Discrete Linear Interpolatory Operators

J. Szabados

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Abstract. This is a survey on discrete linear operators which, besides approximating in Jackson or near-best order, possess some interpolatory property at some nodes. Such operators can be useful in numerical analysis.

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A central problem in approximation theory is the construction of simple functions that approximate well a given set of functions. Traditionally, by "simple" we mean polynomials or rational functions as they are easily implemented on computers, and the set of functions is generally characterized by continuity or belonging to some L^p class. These functions may be defined on finite or infinite intervals, or in complex domains. Another characteristic is the measure of error which can be the supremum or L^p norm, etc., with a weight usually introduced in the case of non-compact domains.

But what are the available data to construct such approximations? From a practical point of view, this is a crucial question. For example, convolution integrals are very useful in proving Jackson's theorem, but to actually calculate them one needs complete knowledge of the function. Obviously, this precludes any numerical application. In practice, we are generally given a *discrete* set of data (like function values at certain points) from which we wish to reconstruct the function.

Another feature which is important to us is that we prefer to construct *linear* operators since they are easier to handle. For example, the so-called "best approximation" is not at all "best" for our purposes, since (except in inner product spaces) it represents a non-linear operator which may be very difficult to calculate.

A natural candidate that satisfies the above requirements is some kind of interpolation operator. It is based on a discrete set of data, and additionally gives a zero error at an increasing number of well defined points. Depending on the nature of the interpolation, it is relatively easy to construct such interpolation operators.

However, there is a significant drawback to these operators. In the case of Lagrange interpolation, for example, no matter how we choose the n nodes we get at least an extra $O(\log n)$ factor compared to the optimal Jackson order of convergence. And if we try to avoid this problematic situation by considering Hermite–Fejér interpolation then, although we obtain the Jackson order of convergence in some cases, this process is saturated. That is, it will not give the Jackson order of approximation, for example, for Lip 1 classes, let alone for classes of functions with higher order of smoothness.

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It transpires that the reason for this negative behavior of interpolating polynomials is the strict restriction on the degree. In 1963, at an Oberwolfach conference, Paul Butzer raised the following question: Is it possible to construct an interpolation process that gives the Jackson order of convergence? Of course, this was meant in the sense that we should now allow interpolating polynomials of degree *higher* than Lagrange or Hermite–Fejér interpolation, say polynomials of degree $\leq an$ for some constant a. It was Géza Freud [11] who answered this question affirmatively by constructing polynomials p_n of degree at most $4n-3$, interpolating at $\sim n/3$ nodes and approximating to the Jackson order

$$\max_{|x|\leq 1}|f(x)-p_n(x)| \leq c\,\omega(f,1/n), \qquad |x|\leq 1,$$

where ω is the ordinary modulus of continuity. This was later improved upon by Freud and Vértesi [14] who constructed a sequence of discrete linear polynomial operators (DLPO, in what follows) $J_n(f,x)$ of degree at most $4n-2$ which interpolate at the Chebyshev nodes $\cos((2k-1)/(2n))\pi$, $k=1,\ldots,n$, and provide a Timan type (pointwise) estimate

$$|f(x)-J_n(f,x)| \leq c[\omega(f,\frac{\sqrt{1-x^2}}{n})+\omega(f,\frac{1}{n})], \qquad |x|\leq 1. \tag{1}$$

Freud and Sharma [12, 13] further extended this result to operators based on general Jacobi nodes, and also succeeded in decreasing the degree of the polynomial to $n(1+\varepsilon)$, for an arbitrary $\varepsilon>0$. (Of course, the c in the above then depends upon ε.)

Freud's work [11] initiated a substantial series of papers exhibiting constructions of a similar character. R. B. Saxena [19] succeeded in constructing a sequence of DLPOs $J_n^\star$ which realized the even stronger Telyakovskii–Gopengauz estimate

$$|f(x)-J_n^\star(f,x)| \leq c\,\omega(f,\frac{\sqrt{1-x^2}}{n}), \qquad |x|\leq 1.$$

Furthermore, for 2π-periodic continuous functions, O. Kis and P. Vértesi [17] constructed a very simple interpolating process: Let $\ell_k(x)$ be the fundamental functions of trigonometric interpolation based on the equidistant nodes $x_k=2k\pi/(2n+1)$, $k=0,\ldots,2n$, i.e., let $\ell_k(x)$ be that trigonometric polynomial of order n for which $\ell_k(x_j)=\delta_{jk}$, $j,k=0,\ldots,2n$, and consider the sequence of operators

$$U_n(f,x)=\sum_{k=0}^{n} f(x_k)[4\ell_k(x)^3-3\ell_k(x)^4].$$

Then, for every 2π-periodic continuous functions f, we have

$$\max_{x\in\mathbf{R}}|f(x)-U_n(f,x)| \leq c\,\omega(f,\frac{1}{n}),$$

and, for this trigonometric polynomial $U_n(f,x)$ of degree at most $4n$, evidently

$$U_n(f,x_k)=f(x_k), \qquad k=0,\ldots,2n,$$

because of the nice identity

$$\sum_{k=0}^{n}[4\ell_k(x)^3-3\ell_k(x)^4]=1$$

(cf. Turetskii [27]). With a proper transformation, Kis and Vértesi [17] also constructed an operator that realized the Timan estimate (cf. (1)) and interpolates at some nodes. In that construction, the limit of the ratio of the degree of the polynomial and the number of points of interpolation was greater than 1. Later, O. Kis and J. Szabados [16] sharpened this to the following: *Given* $0 < \varepsilon \le 1$, *let* $n \ge 20/\varepsilon^2$ *and* $f \in C[-1,1]$. *There exists a sequence of DLPOs* p_n *of degree at most* $n(1+\varepsilon)$ *such that* p_n *interpolates* f *in at least* n *points and*

$$|f(x) - p_n(x)| \le \frac{13}{\varepsilon^2}\,\omega(f, \frac{\pi\sqrt{1-x^2}}{2n}), \qquad |x| \le 1.$$

This answers a question raised by Freud and Sharma [12] about the construction of linear operators of minimal degree compared to the number of points of interpolation, that at the same time realize the Telyakovskii–Gopengauz estimate.

In the algebraic case, the above mentioned operators are based on nodes of orthogonal polynomials which is not the best choice from a numerical point of view. Usually, data about an unknown function are given at *equidistant* nodes. Since Lagrange interpolation at equidistant nodes behaves very poorly (the norm of the operator grows exponentially with the number of nodes), it was not at all obvious if convergent operators based on equidistant nodes could be constructed. On p. 581 of [3], R. DeVore raised this question while requesting a Jackson order of convergence. One possible solution is the following: As an intermediate approximation, we take the rational functions

$$R_n(f,x) = \sum_{k=-n}^{n} \frac{f(k/n)}{(x-k/n)^4} \Bigg/ \sum_{k=-n}^{n} \frac{1}{(x-k/n)^4}\ , \tag{2}$$

based on equidistant nodes. These are a special case of the so-called Shepard operators which approximate to Jackson order (for details, see later). Then, to approximate (2) by polynomials, we can use any standard constructive operator (for example, the one developed by Bojanic and DeVore [2]). An easy calculation shows that the moduli of continuity of f and (2) are equivalent, thus solving the problem (for details, see [24]). Also, if the operator obtained in this way is of degree n, then for some c, $0 < c < 1$, it is possible to modify it so that the new operator will interpolate at the equidistant nodes $k/(cn)$, $k = 0, \pm 1, \dots, \pm cn$ (cf. [25], Theorem 3).

With the exception of Lagrange interpolation, all of the above mentioned operators are *saturated*, i.e., the order of convergence cannot be improved upon beyond a certain limit, even if the function has better and better structural properties. The question then becomes: Can we construct discrete linear operators that are not saturated, and approximate "close" to the order of best approximation $E_n(f)$? Of course, by Korovkin's theorem, such operators cannot be positive. In the case of 2π-periodic functions, the well-known de la Vallée Poussin means do have the desired approximation order, but they are not discrete. In [23], the following family of operators was constructed. Let j, n be positive and k, ℓ nonnegative integers such that $\frac{1}{2}(jn + kn - k + \ell - 1)$ is a nonnegative integer. Set

$$s_{jk\ell n}(t) = \frac{\sin\frac{jnt}{2}(\sin\frac{nt}{2})^k(\cos\frac{t}{2})^\ell}{j(n\sin\frac{t}{2})^{k+1}},$$

$$t_\nu = \frac{2\pi\nu}{jn}, \qquad \nu = 0, \pm 1, \pm 2, \dots,$$

and let

$$S_{jk\ell n}(g,t) = \sum_{\nu=0}^{jn-1} g(t_\nu) s_{jk\ell n}(t),$$

be a discrete linear operator defined on all 2π-periodic continuous functions g. This is a generalization of many of the classical kernels and operators. $s_{100n}(t)$ (n odd) is the Dirichlet kernel, $s_{110n}(t)$ is the Fejér kernel, $s_{310n}(t)$ is the de la Vallée Poussin kernel, and $s_{130n}(t)$ is the Jackson kernel. Correspondingly, $S_{100n}(g,t)$ (n odd) and $S_{101n}(g,t)$ (n even) are ordinary interpolation polynomials, $S_{110n}(g,t)$ is the Jackson polynomial and $S_{310n}(g,t)$ is the discrete version of the de la Vallée Poussin operator (see [16] and [23]). These sequences of operators converge if the so-called Lebesgue constant

$$\|L_{jk\ell n}\| = \max_{t\in\mathbf{R}} \sum_{\nu=0}^{jn-1} |s_{jk\ell n}(t-t_\nu)|$$

remains bounded as $n \to \infty$ (the other parameters being considered as constants). Namely, the following typical error estimate holds:

$$\max_{t\in\mathbf{R}} |S_{jk\ell n}(g,t) - g(t)| \le (1 + \|L_{jk\ell n}\|) E_n^T(g), \tag{3}$$

where $E_n^T(\cdot)$ is the error in the best approximation by trigonometric polynomials. For some special choices of the parameters j, k, ℓ, the following relations hold:

$$\|L_{j10n}\| = \frac{1}{j}\sum_{\nu=1}^{j} \cot\frac{2\nu-1}{4j}\pi = \frac{2}{\pi}(\log\frac{8\pi}{j} + \gamma) + \alpha_j$$

where $\gamma = 0.5772\cdots$ is the Euler constant and $0 < \alpha_j < \frac{\pi}{72j^2}$;

$$\|L_{j11n}\| \le \frac{2}{\pi}\log j + 2.283 \qquad \text{if } j \text{ is even, } n \text{ is odd;}$$

$$\|L_{221n}\| \le \frac{2}{\sqrt{3}};$$

$$\|L_{332n}\| \le \frac{11}{9}.$$

In all of these cases, (3) yields the order of best approximation. In addition, evidently

$$S_{jk\ell n}(g, t_\nu) = g(t_\nu), \qquad \nu = 1, \ldots, jn,$$

i.e., the ratio of the order of the operator to the number of interpolation points is

$$\frac{jn + kn + \ell - k + 1}{2jn},$$

which is fairly close to the optimal ratio 1/2 (which corresponds to the minimal degree interpolation) provided j is large compared with k.

The obvious transformation $x = \cos t$ converts these results into estimates for continuous functions on the interval $[-1, 1]$. Let

$$P_{jk\ell n}(f,x) = S_{jk\ell n}(f(\cos t), t)$$

and

$$x_\nu = \cos t_\nu, \qquad \nu = 0, \ldots, jn-1.$$

This new polynomial operator will have the analogous properties

$$P_{jk\ell n}(f, x_\nu) = f(x_\nu), \qquad \nu = 0, \ldots, jn-1,$$

and

$$\max_{|x|\le 1} |f(x) - P_{jk\ell n}(f,x)| \le (1 + \|L_{jk\ell n}\|) E_n(f),$$

where $E_n(\cdot)$ is the error in the best approximation by algebraic polynomials of degree at most n.

Let the number $r_n = m_n/n$ be the ratio of the degree m_n of a polynomial operator to the number n of nodes where it interpolates. For Lagrange interpolation we always have $\lim_{n\to\infty} r_n = 1$. In the case of (convergent) Hermite–Fejér interpolation, $\lim_{n\to\infty} r_n = 2$. A classical result of Bernstein [1] says that if $\varepsilon > 0$ is arbitrary, then there exists a linear polynomial operator with bounded norm such that

$$\lim_{n\to\infty} r_n = 1 + \varepsilon. \tag{4}$$

In general, it is desirable to investigate the relation between the norm $\|L_n\|$ of a (not necessarily linear) polynomial operator of degree m_n and the number of nodes n where it interpolates. Generalizing the results just mentioned, it can be shown that for such operators

$$\lim_{n\to\infty} \frac{\|L_n\|}{\log \frac{n}{m_n - n + 2}} > 0.$$

This was proved first for Chebyshev nodes [26], and later generalized by B. Shekhtman [20] to an arbitrary system of nodes. We emphasize that this result is true without assuming linearity of the operator. It holds, for example, for operators of the form

$$L_n(f,x) = p_n(f,x) + \sum_{k=1}^{n} [f(x_k) - p_n(f,x_k)] q_k(x),$$

where p_n is the best approximating polynomial of degree m_n and the q_k are polynomials of the same degree such that $q_k(x_j) = \delta_{kj}$, $j,k = 1,\ldots,n$. Sequences of operators having property (4) for general systems of nodes $x_k = \cos t_k$, $k = 1, \ldots, n$, and approximating to the order $O(E_{n(1+\varepsilon)})$ were constructed by Erdős, Kroó and Szabados [10] who showed that such approximation is possible if and only if

$$\limsup_{n\to\infty} \frac{N_n(I_n)}{n|I_n|} \le \frac{1}{\pi} \quad \text{whenever} \quad \lim_{n\to\infty} n|I_n| = \infty,$$

and

$$\lim_{n\to\infty} n \cdot \min_{1\le k\le n-1} (t_k - t_{k+1}) > 0,$$

where $N_n(I_n)$ is the number of t_k's in the interval $I_n \subset [0,\pi]$.

We have, until now, dealt with polynomial operators. From a numerical point of view, *rational functions* are just as useful as polynomials, and as we shall see, they enjoy certain advantages over polynomial operators. We have already briefly mentioned the operator (2). With an exponent 2 instead of 4, and in the case of bivariate functions, this operator was originally introduced by D. Shepard [21]. His interesting paper went unnoticed for some time, but his operator was rediscovered in 1976 by J. Balázs (oral communication). It was then proved in [24] that for the sequence of operators

$$R_n(f,x) = \sum_{k=-n}^{n} \frac{f(k/n)}{|x-k/n|^p} \Bigg/ \sum_{k=-n}^{n} \frac{1}{|x-k/n|^p} \ , \tag{5}$$

in the case $p=4$, we have

$$\max_{|x|\le 1} |f(x) - R_n(f,x)| \le c\,\omega(f, \frac{1}{n}),$$

i.e., a Jackson order of convergence. The major advantage of this sequence of discrete linear operators is that it is based on equidistant nodes (and not on the roots of some orthogonal polynomials). In addition, these operators (for $p=2$) have the properties

$$R_n(f,k/n) = f(k/n), \quad R_n'(f,k/n) = 0, \qquad k = 0,\ldots,\pm n,$$

i.e., they behave like Hermite–Fejér interpolating polynomials.

Balázs' rediscovery of Shepard's operators initiated a series of papers dealing with further problems and generalizations. We mention only a few significant papers and results. G. Somorjai [22] proved that, for $p>2$,

$$\max_{|x|\le 1} |f(x) - R_n(f,x)| \le \frac{c}{n} \quad \Leftrightarrow \quad f \in \mathrm{Lip}\,1,$$

and

$$\max_{|x|\le 1} |f(x) - R_n(f,x)| = o\left(\frac{1}{n}\right) \quad \Leftrightarrow \quad f = \text{const.},$$

i.e., the saturation property of the operator R_n. The important (and much more difficult) case $p=2$ was handled by Della Vecchia, Mastroianni and Totik [4]. They proved that for $p=2$ the trivial class (for which the approximation order is $o(1/n)$) is the set of constant functions, and the saturation class (for which the order of approximation is $O(1/n)$) is contained in $\cap_{\alpha<1}\mathrm{Lip}\,\alpha$. They also proved the surprising fact that, for arbitrary *subsequences* of the Shepard operators, arbitrarily fast convergence may be achieved for some nonconstant functions.

The case $0<p\le 2$ was handled by X. L. Zhou [28]. With the notation

$$T_{\varepsilon,p}f := \max_{0\le x\le 1} \left| \int_{0\le t\le 1,\, |t-x|\ge\varepsilon} \frac{f(t)-f(x)}{|t-x|^p}\,dt \right|,$$

he proved that the saturation order is n^{1-p} or $1/\log n$, and the saturation class is

$$\{f : f \in \mathrm{Lip}\,(p-1),\ \sup_{\varepsilon>0} T_{\varepsilon,p}f < \infty\}$$

or

$$\{f : \omega(f,t) = O(|\log t|^{-1}),\ \sup_{\varepsilon>0} T_{\varepsilon,1} f < \infty\}$$

according as $1 < p \le 2$ or $p = 1$, respectively. He also showed that the case $0 < p < 1$ is not interesting, since then

$$\max_{0\le x\le 1} |f(x) - R_n(f,x)| \to 0 \qquad \Leftrightarrow \qquad f = \text{const.}$$

Many of the above mentioned results have been generalized to weighted approximation (see [6]–[9], [18]), as well as to general systems of nodes (see [5]).

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J. Szabados
Alfréd Rényi Institute of Mathematics
P. O. Box 127
Budapest H-1364
Hungary
szabados@renyi.hu