to 0$. Then $f_j := g_j - G_j \in \mathcal{O}(\Omega_j)$ and $f_j \to f$ uniformly on $\bar{\Omega}$.

In higher dimensions, the smooth approximation step works fine. However, things get tricky in step 2 for two reasons:

(i) we need to solve a $\bar{\partial}$-equation; more precisely,
(ii) we need to solve a $\bar{\partial}$-equation *with uniform (sup-norm) estimates*.

Let's make this precise. Given a C^1 function u, the 1-form du can be written as $du = \partial u + \bar{\partial} u$ where

$$\partial u := \sum_{j=1}^{N} \frac{\partial u}{\partial z_j} dz_j$$

is a form of bidegree $(1,0)$ and

$$\bar{\partial} u := \sum_{j=1}^{N} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j$$

is a form of bidegree $(0,1)$. In general, a differential form ϕ of bidegree (p,q) is a sum

$$\phi = \sum_{|I|=p,\ |J|=q}{}' c_{I,J} dz^I \wedge d\bar{z}^J$$

where $c_{I,J}$ are functions (0-forms) and

$$dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p};\ d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q};$$

the prime means the indices are increasing. We define

$$\bar{\partial}\phi = \sum_{|I|=p,\ |J|=q}{}' \bar{\partial} c_{I,J} \wedge dz^I \wedge d\bar{z}^J;$$

this is a form of bidegree $(p, q+1)$.

We can extend the operator ∂ to (p,q)-forms as well (the result is a $(p+1,q)$-form). Since any form ω of degree $r \in \{0,1,\ldots,2N\}$ can be written as a sum of forms $\omega_{p,q}$ of bidegree (p,q) where $0 \le p,q \le r$ and $p+q=r$, we extend $\bar{\partial}$ and ∂ to general forms by linearity. Note then as differential operators on the space of smooth forms, we have

$$d^2 = d \circ d = 0 = (\partial + \bar{\partial}) \circ (\partial + \bar{\partial}) = \partial^2 + \partial \circ \bar{\partial} + \bar{\partial} \circ \partial + \bar{\partial}^2;$$

by bidegree considerations

$$\partial^2 = \bar{\partial}^2 = 0; \quad \partial \circ \bar{\partial} = -\bar{\partial} \circ \partial$$

(e.g., if ϕ is a (p,q) form, $d^2\phi = 0$ is a form of total degree $p+q+2$; $\bar{\partial}^2\phi$ is of bidegree $(p,q+2)$ and there are no other terms with this bidegree). In particular, let $\phi = \sum_{j=1}^N \phi_j d\bar{z}_j$ be a smooth $(0,1)$ form on a domain Ω in $\mathbb{C}^N$. *If we want to be able to find a function $u \in C^\infty(\Omega)$ with $\bar{\partial} u = \phi$ in Ω, a necessary condition is that $\bar{\partial}\phi = 0$.* This condition is vacuous if $N=1$ (there are no $(0,2)$ forms in $\mathbb{C}$). Note that the inhomogeneous Cauchy-Riemann equation $\bar{\partial} u = \phi$ is a system of $2N$ (real) partial differential equations in $\mathbb{R}^{2N}$ for the (two) unknown functions $\Re u$ and $\Im u$. This is an overdetermined system if $N > 1$.

Are there integral formulas providing solutions to $\bar{\partial}$? For $N = 1$, define, for $z \in \mathbb{C}$, the $(1,0)$-form in ζ

$$\omega_{BM}(\zeta - z) := \frac{1}{2\pi i} \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} \frac{\bar{\zeta} - \bar{z}}{|\zeta - z|^2} d\zeta.$$

Then if $\Omega \subset \mathbb{C}$ is a bounded domain with C^1-boundary and $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$, we have

$$f(z) = \int_{\partial\Omega} f(\zeta)\omega_{BM}(\zeta - z)$$

for $z \in \Omega$. This is the Cauchy integral formula (22). In SCV, we have the following.

Proposition (Bochner-Martinelli formula). *Define the $(N, N-1)$-form*

$$\omega_{BM}(\zeta - z) := \frac{(N-1)!}{(2\pi i)^N} \sum_{j=1}^{N} \frac{(-1)^{j-1}(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2N}} d\bar{\zeta}[j] \wedge d\zeta.$$

If $\Omega \subset \mathbb{C}^N$ is a bounded domain with C^1-boundary and $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$, then

$$f(z) = \int_{\partial\Omega} f(\zeta)\omega_{BM}(\zeta - z) \tag{24}$$

for $z \in \Omega$.

Here $d\bar{\zeta}[j] = d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge \cdots \wedge d\bar{\zeta}_N$ (omit $d\bar{\zeta}_j$) and $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_N$. The reader will note that the ζ_j partial derivative of $\frac{1}{|\zeta - z|^{2N-2}}$, a fundamental solution for the Laplacian in $\mathbb{R}^{2N}$, is

$$\frac{\partial}{\partial \zeta_j} \frac{1}{|\zeta - z|^{2N-2}} = \frac{(1-N)(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2N}},$$

which is, up to a constant, the coefficient of $d\bar{\zeta}[j] \wedge d\zeta$ in $\omega_{BM}(\zeta - z)$. As a generalization of the Cauchy-Green formula (21), for any $f \in C^1(\bar{\Omega})$ we have

$$f(z) = \int_{\partial\Omega} f(\zeta)\omega_{BM}(\zeta - z) - \frac{(N-1)!}{\pi^N} \int_{\Omega} \sum_{j=1}^{N} \frac{\partial f}{\partial \bar{z}_j} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2N}} dA(\zeta) \tag{25}$$

for $z \in \Omega$. Here $dA(\zeta)$ is Lebesgue measure in $\mathbb{C}^N$, i.e.,

$$dA(\zeta) = (i/2)^N d\zeta_1 \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\zeta_N \wedge d\bar{\zeta}_N.$$

However, *only if $N = 1$ are the coefficients of this Bochner-Martinelli kernel $\omega_{BM}(\zeta - z)$ holomorphic in z*; thus only for $N = 1$ can this formula be used to construct solutions to $\bar{\partial}u = \phi$. That is, given a smooth $(0, 1)$ form $\phi = \sum_{j=1}^N \phi_j d\bar{z}_j$ on $\Omega \subset \mathbb{C}^N$ with $\bar{\partial}\phi = 0$, from (25) we'd *like* to define

$$u(z) = -\frac{(N-1)!}{(2\pi i)^N} \int_{\Omega} \sum_{j=1}^{N} \phi_j \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2N}} d\bar{\zeta} \wedge d\zeta$$

to solve $\bar{\partial}u = \phi$ in Ω, but since $(\bar{\zeta}_j - \bar{z}_j)/|\zeta - z|^{2N}$ is not holomorphic in z if $N > 1$, this doesn't work. A major area of research in SCV was the attempted construction of integral formulas with *holomorphic* kernels. For *strictly pseudoconvex domains* (recall section 3), this was done by Henkin and Ramirez. The article [He] of Henkin is a nice historical survey on the subject.

We say that a domain Ω has the **Mergelyan property** if every $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ by functions in $\mathcal{O}(\bar{\Omega})$. This definition is natural, given the one-variable Mergelyan theorem, Theorem (Me), from section 1.

Theorem. *A smoothly bounded, strictly pseudoconvex domain satisfies the Mergelyan property.*

The reason is that Henkin, Kerzman and Lieb showed that one can solve $\bar{\partial}$ with *uniform* estimates by constructing holomorphic kernels in this case.

What if D is pseudoconvex, but not strictly pseudoconvex? Let

$$D = \{(z,w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\},$$

the so-called **Hartogs triangle** (draw a picture in $|z|, |w|-$space to explain the terminology). Then D is pseudoconvex. However, any function holomorphic on a neighborhood of $\bar{D}$ must necessarily extend holomorphically to the unit bidisk $P = \Delta \times \Delta$ (exercise; or see [Sha] sections 7 and 40). Now consider the function $f(z,w) := z^2/w$. Note that

$$\limsup_{(z,w)\to(0,0),\ (z,w)\in D} |f(z,w)| \leq \limsup_{(z,w)\to(0,0),\ (z,w)\in D} |w|^2/|w| = 0$$

so that f extends continuously to $(0,0)$. Hence $f \in \mathcal{O}(D) \cap C(\bar{D})$. We show that f is not uniformly approximable on $\bar{D}$ by holomorphic functions on $\bar{D}$. For suppose $\{f_j\} \subset \mathcal{O}(\bar{D})$ converge uniformly to f on $\bar{D}$. In particular, uniform convergence of $\{f_j\}$ on

$$T^2 = \partial\Delta \times \partial\Delta = \{(z,w) : |z| = |w| = 1\}$$

implies uniform convergence of $\{f_j\}$ on $\bar{P}$ (recall the multivariate Cauchy integral formula (3) from section 1). This limit function, call it g, is necessarily holomorphic on P; in particular, it is holomorphic at $(0,0)$. Moreover, g must coincide with f on D. But f does not extend holomorphically to $(0,0)$.

There are examples of smoothly bounded pseudoconvex domains which do not satisfy the Mergelyan property. It is conjectured that *if D is a smoothly bounded pseudoconvex domain, then D has the Mergelyan property if and only if there are pseudoconvex domains D_j with $\bar{D} \subset D_j$ such that $\bar{D} = \cap_j D_j$.* This latter property fails for the Hartogs triangle. It also fails in all of the known examples of smoothly bounded pseudoconvex domains which fail to satisfy the Mergelyan property. We refer the reader to the article of Bedford and Fornaess [BF] for a more detailed discussion.

8 Approximation on totally real sets

Recall from section 1 the example of the two polynomially convex disks K_1 and K_2 in $\mathbb{C}^2$ defined as

$$K_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \text{ and}$$
$$K_2 := \{(z_1, 0) : |z_1| \leq 1\}.$$

Here $P(K_1) = C(K_1)$ but $P(K_2) \neq C(K_2)$. The difference is that K_1 lies in the totally real submanifold $\mathbb{R}^2$ of $\mathbb{C}^2$.

Definition. *Let Σ be a submanifold of class C^1 of an open set $D \subset \mathbb{C}^N$. We say Σ is* **totally real** *if for each $p \in \Sigma$, the tangent space $T_p\Sigma$ contains no complex lines; i.e., no complex linear subspaces of positive dimension.*

In particular, the dimension of such a (real) submanifold is at most N. A top-dimensional example is the torus $T^N = \{(z_1, \ldots, z_N) : |z_j| = 1,\ j = 1, \ldots, N\}$. Returning to the simpler example of $\mathbb{R}^N = \mathbb{R}^N + i0$, note that

$$u(z) := \operatorname{dist}(z, \mathbb{R}^N)^2 = y_1^2 + \cdots + y_N^2$$

is of class C^2 and strictly psh (compute the complex Hessian!). Of course, directly from the definition, $\mathbb{R}^N = \{z \in \mathbb{C}^N : u(z) = 0\}$. Indeed, *if Σ is a totally real submanifold of class C^2 of an open set $D \subset \mathbb{C}^N$, then there exists a neighborhood ω of Σ such that $u(z) = \operatorname{dist}(z, \Sigma)^2$ is of class C^2 and strictly psh on ω* (cf. [AW], Lemma 17.2). The main result on approximation on totally real submanifolds is the following generalization of the real Stone-Weierstrass theorem due to Harvey and Wells [HW]. It was first proved under stronger regularity hypotheses by Hörmander and Wermer [HöW]. A very enlightening proof has recently been given by Berndtsson [Ber].

Theorem (HW). *Let Σ be a totally real submanifold of class C^1 in an open set in $\mathbb{C}^N$ and let $K \subset \Sigma$ be compact and polynomially convex. Then $P(K) = C(K)$.*

This is related to the question stated towards the end of section 2: *Give a "nice" condition (C) on a compact set $K \subset \mathbb{C}^N$ so that if $K = \hat{K}$ then K satisfies (C) if and only if $P(K) = C(K)$.* In Lavrentiev's theorem, Theorem (La), in the complex plane, $K^o = \emptyset$ was a necessary and sufficient condition for a polynomially convex compact set K to have this property. Since uniform limits of holomorphic objects like (holomorphic) polynomials should be, in some sense, holomorphic, we seek a condition (C) which prohibits K from having any type of "analytic structure". Note that $P(K)$ is a uniform algebra, i.e., a closed subalgebra of $C(K)$. There was a famous conjecture known as the **peak point conjecture**: Suppose A is a uniform algebra on its maximal ideal space X such that every point $x \in X$ is a peak point for A, i.e., there exists $f \in A$ such that $f(x) = 1$ and $|f(y)| < 1$ for all $y \neq x$. Does it follow that A coincides with the algebra $C(X)$? In case X is a polynomially convex compact set in $\mathbb{C}^N$ and $A = P(X)$, we are asking if this "peak point property" suffices as a condition (C). A counterexample given by Cole in 1968 shows that the answer to the general peak point conjecture is no. Anderson, Izzo and Wermer [AIW1], [AIW2] have shown that *if Σ is a compact polynomially convex* **real analytic** *variety in $\mathbb{C}^N$ such that every point in Σ is a peak point for $P(\Sigma)$, then $P(\Sigma) = C(\Sigma)$.* Recall that a relatively closed subset V of an open set U in $\mathbb{C}^N$ is a real analytic subvariety of U if for each $z_0 \in V$ there exists a neighborhood $U' \subset U$ of z and real valued, real analytic functions $f_1, ..., f_m$ in U' with

$$V \cap U' = \{z \in U' : f_1(z) = \cdots = f_m(z) = 0\}.$$

The unit sphere in $\mathbb{C}^N$ for $N > 1$ is a smooth submanifold in $\mathbb{C}^N$ which definitely *has* complex tangents (i.e., is *not* totally real) as the dimension of the sphere is $2N - 1 > N$; however, it is straightforward to see that for any compact subset K of the sphere, $P(K)$ has the peak point property (e.g., at $(1, 0, \ldots, 0)$, take $f(z) = z_1$). Despite this, Izzo [I] has constructed examples of the following:

1. *There exists a compact polynomially convex subset K of the unit sphere in $\mathbb{C}^3$ such that $P(K) \neq C(K)$.*
2. *There exists a C^∞-embedding $F : \mathbb{C}^2 \to \mathbb{C}^5$ such that the set $K = F(\{(z, w) : |z| \leq 1, |w| = 1\})$ is a compact polynomially convex subset of the unit sphere in $\mathbb{C}^5$ which satisfies $P(K) \neq C(K)$.*

Note this last example is a compact, polynomially convex C^∞ submanifold K for which every point is a peak point for $P(K)$ but $P(K) \neq C(K)$; the Anderson, Izzo and Wermer theorem shows that such an example cannot occur in the real analytic category. Stout has recently strengthened the Anderson, Izzo and Wermer theorem to eliminate the hypothesis on peak points:

Theorem ([St2]). *Let K be a compact, polynomially convex real analytic subvariety of $\mathbb{C}^N$. Then $P(K) = C(K)$.*

The aforementioned results of Harvey-Wells and/or Hörmander-Wermer essentially reduce questions of approximation on subsets of real submanifolds on $\mathbb{C}^N$ to approximation on the points where the tangent space to the manifold contains a complex line. The Hörmander-Wermer approach to Theorem (HW), which requires some additional regularity hypotheses on Σ, can be summarized as follows: given $f \in C(K)$, we can clearly approximate f uniformly on K by a global smooth function; i.e., we may assume $f \in C^\infty(\mathbb{C}^N)$. From Theorem (OW), since $K = \hat{K}$, it suffices now to approximate f uniformly on K by functions holomorphic on a neighborhood of K. It is straightforward to construct a function F of class C^1 on $\mathbb{C}^N$ which agrees with f on K and such that

$$|\frac{\partial F}{\partial \bar{z}_j}(z)| = O(\mathrm{dist}(z,\Sigma)^m),\ j = 1,\dots,N$$

if, say, Σ is of class C^{2m+1}. Next, using the function $u(z) = \mathrm{dist}(z,\Sigma)^2$, we can construct a bounded, pseudoconvex neighborhood ω of K in $\mathbb{C}^N$ to which we can apply standard several complex variables machinery – solvability of $\bar{\partial}$ in ω – to construct a function G in ω with

$$\frac{\partial G}{\partial \bar{z}_j} = \frac{\partial F}{\partial \bar{z}_j}, \quad j = 1,\dots,N$$

and with $|G|$ very small in ω. Then $G - F$ is holomorphic in ω and approximates f very well on K.

The $\bar{\partial}$ machinery utilized in the previous paragraph is the Hörmander L^2-theory. If D is a smoothly bounded, pseudoconvex domain in $\mathbb{C}^N$, then there is a constant C depending only on D such that for any $(0,1)$-form $\phi = \sum_{j=1}^N \phi_j d\bar{z}_j$ with $L^2(D)$-coefficients satisfying $\bar{\partial}\phi = 0$, there exists $u \in L^2(D)$ with $\bar{\partial}u = \phi$ in D and

$$\int_D |u|^2 dA \leq C \int_D \sum_{j=1}^N |\phi_j|^2 dA$$

(cf., [Hö], Chapter 4 or [AW] section 16). Note that this is a global L^2-norm estimate. From this, one gets local interior regularity of solutions sufficient to derive the required estimate on G in the previous paragraph. Berndtsson uses a "weighted" version of the global L^2-norm estimate in his work.

The Harvey-Wells approach uses integral kernels to solve $\bar{\partial}$ and is at least similar in spirit to our outline of the proof of Theorem (OW) using the Weil integral formula. Extensions of the Harvey-Wells result have been made by Range and Siu [RS] as well as by Bruna and Burgés [BB]. These papers deal with approximation in Hölder norms on a totally real *compact* subset $X \subset \mathbb{C}^N$: we assume there exists a strictly plurisubharmonic C^2 function in a neighborhood of X whose zero set is X.

Using a generalization of the Bochner-Martinelli kernel, a suitably constructed *Cauchy-Fantappiè-Leray kernel* (see section 10), Weinstock [Wei] has proved an interesting perturbation of the Stone-Weierstrass theorem. We discuss this briefly. Given any compact set $K \subset \mathbb{C}^N$ and functions $f_1, .., f_m \in C(K)$, let $[f_1,\dots,f_m]$ denote the algebra generated by these functions. Note then we always have $[z_1,\dots,z_N,\bar{z}_1,\dots,\bar{z}_N]$ is dense in $C(K)$. Suppose N functions $R_1,\dots,R_N$ are given. Let $A = [z_1,\dots,z_N,\bar{z}_1 + R_1,\dots,\bar{z}_N + R_N]$. Under what conditions on $R := (R_1,\dots,R_N)$ is A dense in $C(K)$? Assume each R_j is defined and continuous in a neighborhood U of K.

Theorem (PSW). *If there exists* $0 \leq k < 1$ *with*

$$|R(z) - R(z')| \leq k|z - z'| \quad \textit{for} \quad z, z' \in U,$$

then A is dense in $C(K)$.

For $N = 1$ this result is due to Wermer [We]. Even in this case, it is the Lipschitz norm of the perturbation R that matters, *not* the supremum norm. For example, if $K = \bar{\Delta}$, the closed unit disk in $\mathbb{C}$, and

$$R(z) := -\bar{z} \text{ if } |z| \leq \epsilon;\ R(z) := \frac{-\epsilon\bar{z}}{|z|}\text{if } \epsilon \leq |z| \leq 1,$$

then $|R(z)| \leq \epsilon$ on $\bar{\Delta}$ but $\bar{z} + R(z) \equiv 0$ on the disk $\{z : |z| \leq \epsilon\}$. Thus each function in A must be holomorphic on $\{z : |z| < \epsilon\}$. A nice exposition of the one and several variable results can be found in chapter 14 of [AW].

9 Lagrange interpolation and orthogonal polynomials

A natural way to construct polynomials which approximate a given function is to use interpolating polynomials. Let K be a polynomially convex L-regular compact subset of $\mathbb{C}^N$. Let $m_n = \binom{N+n}{n}$ denote the dimension of the complex vector space $\mathcal{P}_n$ of polynomials in N complex variables of degree at most n.

For each integer $n \geq 1$, let $a_{n1}, \ldots, a_{nm_n} \in K$. Thus we have a doubly indexed array $(a_{nj})_{n=1,\ldots;\ j=1,\cdots,m_n}$ of points in K. Given a function f holomorphic in a neighborhood of K, under what conditions on the array do the Lagrange polynomials $L_n f$ interpolating f at the points $(a_{nj})_{j=1,\cdots,m_n}$ converge uniformly to f on K?

In one variable, Walsh gave a necessary and sufficient condition on the array in order to guarantee uniform convergence of $\{L_n f\}$ to f on K for *all* such f. In several variables ($N \geq 2$), much less is known because there is no analogue of the Hermite remainder formula used in the proof of Walsh. We remind the reader of the Hermite remainder formula for interpolation of a holomorphic function of one variable. This is a simple consequence of the Cauchy integral formula. Let $z_1, \ldots, z_n$ be n distinct points in the plane and let f be a function which is defined at these points. The functions

$$l_j(z) := \prod_{k \neq j} (z - z_k)/(z_j - z_k), \quad j = 1, \ldots, n,$$

are polynomials of degree $n-1$ with $l_j(z_k) = \delta_{jk}$, called the **fundamental Lagrange interpolating polynomials** associated to $z_1, \ldots, z_n$. Then $(L_n f)(z) := \sum_{j=1}^n f(z_j) l_j(z)$ is the unique polynomial of degree at most n satisfying $(L_n f)(z_j) = f(z_j),\ j = 1, \ldots, n$; we call it the **Lagrange interpolating polynomial** associated to $f, z_1, \ldots, z_n$. If Γ is a rectifiable Jordan curve such that the points $z_1, \ldots, z_n$ are inside Γ, and f is holomorphic inside and on Γ, we can estimate the error in our approximation of f by $L_n f$ at points inside Γ using the following formula.

Lemma (Hermite Remainder Formula). *For any z inside Γ,*

$$f(z) - (L_n f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(z)}{\omega(t)} \frac{f(t)}{(t-z)}\, dt, \tag{26}$$

where $\omega(z) := \prod_{k=1}^n (z - z_k)$.

Note that if $f(z) = 1/(t-z)$, then

$$f(z) - (L_n f)(z) = \frac{\omega(z)}{\omega(t)} \frac{1}{t-z}. \tag{27}$$

The necessary and sufficient condition of Walsh on the univariate array $(a_{nj}) \subset K \subset \mathbb{C}$ so that $L_n f \to f$ uniformly on K for any f holomorphic on K is that the polynomials $\omega_{n+1}(z) := \prod_{j=1}^{n+1} (z - a_{nj})$ satisfy

$$\lim_{n \to \infty} \|\omega_{n+1}\|_K^{\frac{1}{n+1}} = \mathrm{cap}(K);$$

equivalently, the subharmonic functions

$$u_n(z) := \frac{1}{n+1} \log \frac{|\omega_{n+1}(z)|}{\|\omega_{n+1}\|_K}$$

converge locally uniformly to g_K on $\mathbb{C} \setminus K$.

For a survey of some results in the several variable case, we refer the reader to [BBCL], [Bl3] and [BlL1]. We outline the elementary positive results. Let $e_1, \dots, e_{m_n}$ form a basis for $\mathcal{P}_n$. Given $A_n = \{a_{n1}, \dots, a_{nm_n}\} \subset K$ we form the **generalized Vandermonde determinant**

$$V_n(A_n) := \det[e_i(a_{nj})]_{i,j=1,\dots,m_n}.$$

If $V_n(A_n) \neq 0$, we can form the polynomials

$$l_{nj}(z) := \frac{V_n(a_{n1}, \dots, z, \dots, a_{nm_n})}{V_n(A_n)}, \qquad j = 1, \dots, m_n$$

satisfying $l_{nj}(a_{ni}) = \delta_{ji}$. We call

$$\Lambda_n := \sup_{z \in K} \sum_{j=1}^{m_n} |l_{nj}(z)|$$

the n-th **Lebesgue constant** for K, A_n. For f defined on K,

$$(L_n f)(z) := \sum_{j=1}^{m_n} f(a_{nj}) l_{nj}(z)$$

is the Lagrange interpolating polynomial for f at the points A_n. We say that K is **determining** for $\bigcup \mathcal{P}_n$ if whenever $h \in \bigcup \mathcal{P}_n$ satisfies $h = 0$ on K, it follows that $h \equiv 0$. For these sets we can find point sets A_n for each n with $V_n(A_n) \neq 0$. We have the following elementary result.

Proposition. *Let K be determining for $\bigcup \mathcal{P}_n$ and let $A_n \subset K$ be sets of points satisfying $V_n(A_n) \neq 0$ for each n. Given f bounded on K, if* $\limsup \Lambda_n^{1/n} = 1$, *then* $\limsup \|f - L_n f\|_K^{1/n} = \limsup d_n^{1/n}$ *where*

$$d_n = d_n(f, K) = \inf\{\|f - p_n\|_K : p_n \in \mathcal{P}_n\}.$$

Proof. Fix $\epsilon > 0$ and choose, for each n, a polynomial $p_n \in \mathcal{P}_n$ with

$$\|f - p_n\|_K^{1/n} \leq d_n^{1/n} + \epsilon.$$

Since $p_n \in \mathcal{P}_n$, we have $L_n p_n = p_n$ and

$$\begin{aligned} \|f - L_n f\|_K &= \|f - p_n + L_n p_n - L_n f\|_K \\ &\leq \|f - p_n\|_K + \Lambda_n \|f - p_n\|_K = (1 + \Lambda_n)\|f - p_n\|_K. \end{aligned}$$

Using the hypothesis $\limsup \Lambda_n^{1/n} = 1$, we obtain the conclusion. $\square$

Let $p(D) := p(\partial/\partial x_1, \dots, \partial/\partial x_n)$ be an elliptic partial differential operator as at the end of section 6, and let $\mathcal{L}_n$ be the vector space of polynomials q of degree at most n in N variables which are solutions of the equation $p(D)q = 0$. The same proof shows: *Let $K \subset \mathbb{R}^N$ be determining for $\cup_n \mathcal{L}_n$ and let $A_n \subset K$ satisfy $V_n(A_n) \neq 0$ for each n. Given f bounded on K, if* $\limsup \Lambda_n^{1/n} = 1$, *then* $\limsup \|f - L_n f\|_K^{1/n} = \limsup d_n^{1/n}$. Here, we replace m_n by $\tilde{m}_n := \dim \mathcal{L}_n$, $A_n = \{a_{n1}, \dots, a_{n\tilde{m}_n}\} \subset K$ and d_n is defined just as before, but now with respect to $\mathcal{L}_n$.

Arrays of points $\{A_n\}, n = 1, 2, \ldots$ satisfying $\limsup \Lambda_n^{1/n} = 1$ can be constructed by taking, e.g., A_n to be a set of **n-Fekete points for** K: for each n, choose $A_n \subset K$ so that

$$\max_{X_n \subset K} |V_n(X_n)| = |V_n(A_n)|.$$

Since $|V_n(X_n)|$ is a continuous function on K^{m_n}, such points exist. Moreover, from the definition of Fekete points, $\|l_{nj}\|_K = 1$ so that $\Lambda_n \leq m_n$. It is easy to see that $\lim m_n^{1/n} = 1$. The problem is that in several variables these points are essentially impossible to construct. The first known *explicit* example of an array $\{A_n\}, n = 1, 2, \ldots$ satisfying $\limsup \Lambda_n^{1/n} = 1$ associated to a compact set $K \subset \mathbb{C}^N$, $N > 1$, has been recently discovered by Bos, et al [BCDVX]. The set K is the unit square $[-1, 1] \times [-1, 1]$ in $\mathbb{R}^2$. In this example, the Lebesgue constants have minimal possible asymptotic growth: $\Lambda_n = O([\log n]^2)$.

In CCV, for a non-polar compact set $K \subset \mathbb{C}$, the normalized counting measures associated to Fekete arrays satisfy

$$\mu_n := \frac{1}{n+1} \sum_{j=1}^{n+1} \delta_{A_{nj}} \to \frac{1}{2\pi} \Delta g_K$$

in the weak*-topology as measures. Here, Δg_K, the Laplacian of g_K, is to be interpreted as a positive distribution, i.e., a positive measure. Indeed, for any array $\{A_n\}, n = 1, 2, \ldots$ satisfying $\limsup \Lambda_n^{1/n} = 1$ the same conclusion holds (cf. [BBCL]). In SCV, for $K \subset \mathbb{C}^N$ nonpluripolar, there is a conjecture that for Fekete arrays, the normalized discrete measures

$$\mu_n := \frac{1}{m_n} \sum_{j=1}^{m_n} \delta_{A_{nj}}$$

converge weak-* to the **Monge-Ampére measure** $\mu_K := (dd^c V_K^*)^N$ of the L-extremal function V_K^*. For a discussion of the complex Monge Ampére operator $(dd^c \cdot)^N$, see the appendix. To this date, *nothing* is known if $N > 1$. Some special situations, which really reduce to one-variable problems, can be found in [GMS] and [BlL2].

A widely studied topic in classical approximation theory is the study of orthogonal polynomials. Let μ be a positive Borel measure with compact support $K = \text{supp}(\mu) \subset \mathbb{C}^N$. Assume that the set K is determining for $\cup \mathcal{P}_n$. If $N = 1$, this just means that K contains infinitely many points; for $N > 1$, K being nonpluripolar is sufficient (but not necessary). Then the standard basis monomials $\{e_\alpha(z) := z_1^{\alpha_1} \cdots z_N^{\alpha_N}\}$ are linearly independent in $L^2(\mu)$ and one can form the orthonormal polynomials $\{p_\alpha(z, \mu)\}$. For an introduction to this topic in $\mathbb{C}^N$, we recommend Tom Bloom's paper [Bl5] which concerns the relationship between the so-called n-th root asymptotic behavior of the orthonormal polynomials $\{p_\alpha(z, \mu)\}$ and the "pluripotential theory" of the set K. This is presented in a systematic manner analogous to the one-variable study developed in the book of H. Stahl and V. Totik [ST]. See also [Bl8].

The pair (K, μ) is said to have the **Bernstein-Markov property** if for each $\epsilon > 0$ there exists a positive constant $M = M(\epsilon)$ such that

$$\|p\|_K \leq M(1+\epsilon)^{\deg p} \|p\|_{L^2(\mu)}$$

for all polynomials $p = p(z)$. For an L-regular compact set K, such measures always exist; e.g., the Monge-Ampère measure $\mu_K = (dd^c V_K)^N$ (this is the Laplacian Δg_K if $N = 1$). One can even find such a measure μ which is rather "sparse" in the sense that there exists a countable subset $E \subset K$ with $\mu(E) = \mu(K)$. Returning to the setting of the Bernstein-Walsh theorem, given such a measure, best $L^2(\mu)$-approximants to $f \in C(K)$ have optimal behavior.

Proposition. *Let K be a polynomially convex L-regular compact set in $\mathbb{C}^N$ and let μ be a measure supported on K such that (K,μ) satisfies the Bernstein-Markov property. If $f \in C(K)$ satisfies $\limsup_{n\to\infty} d_n(f,K)^{1/n} = \rho < 1$, and if $\{p_n\}$ is a sequence of best $L^2(\mu)$-approximants to f, then $\limsup_{n\to\infty} \|f-p_n\|_K^{1/n} = \rho$.*

The proof follows trivially from the fact that if $\rho < r < 1$ and $\{q_n\}$ are polynomials with $\|f-q_n\|_K \le Mr^n$ for some M (independent of n), then

$$\|f-p_n\|_{L^2(\mu)} \le \|q_n - f\|_{L^2(\mu)} \le \|q_n - f\|_K \mu(K)^{1/2} \le Mr^n\mu(K)^{1/2}.$$

For simplicity we take $\mu(K)=1$. Then we have $\|p_n - p_{n-1}\|_{L^2(\mu)} \le Mr^n(1+1/r)$ which shows that $p_o + \sum_{n=1}^\infty (p_n - p_{n-1})$ converges to f in $L^2(\mu)$ and pointwise μ-a.e. to f on K. By the Bernstein-Markov property, for each $\epsilon < 1/r - 1$ there exists $\tilde{M} > 0$ with

$$\|p_n - p_{n-1}\|_K \le \tilde{M}(1+\epsilon)^n \|p_n - p_{n-1}\|_{L^2(\mu)} \le \tilde{M}[(1+\epsilon)r]^n M(1+1/r)$$

showing that $p_o + \sum_{n=1}^\infty (p_n - p_{n-1})$ converges uniformly to a continuous function g on K (holomorphic on the interior of K). Since f and g are continuous and $g=f$ μ-a.e. on K, $g=f$ on K. Then

$$\|f-p_n\|_K = \|\sum_{k=n+1}^{\infty}(p_k - p_{k-1})\|_K \le \tilde{M}[(1+\epsilon)r]^{n+1} M \frac{(1+1/r)}{[1-(1+\epsilon)r]}$$

showing that $\limsup_{n\to\infty}\|f-p_n\|_K^{1/n} \le (1+\epsilon)r$. Again, a similar result holds in the elliptic partial differential operator case if $K \subset \mathbb{R}^N$.

10 Kergin interpolation

A more promising type of interpolation procedure has been successfully applied to many approximation problems by Tom Bloom and his collaborators. A natural extension of Lagrange interpolation to $\mathbb{R}^s$, $s>1$ was discovered by P. Kergin (a student of Bloom) in his thesis. Indeed, Kergin interpolation acting on ridge functions (a univariate function composed with a linear form) is Lagrange interpolation. The Kergin interpolation polynomials generalize to the case of C^m functions in $\mathbb{R}^N$ both the Lagrange interpolation polynomials and those of Hermite.

As brief motivation, given $f \in C^m([0,1])$, say, and given $m+1$ points $t_0 < \cdots < t_m \in [0,1]$, if one constructs the Lagrange interpolating polynomial $L_m f$ for f at these points, then there exist (at least) $m-1$ points between pairs of successive t_j at which f' and $(L_m f)'$ agree; then there exist (at least) $m-2$ points between triples of successive t_j at which f'' and $(L_m f)''$ agree, etc. Given a set $A = [A_0, A_1, \ldots, A_m] \subset \mathbb{R}^N$ of $m+1$ points and f a function of class C^m on a neighborhood of the convex hull of these points, there exists a unique polynomial $\mathcal{K}_A(f) = \mathcal{K}_A(f)(x_1,\ldots,x_N)$ of total degree m such that $\mathcal{K}_A(f)(A_j) = f(A_j)$, $j=0,1,\ldots,m$, and such that for every integer r, $0 \le r \le m-1$, every subset J of $\{0,1,\ldots,m\}$ with cardinality equal to $r+1$, and every homogeneous differential operator Q of order r with constant coefficients, there exists ξ belonging to the convex hull of the (A_j), $j \in J$, such that $Qf(\xi) = Q\mathcal{K}_A(f)(\xi)$. In [Bl1], Bloom gives a proof of this result by using a formula due to Micchelli and Milman [MM] which gives an explicit expression for $\mathcal{K}_A(f)$. If $f = u+iv$ is holomorphic in a convex region D in $\mathbb{C}^N$, and if $A = [A_0, A_1, \ldots, A_m] \subset D \subset \mathbb{C}^N = \mathbb{R}^{2N}$, then we can construct $\mathcal{K}_A(u)$ and $\mathcal{K}_A(v)$. It turns out (cf. [Bo2]) that $\mathcal{K}_A(u) + i\mathcal{K}_A(v)$ is a holomorphic polynomial.

An alternate description, which we give in the holomorphic setting, is as follows (cf. [BC2]). Let D be a $\mathbb{C}$-convex domain in $\mathbb{C}^N$, i.e., the intersection of D with any complex line is connected and simply connected. Note that in $\mathbb{R}^N$ this is the same condition as convexity if we replace "complex line" by "real line." For any set $\mathcal{A} = [A_0, \ldots, A_d]$ of (not necessarily distinct) $d+1$ points in D there exists a unique linear projector $\mathcal{K}_{\mathcal{A}} : \mathcal{O}(D) \to \mathcal{P}_d$ (recall that $\mathcal{O}(D)$ is the space of holomorphic functions on D and $\mathcal{P}_d$ is the space of polynomials of N complex variables of degree less than or equal to d) such that

(i) $\mathcal{K}_{\mathcal{A}}(f)(A_j) = f(A_j)$ for $j = 0, \cdots, d$,
(ii) $\mathcal{K}_{\mathcal{A}}(g \circ \lambda) = \mathcal{K}_{\lambda(\mathcal{A})}(g) \circ \lambda$ for every affine map $\lambda : \mathbb{C}^N \to \mathbb{C}$ and $g \in \mathcal{O}(\lambda(D))$, where $\lambda(\mathcal{A}) = (\lambda(A_0), \ldots, \lambda(A_d))$,
(iii) $\mathcal{K}_{\mathcal{A}}$ is independent of the ordering of the points in $\mathcal{A}$, and
(iv) $\mathcal{K}_{\mathcal{B}} \circ \mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{B}}$ for every subsequence $\mathcal{B}$ of $\mathcal{A}$.

The operator $\mathcal{K}_{\mathcal{A}}$ is called the Kergin interpolating operator with respect to $\mathcal{A}$.

Set $\mathcal{K}_d := \mathcal{K}_{\mathcal{A}_d}$ with $\mathcal{A}_d = [A_{d0}, \ldots, A_{dd}]$ and A_{dj} in a compact subset K of $D \subset \mathbb{C}^N$ for every $j = 0, \ldots, d$ and $d = 1, 2, 3, \ldots$. *Under what conditions on the array $\{\mathcal{A}_d\}_{d=1,2,\ldots}$ is it true that $\mathcal{K}_d(f)$ converges to f uniformly on K as $d \to \infty$ for every function f holomorphic in some neighborhood of $\bar{D}$?* Bloom and Calvi [BC2] attacked this problem with the aid of an integral representation formula for the remainder $f - \mathcal{K}_d(f)$ proved by M. Andersson and M. Passare [AP]. Their solution reads as follows. Assume that the measures $\mu_d = (d+1)^{-1} \sum_{j=0}^{d} \delta_{A_{dj}}$ converge weak-* as $d \to \infty$ to a measure μ. In one variable, the answer comes from potential theory: one considers the logarithmic potential

$$V_\mu(u) := \int_K \log|u-t| d\mu(t)$$

and the required condition is that

$$\{u \in \mathbb{C} : V_\mu(u) \le \sup_K V_\mu\} \subset D.$$

For $N > 1$, given a linear form $p : \mathbb{C}^N \to \mathbb{C}$, define $\mu^p = p_* \mu$ as the push-forward of μ to $\mathbb{C}$ via p, i.e., for $f \in C_0(\mathbb{C})$,

$$\mu^p(f) := \int_{\mathbb{C}} f d\mu^p = \mu(f \circ p) := \int_{\mathbb{C}^N} (f \circ p) d\mu.$$

Set

$$\Psi_\mu(p, u) := \mu^p(\log|u - \cdot|) = \int_{\mathbb{C}} \log|u - \zeta| d\mu^p(\zeta),$$

and let $M_\mu(p)$ be the maximum of $u \mapsto \Psi_\mu(p, u)$ on $p(K)$. *If D has C^2 boundary and $\{u \in \mathbb{C} : \Psi_\mu(p, u) \le M_\mu(p)\} \subset p(D)$ for every linear form p on $\mathbb{C}^N$, then $\mathcal{K}_d(f)$ converges to f uniformly on K as $d \to \infty$ for every function f holomorphic in some neighborhood of $\bar{D}$.*

We call an array $\{\mathcal{A}_d\}_{d=1,2,\ldots}$ **extremal** for K if $\mathcal{K}_d(f)$ converges to f uniformly on K for **each** f holomorphic in a neighborhood of K. In the setting of subsets K of $\mathbb{R}^N$, Bloom and Calvi proved the following striking result.

Theorem ([BC3]). *Let $K \subset \mathbb{R}^N$, $N \ge 2$, be a compact, convex set with nonempty interior. Then K admits extremal arrays if and only if $N = 2$ and K is the region bounded by an ellipse.*

For the Andersson-Passare remainder formula one needs an integral formula with

1. a *holomorphic* kernel; moreover, one with
2. a kernel that is the composition of a univariate function with an affine function.

Together with property (ii) of the Kergin interpolating operator, this allows a reduction of the multivariate problem to a univariate setting.

For $a, b \in \mathbb{C}^N$, we write $\langle a, b\rangle := \sum_{j=1}^N a_j b_j$. Let $D \subset \mathbb{C}^N$ be a bounded domain with smooth boundary and fix N functions $w_j(\zeta),\ j = 1, \ldots, N$ which are defined and smooth on ∂D and satisfy

$$\langle w(\zeta), \zeta - z\rangle = \sum_{j=1}^N w_j(\zeta)(\zeta_j - z_j) \neq 0 \tag{28}$$

for all $z \in D$ and $\zeta \in \partial D$. We give examples of such w_j below. Define

$$\Omega(s,t) := \frac{(N-1)!}{(2\pi i)^N} \sum_{j=1}^N \frac{(-1)^{j-1} t_j}{\langle s, t\rangle^N} dt[j] \wedge ds.$$

Here $dt[j] = dt_1 \wedge \cdots \wedge \hat{dt_j} \wedge \cdots \wedge dt_N$ (omit dt_j) and $ds = ds_1 \wedge \cdots \wedge ds_N$. Note that for fixed z, $\Omega(\zeta - z, \bar\zeta - \bar z)$ is simply the Bochner-Martinelli kernel

$$\omega_{BM}(\zeta - z) := \frac{(N-1)!}{(2\pi i)^N} \sum_{j=1}^N \frac{(-1)^{j-1}(\bar\zeta_j - \bar z_j)}{|\zeta - z|^{2N}} d\bar\zeta[j] \wedge d\zeta$$

which is used in the Bochner-Martinelli formula (24) from section 7. If $f \in \mathcal{O}(D) \cap C(\bar D)$, we have the following generalization of (24):

$$f(z) = \int_{\partial D} f(\zeta)\Omega(\zeta - z, w(\zeta))$$

$$= \frac{(N-1)!}{(2\pi i)^N} \int_{\partial D} \frac{f(\zeta)}{[\sum_{j=1}^N w_j(\zeta) \cdot (\zeta_j - z_j)]^N} \cdot \sum_{j=1}^N (-1)^{j-1} w_j(\zeta) dw[j] \wedge d\zeta \tag{29}$$

for $z \in D$. Note here $dw[j] = dw_1 \wedge \cdots \wedge \hat{dw_j} \wedge \cdots \wedge dw_N$; thus it is the $(0,1)-$piece of each $1-$form $dw_j = dw_j(\zeta)$ that is important. This is known as a **Cauchy-Fantappiè-Leray (CFL) formula**. Weinstock's proof of Theorem (PSW) in section 8 hinged on a judicious choice of the w_j's. The Henkin, Kerzman and Lieb results mentioned in section 7 also utilize CFL-type kernels.

Let $D = \{\zeta \in \mathbb{C}^N : \rho(\zeta) < 0\}$ where $\rho \in C^1(\bar D)$ with $d\rho \neq 0$ on ∂D. Suppose that at each point $\zeta \in \partial D$ the complex tangent plane $T_p^{\mathbb{C}}(\partial D)$ lies outside of D, i.e.,

$$\sum_{j=1}^N \frac{\partial\rho}{\partial\zeta_j}(\zeta) \cdot (\zeta_j - z_j) \neq 0$$

for $\zeta \in \partial D$ and $z \in D$. Such a domain is called *lineally convex*; convex domains are special examples. In the smoothly bounded category, lineally convex domains are the same as $\mathbb{C}-$convex domains (cf. [APS], Chapter 2). The functions

$$w_j(\zeta) = \frac{\partial\rho}{\partial\zeta_j}(\zeta)$$

satisfy (28) and we obtain the following special case of the CFL formula (29):

$$f(z) = \frac{(N-1)!}{(2\pi i)^N} \int_{\partial D} \frac{f(\zeta)}{[\sum_{j=1}^N \frac{\partial \rho}{\partial \zeta_j}(\zeta) \cdot (\zeta_j - z_j)]^N} \cdot \sum_{j=1}^N \frac{\partial \rho}{\partial \zeta_j}(\zeta) d\bar{\zeta}[j] \wedge d\zeta. \tag{30}$$

(see [Sha], chapter III for details). For example, if D is the unit ball and $\rho(\zeta) = \sum_{j=1}^N \zeta_j \bar{\zeta}_j - 1$, we have $w_j(\zeta) = \bar{\zeta}$ and we get

$$f(z) = \frac{(N-1)!}{(2\pi i)^N} \int_{\partial D} \frac{f(\zeta)}{[1 - \sum_{j=1}^N \bar{\zeta}_j z_j]^N} \cdot \sum_{j=1}^N \bar{\zeta}_j d\bar{\zeta}[j] \wedge d\zeta.$$

Let's write $\rho'(\zeta) := (\frac{\partial \rho}{\partial \zeta_1}, \ldots, \frac{\partial \rho}{\partial \zeta_N})$. The Andersson-Passare remainder formula reads as follows.

Theorem ([AP]). *Let $D = \{z \in \mathbb{C}^N : \rho(z) < 0\}$ be a $\mathbb{C}-$convex domain with C^2-boundary and let $f \in \mathcal{O}(D) \cap C(\bar{D})$. Let $p_0, \ldots, p_d$ be $d+1$ points in D. Then*

$$(f - \mathcal{K}_d f)(z) = \frac{1}{(2\pi i)^N} \int_{\partial D} \sum_{|\alpha|+\beta = N-1} (\prod_{j=0}^d \frac{\langle \rho'(\zeta), z - p_j \rangle}{\langle \rho'(\zeta), \zeta - p_j \rangle})$$

$$\times \frac{f(\zeta) \partial \rho(\zeta) \wedge (\bar{\partial} \partial \rho(\zeta))^{N-1}}{[\prod_{j=0}^d \langle \rho'(\zeta), \zeta - p_j \rangle^{\alpha_j}] \langle \rho'(\zeta), \zeta - z \rangle^{\beta+1}} \tag{31}$$

for $z \in D$ where $\alpha = (\alpha_0, \ldots, \alpha_d)$ is a multiindex and β is a nonnegative integer.

Here, $\bar{\partial}\partial\rho$ is a $(1,1)$-form and

$$(\bar{\partial}\partial\rho)^{N-1} = \bar{\partial}\partial\rho \wedge \cdots \wedge \bar{\partial}\partial\rho \ (N-1 \text{ times}).$$

Thus $\partial\rho(\zeta) \wedge (\bar{\partial}\partial\rho(\zeta))^{N-1}$ is an $(N, N-1)$-form. Equation (31) follows from (30) in a manner analogous to that of obtaining the Hermite Remainder Formula (26) from the Cauchy Integral Formula and an explicit formula for the remainder between the Cauchy kernel and its Lagrange interpolant, formula (27). One explicitly computes the Kergin interpolant of

$$\left[\sum_{j=1}^N \frac{\partial \rho}{\partial \zeta_j}(\zeta) \cdot (\zeta_j - z_j)^N \right]^{-1},$$

the portion of the CFL kernel depending on the z-variables, using the fact that this is the composition of a univariate function with an affine function on $\mathbb{C}^N$.

Bloom and Calvi [BC1] also considered what happens to multivariate Lagrange (or Hermite) interpolants $L_d f$ to a given function f of some minimal smoothness fixing the degree d and letting the interpolation points coalesce. They give both a geometric condition and an algebraic condition sufficient for the interpolants to converge to the Taylor polynomial of the function at the point of coalescence. The proof makes an interesting use of Kergin interpolation.

There has been lots of work done in the holomorphic category; for results on Kergin interpolation of entire functions, see [Bl2] and [Bl4]. We finish this section with an interesting "real" result.

The nature and definition of Kergin interpolants requires C^n smoothness of a real-valued function f in order to construct an interpolant to f associated with $n+1$ points. Bos and Waldron [BW] have observed that for $n+1$ points in $\mathbb{R}^N$ in general position, the Kergin polynomial interpolant of C^n functions may be extended to an interpolant on all functions of class C^{N-1}. In particular, in $\mathbb{R}^2$ one can construct Kergin interpolants of all degrees, provided points are in general position, for any C^1 function. Using $(n+1)$-st "roots of unity" A_n on the unit circle in $\mathbb{R}^2$, Bos and Calvi [BC] proved that *for any $f \in C^2(U)$, where U is a neighborhood of the closed unit disk K in $\mathbb{R}^2$,* $\lim_{d\to\infty} \|f - \mathcal{K}_{A_n}(f)\|_K = 0$. This is a natural generalization of the analogous fact for C^1-functions on the interval using Lagrange interpolants at the Chebyshev nodes.

11 Rational approximation in $\mathbb{C}^N$

Suppose f is holomorphic in a neighborhood of the origin in $\mathbb{C}^N$. We say that a sequence $r_1, r_2, \ldots$ of rational functions (with the degree of r_k not greater than k) **rapidly approximates** f if the kth root of $|f - r_k|$ converges to zero in measure. Let R^0 be the class of all f that admit a rapid approximation near the origin. If $N = 1$ Sadullaev [Sa2] characterized the class R^0 in terms of Taylor coefficients.

Theorem (R^0). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be holomorphic in a neighborhood of the closed unit disk in $\mathbb{C}$. Define*

$$A_{j_1,\ldots,j_k} := |\det[a_{j_n+m}]_{n=1,\ldots,k;\ m=0,\ldots,k-1}|$$

and $V_k := \sup_{j_1,\ldots,j_k} A_{j_1,\ldots,j_k}$. Then $f \in R^0$ if and only if $\lim_{k\to\infty} V_k^{1/k^2} = 0$.

Sadullaev used this condition to show that *a holomorphic function f in a neighborhood of the origin in $\mathbb{C}^N$ for $N > 1$ is rapidly approximable if and only if its restriction to every complex line L through the origin is rapidly approximable.* The idea of the only if direction is very simple and utilizes Hartogs series, which we used back in section 3. Via a preliminary complex-linear transformation, we may assume $f \in R^0(\mathbb{C}^N)$ is holomorphic in a neighborhood of the unit polydisk and that $L = \{(z', z_N) := (z_1, \ldots, z_{N-1}, z_N) : z' = 0\}$. We want to show that $g(z_N) := f(0, \ldots, 0, z_N)$ is in $R^0(L)$. We expand f in a Hartogs series

$$f(z) = \sum_{j=0}^{\infty} a_j(z') z_N^j.$$

We get a sequence of functions $V_k(z')$ defined in the closed polydisk

$$\bar{U}' := \{z' := (z_1, \ldots, z_{N-1}) : |z_j| \leq 1\}$$

in $\mathbb{C}^{N-1}$. Since each V_k is the supremum of the moduli of holomorphic functions in U', each function

$$u_k(z') := \frac{1}{k^2} \log V_k(z')$$

is psh in U'. Since f is holomorphic in a neighborhood of the unit polydisk in $\mathbb{C}^N$ the coefficients a_j are uniformly bounded on $\bar{U}'$; i.e., $|a_j(z')| \leq C$ for $z \in \bar{U}'$ for each $j = 0, 1, \ldots$. Hence the sequence $\{u_k\}$ of psh functions is uniformly bounded above on $\bar{U}'$.

The key step is to show that

$$\lim_{k\to\infty} \int_{U'} u_k(z') dA(z') = -\infty.$$

To be brief, this is achieved using the fact that $f \in R^0(\mathbb{C}^N)$ together with estimates on the size of certain sets involving a notion of a Chebyshev constant $T(K)$ associated to a compact set K. This Chebyshev constant will be defined in section 13. Since u_k is psh in U', it is $\mathbb{R}^{2N-2}$-subharmonic; by the subaveraging property and the fact that $\{u_k\}$ is uniformly bounded above on $\bar{U}'$,

$$u_k(0) \leq \frac{1}{A(U')} \int_{U'} u_k(z')dA(z') \to -\infty \text{ as } k \to \infty.$$

This shows that $\lim_{k\to\infty} V_k(0)^{1/k^2} = 0$; thus by Theorem (R^0), $g(z_N) = f(0, \ldots, 0, z_N) \in R^0(L)$.

Gonchar [G1] showed that if f is rapidly approximable then the maximal region to which f continues analytically is single-sheeted and the rapid approximation persists in this region. Taking this a step further, Sadullaev [Sa2] showed that **every** *holomorphic function on a domain D is rapidly approximable if and only if the complement of the envelope of holomorphy of D is a pluripolar set.* In this setting, the envelope of holomorphy $\tilde{D}$ of D is the smallest domain of holomorphy containing D. In particular, all $g \in \mathcal{O}(D)$ extend holomorphically to $\tilde{D}$.

We remark that Bloom proved that rapid convergence in measure of a sequence $\{r_n\}$ of rational functions to a holomorphic function f on an open set $\Omega \subset \mathbb{C}^N$ implies rapid convergence in relative capacity (this will also be defined in section 13) on the natural domain of definition of f. This has the consequence that for a meromorphic function f on $\mathbb{C}^N$ which is holomorphic on a neighborhood of the origin the Gonchar-Padé approximants $\{\pi_n(z, f, \lambda)\}$ converge rapidly in capacity to f. We refer the reader to [Bl7] for definitions and details.

In [C1], Chirka proved a "meromorphic" version of the Bernstein-Walsh theorem. We first describe the one-variable result. For an open set $D \subset \mathbb{C}$ and a nonnegative integer m, let $\mathcal{M}_m(D)$ denote the class of meromorphic functions in D which have at most m poles (counted with multiplicities). Recall for a compact set K in $\mathbb{C}^N$, $N \geq 1$, and $R > 1$, we write $D_R := \{z \in \mathbb{C}^N : V_K(z) < \log R\}$. For $f \in C(K)$ and nonnegative integers m and n, let

$$r_{m,n} = r_{m,n}(f, K) := \inf\{\|f - p/q\|_K : p \in \mathcal{P}_n,\ q \in \mathcal{P}_m\}.$$

The following result is due to Gonchar [G2]; a special case was proved earlier by Saff [S].

Theorem (RBW1). *Let $K \subset \mathbb{C}$ be a regular compact set and let $R > 1$. Given a continuous function $f : K \to \mathbb{C}$ and a fixed integer $m \geq 0$, the following conditions are equivalent:*
(i) $\limsup_{n\to\infty}(r_{m,n})^{1/n} \leq 1/R$;
(ii) there exists a function $F \in \mathcal{M}_m(D_R)$ with $F|_K = f$.

If $N > 1$, the definitions of $\mathcal{M}_m(D)$ and the approximation numbers $r_{m,n}(f, K)$ need to be modified. We define $\mathcal{M}_m(D)$ to be the class of all functions in D of the form h/q_m where $h \in \mathcal{O}(D)$ and $q_m \in \mathcal{P}_m$.

Theorem (RBWN). *Let $K \subset \mathbb{C}^N$ be compact and $L-$regular and let $R > 1$. Given a continuous function $f : K \to \mathbb{C}$ and a fixed integer $m \geq 0$, for $n \geq 1$ let*

$$r^*_{m,n} = r^*_{m,n}(f, K) := \inf\{\|qf - p\|_K : p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,\ \|q\|_K = 1\}.$$

The following conditions are equivalent:
(i) $\limsup_{n\to\infty}(r^*_{m,n})^{1/n} \leq 1/R$;

(ii) there exists a function $F \in \mathcal{M}_m(D_R)$ with $F|_K = f$.

The proof of (ii) implies (i) is immediate from the standard Bernstein-Walsh theorem, Theorem (BWN) in section 5. Let $f = h/g_m \in \mathcal{M}_m(D_R)$. Since $h = fg_m \in \mathcal{O}_m(D_R)$, by Theorem (BWN) there exists a sequence $\{p_n\}$ of polynomials, $p_n \in \mathcal{P}_n$, with

$$\limsup_{n\to\infty} \|fg_m - p_n\|_K^{1/n} \le 1/R$$

which gives (i). The other implication is much deeper. Much in the spirit of Sadullaev's proof of Theorem (R^0), Chirka needs SCV-type capacity estimates as well as univariate arguments and techniques to achieve his goal.

Chirka constructs some interesting examples in [C1] to explain the difference between Theorems (RBW1) and (RBWN). The first example utilizes the Hartogs triangle from section 7. Precisely, let

$$K := \bar{D} = \{(z,w) \in \mathbb{C}^2 : |z| \le |w| \le 1\}$$

be the closure of the Hartogs triangle $D = \{(z,w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$. The polynomial hull $\hat{K}$ is the closed unit bidisk:

$$\hat{K} = \bar{\Delta} \times \bar{\Delta} = \{(z,w) \in \mathbb{C}^2 : |z| \le 1,\ |w| \le 1\}$$

since $K \subset \bar{\Delta} \times \bar{\Delta}$; K contains the torus $T^2 = \partial\Delta \times \partial\Delta$; and the polynomial hull of the torus T^2 is clearly the closed bidisk $\bar{\Delta} \times \bar{\Delta}$. Thus the $L-$extremal function V_K coincides with that of the bidisk:

$$V_K(z,w) = \max[\log^+|z|, \log^+|w|]$$

so that the sublevel sets D_R are larger bidisks. In particular, the set K is $L-$regular so that we may apply Theorem (RBWN) to the function $f(z,w) := z^2/w$ (recall that $f \in \mathcal{O}(D) \cap C(K)$). By its very definition, $f \in \mathcal{M}_1(D_R)$ for all $R > 1$ so that

$$\limsup_{n\to\infty} (r_{1,n}^*)^{1/n} \le 1/R$$

for all $R > 1$ and hence $(r_{1,n}^*)^{1/n} \to 0$. However, we cannot even uniformly approximate f on K by a rational function p/q with $q \ne 0$ on K, for $p/q \in \mathcal{O}(\bar{D})$ and, as we saw in section 7, f is not uniformly approximable by functions in $\mathcal{O}(\bar{D})$. Thus for each m the sequence $\{r_{m,n}\}$ does not tend to zero.

In this example, the set K is not polynomially convex. However, Chirka constructs another example in which the set K is a small ball $\{z \in \mathbb{C}^N : |z| \le \delta\}$ (and hence $\hat{K} = K$). He constructs a function $f = h/g_m \in \mathcal{M}_m(D_R)$ for certain $m > 1$ and $R > 1$ with the property that there does not exist a sequence $\{p_n/q_m\}$ with $p_n \in \mathcal{P}_n$ and $q_m \in \mathcal{P}_m$ so that

$$\limsup_{n\to\infty} \|f - p_n/q_m\|_K^{1/n} \le 1/R.$$

The problem is that the "pole-set" of f in this example cannot be written in the form $\{z \in D_R : q(z) = 0\}$ for a polynomial q (in this sense, the "pole-set" is "nonalgebraic"). On the other hand, since $h = fg_m \in \mathcal{O}_m(D_R)$, we do have $\limsup_{n\to\infty}(r_{m,n}^*)^{1/n} \le 1/R$.

In [C2] Chirka utilized Jacobi series to prove holomorphic extension results. As a sample, let $g = p/q$ be a rational function in $\mathbb{C}$. Let G_r be a connected component of the set $\{z : |g(z)| \leq r\}$ (G_r is a **rational lemniscate**). If f is holomorphic in a neighborhood of $\bar{G}_r$ then

$$F(z,w) := \frac{1}{2\pi i}\int_{\partial G_r} \frac{f(\zeta)}{g(\zeta)-w} \cdot \frac{g(\zeta)-g(z)}{\zeta - z} d\zeta$$

is a holomorphic function on $G_r \times \{|w| < r\}$ which satisfies $F(z, g(z)) = f(z)$ (from the Cauchy integral formula). Thus we can expand F in a Taylor series in w and set $w = g(z)$ to obtain a **Jacobi series** for f:

$$f(z) := \sum_{k=0}^{\infty} C_k(z)[g(z)]^k$$

where

$$C_k(z) = \frac{1}{2\pi i}\int_{\partial G_r} f(\zeta) \cdot \frac{g(\zeta)-g(z)}{[g(\zeta)]^{k+1}(\zeta - z)} d\zeta$$

are rational functions with poles at the poles of $g(z)$. The following result is proved in [C2].

Theorem. *Let f be holomorphic in the polydisk $U' \times \{|z_N| < r\}$ where U' is a polydisk in $\mathbb{C}^{N-1}$. Suppose for each fixed point $p \in E \subset U'$, where E is nonpluripolar in $\mathbb{C}^{N-1}$, $f(p,\cdot)$ extends to a function holomorphic in $\mathbb{C} = \mathbb{C}_{z_N}$ except perhaps for a finite number of singularities. Then f extends holomorphically to $(U' \times \mathbb{C}) \setminus A$ where A is an analytic variety.*

Far-reaching generalizations of these "extension" results exist throughout the literature. For a start, consult [Iv].

12 Markov inequalities

The classical Bernstein-Markov inequalities say that for $p : \mathbb{R} \to \mathbb{R}$ a real polynomial such that $\|p\|_{[-1,1]} = \sup_{x\in[-1,1]} |p(x)| \leq 1$,

$$\left|\frac{p'(x)}{\sqrt{1-p^2(x)}}\right| \leq (\deg p)\frac{1}{\sqrt{1-x^2}}, \ x \in (-1,1);$$

and, for a uniform estimate,

$$\|p'\|_{[-1,1]} \leq (\deg p)^2 \|p\|_{[-1,1]}.$$

Equivalently, for a trigonometric polynomial $t = t(\theta)$ on the unit circle T,

$$\sup_\theta |t'(\theta)| \leq (\deg t) \sup_\theta |t(\theta)|. \tag{32}$$

These estimates are useful in inverse theorems in univariate approximation theory. More generally, let K be a compact set in $\mathbb{C}^N$. We say that K **satisfies a Markov inequality with exponent** r if there exist constants $r \geq 1$ and $M > 0$ depending only on K such that

$$\|\frac{\partial p}{\partial z_j}\|_K \leq M(\deg p)^r \|p\|_K, \quad j = 1,\ldots,N$$

for all polynomials p. Convex sets in $\mathbb{R}^N \subset \mathbb{C}^N$ satisfy a Markov inequality with $r = 2$ (cf. [BP]) while a closed Euclidean ball in $\mathbb{C}^N$ satisfies a Markov inequality with exponent $r = 1$. This last statement follows from the fact that (recall section 5) $V_K(z) = \max[0, \log|z-a|/R]$ if $K = \{z : |z-a| \leq R\}$ so that $V_K = V_K^*$ is Lipschitz, together with the following observation:

Proposition. *Let $K \subset \mathbb{C}^N$ satisfy a Hölder continuity property (HCP):*

$$\exp[V_K(z)] \leq 1 + M\delta^m \text{ if } dist(z, K) \leq \delta \leq 1$$

where $m, M > 0$ are independent of $\delta > 0$. Then K satisfies a Markov inequality with exponent $r = 1/m$.

Proof. Fix a polynomial p of degree n, say, and let $\|\frac{\partial p}{\partial z_j}\|_K = |\frac{\partial p}{\partial z_j}(a)|$. Applying Cauchy's inequalities on a polydisk P centered at $a \in K$ of (poly-)radius $R > 0$ (recall the Cauchy integral formula (3)), we have

$$\|\frac{\partial p}{\partial z_j}\|_K \leq \|p\|_P / R.$$

From the Bernstein-Walsh inequality (19) we have

$$\|p\|_P \leq \|p\|_K \exp[n \sup_P V_K] \leq \|p\|_K (1 + MR^m)^n.$$

Thus

$$\|\frac{\partial p}{\partial z_j}\|_K \leq \|p\|_K \frac{(1 + MR^m)^n}{R}.$$

Choosing $R = 1/n^{1/m}$ gives the result. □

To this date, *there are no known examples of compact sets in $\mathbb{C}^N$ which satisfy a Markov inequality but which do not satisfy (HCP).*

One of the most beautiful applications of multivariate Markov inequalities is due to Plesniak [Pl1]. Recall that a C^∞ function on a compact set $E \subset \mathbb{R}^N$ is a function $f : E \to \mathbb{R}$ such that there exists $\tilde{f} \in C^\infty(\mathbb{R}^N)$ with $\tilde{f}|_E = f$. We write $f \in C^\infty(E)$. We say that E is C^∞**-determining** if $g \in C^\infty(\mathbb{R}^N)$ with $g|_E = 0$ implies $D^\alpha g|_E = 0$ for all multiindices α. Plesniak [Pl1] has shown the following.

Theorem ([Pl1]). *Let $E \subset \mathbb{R}^N$ be C^∞-determining. Then E satisfies a Markov inequality if and only if there is a continuous linear extension operator*

$$L : (C^\infty(E), \tau_1) \to (C^\infty(\mathbb{R}^N), \tau_0)$$

such that $L(f)|_E = f$ for each $f \in C^\infty(E)$.

Here τ_0 is the standard Fréchet space topology on $C^\infty(\mathbb{R}^N)$ generated by the seminorms $\{\|\tilde{f}\|_K^d := \max_{|\alpha| \leq d} \|D^\alpha \tilde{f}\|_K\}$ where K ranges over compact subsets of $\mathbb{R}^N$ and $d = 0, 1, \ldots$; and τ_1 is the quotient topology on $C^\infty(\mathbb{R}^N)/I(E)$ where $I(E) := \{f \in C^\infty(\mathbb{R}^N) : f|_E = 0\}$. This is proved in [Pl1] using Lagrange interpolation operators corresponding to Fekete points (see section 9): the operator L is of the form

$$L(f) := u_1 L_1(f) + \sum_{d=1}^{\infty} u_d (L_{d+1}(f) - L_d(f))$$

where u_d are standard cut-off functions and $L_d(f)$ is the Lagrange interpolating polynomial of f at a set of d-Fekete points of E.

There is an extensive literature on Markov inequalities. Baran and Plesniak and their students have produced many of the results related to multivariate approximation; cf. [Ba1], [Ba2], [Ba3], [Ba4]; and see [Pl2] for a nice survey article. Markov inequalities have been used to construct natural pseudodistances on compact subsets of $\mathbb{R}^N$ (see [BLW]). As a final application, observe that (32) can be interpreted in the following manner: setting $x = \cos\theta$ and $y = \sin\theta$, for any bivariate polynomial $p(x, y)$ on $\mathbb{R}^2$, the unit tangential derivative $D_\tau p(x, y)$ on the unit circle $T \subset \mathbb{R}^2$ satisfies

$$|D_\tau p(x, y)|_T \leq (\deg p)\|p\|_T, \quad (x, y) \in T.$$

Of course no estimate on normal derivatives is possible as there exist nonzero polynomials (e.g., $p(x, y) = x^2 + y^2 - 1$) which vanish on T. Note that T is an algebraic submanifold of $\mathbb{R}^2$. Using a deep result of Sadullaev [Sa3] on the L-extremal function of compact subsets of algebraic sets in $\mathbb{C}^N$, the following characterization of algebraicity is known:

Theorem ([BLMT]). *Let K be a smooth, m-dimensional submanifold of $\mathbb{R}^N$ without boundary where $1 \leq m \leq N-1$. Then K is algebraic if and only if K satisfies a tangential Markov inequality with exponent one: there exists a positive constant M depending only on K such that for all polynomials p and all unit tangential derivatives D_τ,*

$$|D_\tau p(x_1, \ldots, x_N)| \leq M(\deg p)\|p\|_K, \quad (x_1, \ldots, x_N) \in K.$$

Note that the finite-dimensionality of the vector space of polynomials of degree at most n implies that there is a constant C_n depending on n and K with

$$|D_\tau p(x_1, \ldots, x_N)| \leq C_n\|p\|_K, \quad (x_1, \ldots, x_N) \in K$$

for all such polynomials p. The content of the above theorem is that one can take $C_n = Mn$ where M depends only on K. Indeed, a stronger version of the "if" implication is known: *if K satisfies a tangential Markov inequality*

$$|D_\tau p(x_1, \ldots, x_N)| \leq M(\deg p)^r\|p\|_K, \quad (x_1, \ldots, x_N) \in K$$

with exponent $r < (m + 1)/m$, then K is algebraic. We refer the reader to [BLMT] for details.

13 Appendix on pluripolar sets and extremal psh functions

In CCV, polar sets play an essential role. A subset $E \subset \mathbb{C}$ is **polar** if there exists a *subharmonic* function u defined in a neighborhood of E with $E \subset \{z : u(z) = -\infty\}$; whereas a subset $E \subset \mathbb{C}^N$ is **pluripolar** if there exists a *plurisubharmonic* function u defined in a neighborhood of E with $E \subset \{z : u(z) = -\infty\}$. The neighborhood may be taken to be all of $\mathbb{C}^N$. Apriori, there is a local notion in each case: E is **locally (pluri-)polar** if for each point $z \in E$ there exists an open neighborhood U of z and a (pluri-)subharmonic function u in U such that

$$E \cap U \subset \{z \in U : u(z) = -\infty\}.$$

It is easy if $N = 1$ and much harder if $N > 1$ to verify that the local notions are equivalent to the global ones. For $N > 1$ this was first proved by Josefson [J]. We remark that since the notion of psh function makes sense on a complex manifold M (see section 3), the notion of a locally pluripolar set in M can be defined.

1. *Nonpluripolar sets can be small*: Take a non-polar Cantor set $E \subset \mathbb{R} \subset \mathbb{C}$ of Hausdorff dimension 0 (for the idea behind the construction of such sets, see [Ra] section 5.3). Then $E\times\cdots\times E$ is nonpluripolar in $\mathbb{C}^N$ (in general, $E_1\times\cdots\times E_j \subset \mathbb{C}^{m_1}\times\cdots\times\mathbb{C}^{m_j}$ is nonpluripolar in $\mathbb{C}^{m_1+\cdots+m_j}$ if and only if $E_k \subset \mathbb{C}^{m_k}$ is nonpluripolar in $\mathbb{C}^{m_k}$ for $k = 1,\ldots,j$) and has Hausdorff dimension 0.
2. *Pluripolar sets can be big*: A complex hypersurface $S = \{z : f(z) = 0\}$ associated to a holomorphic function f is a pluripolar set (take $u = \log|f|$) which has Hausdorff dimension $2N-2$. Recall that a psh function is, in particular, subharmonic in the $\mathbb{R}^{2N}$ sense; hence a pluripolar set is Newtonian polar and thus the Hausdorff dimension of a pluripolar set cannot exceed $2N-2$ (cf. [Ca], section IV).
3. *Size doesn't matter*: In $\mathbb{C}^2$, the totally real plane $\mathbb{R}^2 = \{(z_1,z_2) : \Im z_1 = \Im z_2 = 0\}$ is non-pluripolar (why?) but the *complex* plane $\mathbb{C} = \{(z_1,0) : z_1 \in \mathbf{C}\}$ is pluripolar (take $u = \log|z_1|$). Also, there exist C^∞ arcs in $\mathbb{C}^N$ which are not pluripolar (cf. [DF]); while such a *real-analytic* arc must be pluripolar (why?).

One can easily construct examples of nonpluripolar sets $E \subset \mathbb{C}^N$ which intersect every affine complex line in finitely many points (hence these intersections are polar in these lines). Indeed, take

$$E := \{(z_1,z_2) \in \mathbb{C}^2 : \Im(z_1+z_2^2) = \Re(z_1+z_2+z_2^2) = 0\}.$$

Then for any complex line $L := \{(z_1,z_2) : a_1z_1 + a_2z_2 = b\}$, $a_1,a_2,b \in \mathbb{C}$, $E\cap L$ is the intersection of two real quadrics and hence consists of at most four points. However, E is a totally real, two-(real)-dimensional submanifold of $\mathbb{C}^2$ and hence – as is the case with $\mathbb{R}^2 = \mathbb{R}^2 + i0 \subset \mathbb{C}^2$ in 3. – is not pluripolar. Thus pluripolarity cannot be detected by "slicing" with complex lines. In this example, E intersects the one-(complex)-dimensional analytic variety $A := \{(z_1,z_2) : z_1 + z_2^2 = 0\}$ in a nonpolar set. Nevertheless, one can construct a nonpluripolar set E in $\mathbb{C}^N$, $N > 1$, which intersects every one-dimensional complex analytic subvariety in a polar set [CLP].

In certain instances, however, slicing can detect pluripolarity. We define a set $E \subset \mathbb{C}^N$ to be **pseudoconcave** if for each point $p \in E$ there is a neighborhood U of p such that $U\setminus E$ is open and pseudoconvex in $\mathbb{C}^N$ (see section 3). Canonical examples are zero sets of a holomorphic function, or, more generally, zero sets of multiple-valued holomorphic functions; e.g., $\{(z_1,z_2) \in \mathbb{C}^2 : z_2^2 = z_1\}$ is pseudoconcave. This notion is related to the work in section 11 with the class R^0. Sadullaev [Sa1] has shown the remarkable result that if E is a closed, pseudoconcave set in $\mathbb{C}^N\setminus\{0\}$, then E is pluripolar if and only if $E\cap L$ is polar in L for each complex line L passing through 0. Moreover, for this class of pseudoconcave sets, pluripolarity is equivalent to $\mathbb{R}^{2N}$-(Newtonian) polarity!

There are several distinct notions of capacities in SCV. Given a compact set $K \subset \mathbb{C}$, we recall the definition of the extremal psh function $V_K^*(z)$, the usc regularization of

$$V_K(z) := \max\left\{0, \sup_p\left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\}.$$

The **Siciak** or **Robin** capacity of K is the number

$$c(K) := \exp\bigl(-\limsup_{|z|\to\infty}[V_K^*(z) - \log|z|]\bigr).$$

Unlike in CCV, the limit (usually) does not exist. Indeed, the Robin function $\rho_K := \rho_{V_K^*}$ (see section 5) associated to V_K^* provides information on the asymptotic behavior of this function on complex lines through the origin.

The quantity $c(K)$ coincides with a Chebyshev constant $\tilde{T}(K) := \lim_{n\to\infty} \tilde{M}_n(K)^{1/n}$ where

$$\tilde{M}_n(K) := \inf\{\|p_n\|_K : p_n = \hat{p}_n + \text{ lower degree terms }, \ \|\hat{p}_n\|_{\bar{B}} \geq 1\}$$

([Sa4], section 10). Here $\bar{B}$ is the closed unit (Euclidean) ball. Other normalizations may be used to define other Chebyshev constants. For example, defining

$$M_n(K) := \inf\{\|p_n\|_K : \ \|p_n\|_{\bar{B}} \geq 1\},$$

the Chebyshev constant $T(K) := \lim_{n\to\infty} M_n(K)^{1/n}$ coincides with $\exp\big(-\sup_{z\in B} V_K^*(z)\big)$. Although the numbers $T(K)$ and $\tilde{T}(K)$ are, in general, different, they are comparable; i.e. for all compact sets K, $T(K)$ is bounded above and below by a constant multiple of $\tilde{T}(K)$. Note that the subsets of $\mathcal{P}_n$ used to define $M_n(K)$ and $\tilde{M}_n(K)$ are not multiplicative classes (as in the case of univariate monic polynomials); thus an (elementary) argument is needed to verify the existence of the limits $T(K)$ and $\tilde{T}(K)$.

Next we give the definition of the transfinite diameter $d(K)$. Details may be found in Zaharjuta's paper [Z1]. Let $e_1(z), \ldots, e_j(z), \ldots$ be a listing of the monomials $\{e_i(z) = z^{\alpha(i)} = z_1^{\alpha_1} \cdots z_N^{\alpha_N}\}$ in $\mathbb{C}^N$ indexed using a lexicographic ordering on the multiindices $\alpha(i) \in \mathbf{N}^N$, but with $\deg e_i = |\alpha(i)|$ nondecreasing. For $\zeta_1, \ldots, \zeta_n \in \mathbb{C}^N$, let

$$VDM(\zeta_1, \ldots, \zeta_n) = |\det[e_i(\zeta_j)]_{i,j=1,\ldots,n}|$$

and for a compact subset $K \subset \mathbb{C}^N$ let

$$V_n = V_n(K) := \max_{\zeta_1,\ldots,\zeta_n \in K} VDM(\zeta_1, \ldots, \zeta_n).$$

Define $h_d = \#\{i : \deg e_i \leq d\}$ and $l_d = \sum_{i=1}^{h_d} (\deg e_i)$. Then

$$d(K) := \limsup_{d\to\infty} V_{h_d}^{1/l_d} \tag{33}$$

is the **transfinite diameter** of K.

If $N = 1$, it is well-known and trivial that the sequence $\{V_{h_d}^{1/l_d}\}$ is monotone decreasing and hence has a limit; moreover, in this case, $d(K)$ coincides with the logarithmic capacity of K and the Chebyshev constant of K defined back in section 5. Zaharjuta [Z1] proved the highly nontrivial result that the limit in (33) exists in the case when $N > 1$. In this setting the numbers $c(K), T(K)$ and $d(K)$ are not generally equal; however it is the case that *for $K \subset \mathbb{C}^N$ compact, K is pluripolar if and only if $c(K) = T(K) = d(K) = 0$* (see [LT]).

There are several extremal psh functions in SCV. Recall the relative extremal function introduced at the end of section 6: for E a subset of D, define

$$\omega(z, E, D) := \sup\{u(z) : u \text{ psh in } D, \ u \leq 0 \text{ in } D, \ u|_E \leq -1\}.$$

The usc regularization $\omega^*(z, E, D)$ is called the **relative extremal function** of E relative to D.

Proposition (ω). *Either* $\omega^* \equiv 0$ *in* D *or else* ω^* *is a nonconstant psh function in* D. *We have* $\omega^* \equiv 0$ *if and only if* E *is pluripolar.*

Proof. If $\omega^*(z^0) = 0$ at some point $z^0 \in D$, then $\omega^* \equiv 0$ in D by the maximum principle. Hence we can find a sequence $z^j \to z^0,\ z^j \in D$, with $\omega(z^j, E, D) \to 0$. By subaveraging,

$$\omega(z^j, E, D) \leq \frac{1}{\mathrm{vol}(B(z^j, r))} \int_{B(z^j,r)} \omega(z, E, D) dA(z)$$

for r sufficiently small so that $B(z^j, r) \subset D$. We conclude that $\omega(z, E, D) = 0$ a.e. in a neighborhood of z^0. Fix a point z' with $\omega(z', E, D) = 0$ and take a sequence of psh functions u_j in D with $u_j \leq 0$ in D, $u_j|_E \leq -1$, and $u_j(z') \geq -1/2^j$. Then $u(z) := \sum u_j(z)$ is psh in D (the partial sums form a decreasing sequence of psh functions) with $u(z') \geq -1$ (so $u \not\equiv -\infty$) and $u|_E = -\infty$; thus E is pluripolar.

Conversely, if E is pluripolar, there exists u psh in D with $u|_E = -\infty$; since D is bounded we may assume $u \leq 0$ in D. Then $\epsilon u \leq \omega(z, E, D)$ in D for all $\epsilon > 0$ which implies that $\omega(z, E, D) = 0$ at all points $z \in D$ where $u(z) \neq -\infty$. Since pluripolar sets have measure zero (why?), $\omega(z, E, D) = 0$ a.e. in D and hence $\omega^*(z, E, D) \equiv 0$ in D. □

Using Proposition (ω), Bedford and Taylor [BT2] gave a simple proof of Josefson's result that locally pluripolar sets are globally pluripolar. Similar to this proposition, one can show that for a bounded set $E \subset \mathbb{C}^N$, $V_E^* \equiv +\infty$ if and only if E is pluripolar. As mentioned in section 1, a nice introduction to pluripotential theory is the book of Klimek [K]. There is also a developing theory of *weighted* pluripotential theory. We refer the reader to the book of Saff-Totik [SaT] for an introduction to one-variable weighted potential theory in $\mathbb{C}$. An introduction to the SCV setting can be found in the appendix of [SaT] written by Tom Bloom (see also [BlL3]).

We remark that the quantity

$$C(E, D) := \sup\{\int_E (dd^c u)^N : u \text{ psh in } D,\ 0 \leq u \leq 1 \text{ in } D\}$$

is called the **relative capacity** of E relative to D. The precise definition of this complex Monge-Ampère operator $(dd^c u)^N$ will be given in the next section. Alexander and Taylor [AT] proved the following comparison between the relative capacity $C(K, D)$ and the Chebyshev constant $T(K)$ for a compact set K:

Theorem ([AT]). *Let* K *be a compact subset of the unit ball* $B \subset \mathbb{C}^N$. *We have*

$$\exp\bigl[-A/C(K, B)\bigr] \leq T(K) \leq \exp\bigl[-(c_N/C(K, B))^{1/N}\bigr]$$

where the right-hand inequality holds for all $K \subset B$ *and the left-hand inequality holds for all* $K \subset B(0, r) := \{z : |z| < r\}$ *where* $r < 1$ *and* $A = A(r)$ *is a constant depending only on* r.

Here, c_N is a dimensional constant. As a corollary, Alexander and Taylor construct normalized polynomials that are small where a given holomorphic function is small; or, more generally, where a psh function is very negative:

Proposition ([AT]). *Let* u *be a negative psh function in the unit ball* B *with* $u(0) \geq -1$. *For* $r < 1$ *and* $A > 1$, *let* K *be a compact subset of*

$$\{z \in \mathbb{C}^N : |z| \leq r,\ u(z) < -A\}.$$

Then there exists a sequence $\{p_d\}$ of polynomials with $degp_d \leq d$ and $\|p_d\|_B = 1$ such that

$$\|p_d\|_K \leq \exp\big[-C(A)^{1/N} d\big]$$

where C depends only on r and N.

The proposition is proved in a much more complicated manner in [J]; there, Padé-type approximants are constructed. This is the key ingredient in the original proof of Josefson's theorem that locally pluripolar sets are globally pluripolar.

14 Appendix on complex Monge-Ampère operator

For simplicity, we work in $\mathbb{C}^2$ with variables (z,w). We use the notation $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$ where, for a C^1 function u,

$$\partial u := \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial w}dw, \quad \bar\partial u := \frac{\partial u}{\partial \bar z}d\bar z + \frac{\partial u}{\partial \bar w}d\bar w$$

(recall section 7) so that $dd^c = 2i\partial\bar\partial$. For a C^2 function u,

$$(dd^c u)^2 = 16\big[\frac{\partial^2 u}{\partial z\partial \bar z}\frac{\partial^2 u}{\partial w\partial \bar w} - \frac{\partial^2 u}{\partial z\partial \bar w}\frac{\partial^2 u}{\partial w\partial \bar z}\big]\frac{i}{2}dz\wedge d\bar z \wedge \frac{i}{2}dw\wedge d\bar w$$

is, up to a positive constant, the determinant of the complex Hessian of u times the volume form on $\mathbb{C}^2$. Thus if u is also psh, $(dd^c u)^2$ is a positive measure which is absolutely continuous with respect to Lebesgue measure. If u is psh in an open set D and locally bounded there, then $(dd^c u)^2$ is a positive measure in D (cf. [BT1]).

To see this, we first recall that a psh function u in D is an usc function u in D which is subharmonic on components of $D \cap L$ for complex affine lines L. In particular, u is a locally integrable function in D such that

$$dd^c u = 2i\big[\frac{\partial^2 u}{\partial z\partial \bar z}dz\wedge d\bar z + \frac{\partial^2 u}{\partial w\partial \bar w}dw\wedge d\bar w + \frac{\partial^2 u}{\partial z\partial \bar w}dz\wedge d\bar w + \frac{\partial^2 u}{\partial \bar z\partial w}d\bar z\wedge dw\big]$$

is a positive $(1,1)$ current (dual to $(1,1)$ forms); i.e., a $(1,1)$ form with distribution coefficients. The derivatives are to be interpreted in the distribution sense and are actually measures; i.e., they act on compactly supported continuous functions. Here, a $(1,1)$ current T on a domain D in $\mathbb{C}^2$ is positive if T applied to $i\beta\wedge\bar\beta$ is a positive distribution for all $(1,0)$ forms $\beta = adz + bdw$ with $a, b \in C_0^\infty(D)$ (smooth functions having compact support in D). Writing the action of a current T on a form ψ as $\langle T, \psi\rangle$, this means that

$$\langle T, \phi(i\beta\wedge\bar\beta)\rangle \geq 0 \quad \text{for all } \phi \in C_0^\infty(D) \text{ with } \phi \geq 0.$$

For a discussion of currents and the general definition of positivity, we refer the reader to Klimek [K], section 3.3.

Following [BT1], we now define $(dd^c v)^2$ for a psh v in D if $v \in L^\infty_{loc}(D)$ using the fact that $dd^c v$ is a positive $(1,1)$ current with measure coefficients. First note that if v were of class C^2, given $\phi \in C_0^\infty(D)$, we have

$$\begin{aligned}\int_D \phi(dd^c v)^2 &= -\int_D d\phi\wedge d^c v\wedge dd^c v \\ &= \text{(exercise!)} \quad -\int_D dv\wedge d^c\phi\wedge dd^c v = \int_D v dd^c\phi\wedge dd^c v\end{aligned}$$

since all boundary integrals vanish. The applications of Stokes' theorem are justified if v is smooth; for arbitrary psh v in D with $v \in L^\infty_{loc}(D)$, these formal calculations serve as motivation to *define* $(dd^c v)^2$ as a positive measure (precisely, a positive current of bidegree $(2,2)$ and hence a positive measure) via

$$\langle (dd^c v)^2, \phi \rangle := \int_D v dd^c \phi \wedge dd^c v.$$

This defines $(dd^c v)^2$ as a $(2,2)$ current (acting on $(0,0)$ forms; i.e., test functions) since $vdd^c v$ has measure coefficients. We refer the reader to [BT1] or [K] (p. 113) for the verification of positivity of $(dd^c v)^2$.

The next result shows that for a nonpluripolar compact set $K \subset \mathbb{C}^2$, the Monge-Ampère measure $(dd^c V_K^*)^2$ associated to the L-extremal function V_K^* of K plays the role of the equilibrium measure Δg_K associated to the Green function g_K of a nonpolar compact set $K \subset \mathbb{C}$. A plurisubharmonic function u in a domain D satisfying the property that *for any D' relatively compact in D, and any v psh in $\bar{D}'$, if $u \geq v$ on $\partial D'$, then $u \geq v$ on D'*, is called **maximal** in D. Bedford and Taylor showed that for locally bounded psh u, u is maximal in D if and only if $(dd^c u)^2 = 0$ in D. Thus, maximal psh functions are the "correct" analogue of harmonic functions in $\mathbb{C}$. The following result (cf. [BT2], Corollary 9.4) shows that V_K^* is maximal outside of $\hat{K}$.

Proposition. *Let K be a nonpluripolar compact set in $\mathbb{C}^N$. Then we have $(dd^c V_K^*)^N = 0$ outside of K.*

Similarly, if D is a bounded open neighborhood of K, the relative extremal function satisfies $(dd^c \omega^*(\cdot, K, D))^N = 0$ in $D \setminus K$.

15 A few open problems

Here are a few open problems.

1. (Section 6) *Theorem ([A], [Bl2]) has a sharp version for balls; e.g., if K is the closed unit ball of $\mathbb{R}^N$, then $f \in C(K)$ extends to be harmonic in the ball of radius $R > 1$ if and only if $\limsup_{n\to\infty} d_n(f, K)^{1/n} \leq 1/R$ (cf. [BL1]). For $N > 2$, are there compact sets other than balls for which a sharp version of this theorem holds?*
2. (Section 7) *Let D be a smoothly bounded pseudoconvex domain. Does D possess the Mergelyan property if and only if there are pseudoconvex domains D_j with $\bar{D} \subset D_j$ such that $\bar{D} = \cap_j D_j$?*
3. (Section 8) *Does there exist a compact, polynomially convex subset K of the unit sphere in $\mathbb{C}^2$ such that $P(K) \neq C(K)$?*
4. (Section 9) *For Fekete arrays on a nonpluripolar compact set $K \subset \mathbb{C}^N$, $N \geq 2$, do the normalized discrete measures μ_n converge weak-* to the Monge-Ampére measure $\mu_K := (dd^c V_K^*)^N$ of the L-extremal function V_K^*? Verify this for* **any** *single nonpluripolar compact set K! More generally, is the conclusion true for arrays satisfying*
$$\limsup_{n\to\infty} \Lambda_n^{1/n} \leq 1?$$
5. (Section 10) *Let $K \subset \mathbb{R}^N$, $N \geq 3$, be a compact, convex set with nonempty interior. Which such sets K admit* **harmonic** *extremal arrays, i.e., arrays $\{\mathcal{A}_d\}_{d=1,2,\ldots}$ in K such that $\mathcal{K}_d(f)$ converges to f uniformly on K for each f harmonic in a neighborhood of K?*
6. (Section 12) *If $K \subset \mathbb{C}^N$ satisfies a Markov inequality does K have property (HCP)? Is K necessarily regular? If $N > 1$, is K necessarily nonpluripolar?*

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