Uniform and Pointwise Shape Preserving Approximation by Algebraic Polynomials

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Abstract

We survey developments, over the last thirty years, in the theory of Shape Preserving Approximation (SPA) by algebraic polynomials on a finite interval. In this article, "shape" refers to (finitely many changes of) monotonicity, convexity, or q-monotonicity of a function (for definition, see Section 4). It is rather well known that it is possible to approximate a function by algebraic polynomials that preserve its shape (*i.e.,* the Weierstrass approximation theorem is valid for SPA). At the same time, the degree of SPA is much worse than the degree of best unconstrained approximation in some cases, and it is "about the same" in others. Numerous results quantifying this difference in degrees of SPA and unconstrained approximation have been obtained in recent years, and the main purpose of this article is to provide a "bird's-eye view" on this area, and discuss various approaches used.

In particular, we present results on the validity and invalidity of uniform and pointwise estimates in terms of various moduli of smoothness. We compare various constrained and unconstrained approximation spaces as well as orders of unconstrained and shape preserving approximation of particular functions, etc. There are quite a few interesting phenomena and several open questions.

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1 Scope of the survey

This is a comprehensive survey of *uniform* and *pointwise* estimates of *polynomial* shape-preserving approximation (**SPA**) of the following types: (i) monotone, (ii) convex, (iii) q-monotone, (iv) comonotone, (v) coconvex.

The following types of SPA are not covered: (i) co-q-monotone (no positive results are known for $q \geqslant 3$), (ii) estimates of SPA in the $\mathbb{L}_p$ (quasi)norms, (iii) positive, copositive, onesided and intertwining approximation, (iv) nearly shape preserving approximation, (v) pointwise estimates of SPA with interpolation at the endpoints, (vi) SPA by splines with fixed and free knots, (vii) SPA by rational functions, (viii) simultaneous SPA, (ix) SPA of periodic functions.

2 List of Tables

3 Motivation

The purpose of this section is to provide some motivation to the problems/questions discussed in this survey. Because convex functions are "nicer" than monotone ones (for example, a convex function defined on an open interval is necessarily locally absolutely continuous on that interval), we pick "convexity" as our example of "shape" in the examples appearing in this section. Readers who are not familiar with the notations, definitions and/or symbols that we use here can consult Section 4 "Definitions, Notations and Glossary of Symbols".

With $E_n(f)$ denoting the degree of approximation of f by polynomials of degree $< n$, it was shown by Bernstein in 1912 that

$$E_n(|x|) \sim n^{-1}, \quad n \in \mathbb{N}.$$

(The upper estimate was obtained in 1908 by de la Vallée Poussin.) Since the polynomial of best approximation to f necessarily interpolates f at "many" points it is clear that despite the fact that $|x|$ is convex on $[-1,1]$, its polynomial of best approximation of degree $< n$ is not going to be convex for $n \geqslant 5$.

A natural question is then if one can approximate $|x|$ by a convex polynomial with the error of approximation still bounded by cn^{-1} (the answer is "yes" – we omit almost all references in this section for the reader's convenience, since the results mentioned here are either simple or particular cases of general theorems discussed in detail later in this survey). Is this the case for other functions having the same approximation order? In other words, if a convex function f is such that $E_n(f) = O\left(n^{-1}\right)$, does this imply that there exist convex polynomials providing the same rate of approximation, *i.e.,* $E_n^{(2)}(f) = O\left(n^{-1}\right)$? (The answer is "yes".) Is this the case for other rates of convergence of errors to zero? For example, if a convex function f is such that $E_n(f) = O\left(n^{-\alpha}\right)$ for some $\alpha > 0$, does this imply that $E_n^{(2)}(f) = O\left(n^{-\alpha}\right)$? (The answer is "yes".) Does this mean that the rates of approximation of convex functions by all polynomials and by convex polynomials from $\mathbb{P}_n$ are "about the same"? In other words, because it is clear that $E_n(f) \leqslant E_n^{(2)}(f)$ (since the set of all convex polynomials from $\mathbb{P}_n$ is certainly a proper subset of $\mathbb{P}_n$), is it possible that the converse is also true, *i.e.,* there exists a constant $c(f)$ such that $E_n^{(2)}(f) \leqslant c(f)E_n(f)$ for all n? (The answer is "no".)

So, approximation by convex polynomials cannot simply be reduced to unconstrained approximation.

Going back to a convex function f such that $E_n(f) = O\left(n^{-\alpha}\right)$, we know that this implies that $E_n^{(2)}(f) = O\left(n^{-\alpha}\right)$, but these are asymptotic rates that only deal with the behavior of the quantities $E_n(f)$ and $E_n^{(2)}(f)$ as $n \to \infty$. Thus, we conclude that $E_n^{(2)}(f) \leqslant cn^{-\alpha}$, $n \geqslant m$, for some large m.

What can be said about $E_n^{(2)}(f)$ for small values of n? Clearly, we can say something about it even when we do not have asymptotic information on $E_n(f)$. Indeed,

$$E_1(f) = E_1^{(2)}(f) = (\max_{x\in[-1,1]} f(x) - \min_{x\in[-1,1]} f(x))/2 \leqslant \|f\|$$

implies that, for any $m \in \mathbb{N}$,

$$E_n^{(2)}(f) \leqslant E_1^{(2)}(f) \leqslant (\|f\| m^\alpha) n^{-\alpha}, \quad 1 \leqslant n \leqslant m.$$

Hence, for a convex function f such that $E_n(f) = O(n^{-\alpha})$, we have

$$E_n(f) = O\left(n^{-\alpha}\right) \quad \Longrightarrow \quad E_n^{(2)}(f) \leqslant c(f) n^{-\alpha}, \quad n \in \mathbb{N}. \tag{1}$$

It is easy to see that this implication is no longer valid if we require that the constant c in (1) remain independent of f (for example, for $f_\gamma(x) = \gamma x^2$, it is obvious that $E_n(f_\gamma) = O\left(n^{-\alpha}\right)$ but $E_1(f_\gamma) = E_1^{(2)}(f_\gamma) = |\gamma|/2 = \|f_\gamma\|/2 \to \infty$ as $\gamma \to \infty$).

We should therefore word this question more carefully. Namely, suppose that f is a convex function such that $E_n(f) \leqslant n^{-\alpha}$ for $n \in \mathbb{N}$, does this imply that $E_n^{(2)}(f) \leqslant c(\alpha) n^{-\alpha}$ for all $n \in \mathbb{N}$? (The answer is "yes".) What if we only know that $E_n(f) \leqslant n^{-\alpha}$ for $n \geqslant 2011$, can we say that $E_n^{(2)}(f) \leqslant c(\alpha) n^{-\alpha}$ for $n \geqslant 2011$? (The answer is "no".) So, the fact that $E_n(f) \leqslant n^{-\alpha}$ for $n \geqslant \mathcal{N}$ sometimes implies that $E_n^{(2)}(f) \leqslant c(\alpha) n^{-\alpha}$ for $n \geqslant \mathcal{N}$, and sometimes does not. Do we know precisely for which $\mathcal{N} \in \mathbb{N}$ this implication is valid, and for which it is invalid in general? (The answer is "yes, we do know: if $\alpha \leqslant 4$, this implication is valid for $\mathcal{N} \leqslant 4$, and it is in general invalid in the stated form for $\mathcal{N} \geqslant 5$; if $\alpha > 4$, then this implication is valid for all $\mathcal{N} \in \mathbb{N}$".)

In general, given a function $f \in C^r[-1,1]$, in order to obtain an estimate of the form $E_n(f) \leqslant cn^{-\alpha}$ one can try to apply a Jackson-Stechkin type uniform estimate

$$E_n(f) \leqslant c(k,r) n^{-r} \omega_k\big(f^{(r)}, n^{-1}\big), \quad n \geqslant k + r. \tag{2}$$

For example, since $|x| \in C[-1,1]$ and $\omega(|x|, \delta) = \min\{1, \delta\}$, using (2) with $r = 0$ and $k = 1$, one immediately obtains the above mentioned de la Vallée Poussin estimate $E_n(|x|) \leqslant cn^{-1}$, $n \in \mathbb{N}$. Hence, it is desirable to have analogs of Jackson-Stechkin type estimates for approximation by convex polynomials. So, does one have an estimate $E_n^{(2)}(f) \leqslant c\omega\left(f, n^{-1}\right)$, $n \in \mathbb{N}$, which would then immediately imply that $E_n^{(2)}(|x|) \leqslant cn^{-1}$, $n \in \mathbb{N}$? (The answer is "yes".)

One can naturally try to use the same process for functions having greater smoothness (and, hence, having better approximation rates). For example, it is known (see Timan [Tim94, p. 416], Bernstein [Ber54, pp. 262–272, pp. 402–404] and Ibragimov [Ibr50]) that, for the function $f_{\mu,\lambda}(x) = x^{\mu-1}|x|^{1+\lambda}$, $\mu \in \mathbb{N}$, $|\lambda| < 1$,

$$\lim_{n\to\infty} n^{\mu+\lambda} E_n(f_{\mu,\lambda}) = c(\mu, \lambda) > 0.$$

In particular, this implies that, if $\alpha > 0$ is such that $\alpha/2 \notin \mathbb{N}$ (otherwise, the function becomes a polynomial), then

$$E_n(|x|^\alpha) \overset{\alpha}{\sim} n^{-\alpha}, \quad n \in \mathbb{N}.$$

An upper estimate can be obtained directly from (2) by setting $r = 0$, $k = \lceil \alpha \rceil$, and using the fact that

$$\omega_k(|x|^\alpha, \delta) \leqslant c(\alpha) \min\{1, \delta^\alpha\}.$$

Now, since $|x|^{2011}$ is convex, we would get the estimate $E_n^{(2)}(|x|^{2011}) \leqslant cn^{-2011}$ if the estimate $E_n^{(2)}(f) \leqslant c\omega_{2011}\left(f, n^{-1}\right)$ were true for all convex functions. Unfortunately, this is not the case, and it is known that even the estimate $E_n^{(2)}(f) \leqslant O\left(\omega_5(f, n^{-1})\right)$ is invalid for some convex functions (and the estimate $E_n^{(2)}(f) \leqslant c\omega_4\left(f, n^{-1}\right)$ cannot be valid for all convex functions f and all $n \geqslant \mathcal{N}$ with the constant $\mathcal{N}$ independent of f). In this particular case, one can overcome this obstacle by using the estimate $E_n^{(2)}(f) \leqslant c(k)n^{-2}\omega_k\left(f'', n^{-1}\right)$ which is true for all $k \in \mathbb{N}$ and all convex functions f from $C^2[-1,1]$, and which implies the required result.

Can this approach always be used? In other words, is it enough to use Jackson-Stechkin type estimates in order to get the "right" order of polynomial approximation? Again, unfortunately, the answer is "no". It has been known for a long time that the Jackson-Stechkin type estimates, while producing the "right" orders of approximation for some functions, produce rather weak estimates in some special cases. For example, Ibragimov [Ibr46] verified Bernstein's conjecture (see [Ber37, p. 91]) that, for $g(x) = (1-x)\ln(1-x)$ (we will later call this function $g_{1,1}$),

$$E_n(g) \sim n^{-2}\,.$$

Because $g \notin C^1[-1,1]$, one can only use (2) with $r = 0$, and since, for any $k \geqslant 2$, $\omega_k(g, n^{-1}) \overset{k}{\sim} n^{-1}$, one can only get from (2) the rather weak estimate $E_n(g) \leqslant cn^{-1}$. In particular, this means that, given a function $f \in C[-1,1]$ such that $E_n(f) = O(n^{-2})$, in general we cannot use Jackson-Stechkin type estimates in order to prove $E_n^{(2)}(f) = O(n^{-2})$, and more precise estimates (yielding constructive characterization of function classes) are required. This brings us to the Ditzian-Totik type estimates (see [DT87, Dit07]). In particular, since the following Ditzian-Totik type estimate is valid

$$E_n(f) \leqslant c(k)\omega_k^{\varphi}\left(f, n^{-1}\right), \quad n \geqslant k, \tag{3}$$

and since

$$\omega_2^{\varphi}\left(g, n^{-1}\right) \sim n^{-2},$$

one immediately obtains $E_n(g) \leqslant cn^{-2}$. Now, since g is convex on $[-1,1]$, how does one go about proving the estimate $E_n^{(2)}(g) \leqslant cn^{-2}$ (which, as we mentioned above, has to be valid)? This estimate would immediately follow if the estimate $E_n^{(2)}(f) \leqslant c\omega_2^{\varphi}\left(f, n^{-1}\right)$ were true for all convex f, but is it true? (The answer is "yes".)

Suppose now that instead of the function g we consider a similar function having higher smoothness (and, hence, a better rate of polynomial approximation). For example, Ibragimov [Ibr46, Theorem VII] showed that, for $g_{\mu,\lambda}(x) = (1-x)^{\mu}\ln^{\lambda}(1-x)$, ν, $\lambda \in \mathbb{N}$,

$$E_n(g_{\mu,\lambda}) \overset{\lambda,\mu}{\sim} \frac{(\ln n)^{\lambda-1}}{n^{2\mu}}, \quad n \geqslant 2. \tag{4}$$

In particular, consider $h(x) = 10x^2 - (1-x)^3\ln(1-x)$ which is convex on $[-1,1]$. Since

$$\omega_6^{\varphi}(h, n^{-1}) \sim n^{-6}$$

the estimate (3) implies that $E_n(h) \leqslant cn^{-6}$. Now, a similar estimate for the rate of convex approximation would immediately follow if the estimate $E_n^{(2)}(f) \leqslant c\omega_6^{\varphi}\left(f, n^{-1}\right)$ were valid for all convex f, but is it valid? (The answer is "no".) In the above example for the Jackson-Stechkin type estimates we managed to overcome a similar difficulty by using an estimate involving the modulus of a derivative of f. What if we try a similar approach now? We know that

$$\omega_6^{\varphi}(f,\delta) \leqslant c\delta^2\omega_4^{\varphi}(f'',\delta) \tag{5}$$

and the function h is clearly in $C^2[-1,1]$. Hence, if we show that $\omega_4^{\varphi}(h'',n^{-1}) \leqslant cn^{-4}$, and if the estimate $E_n^{(2)}(f) \leqslant cn^{-2}\omega_4^{\varphi}\left(f'',n^{-1}\right)$ is valid for all convex functions $f \in C^2[-1,1]$, then we will have proved what we want. Now, since $h''(x) = 15 + 5x - 6(1-x)\ln(1-x)$, we have

$$\omega_4^{\varphi}(h'',n^{-1}) = c\omega_4^{\varphi}(g_{1,1},n^{-1}) \leqslant cn^{-2},$$

and the last inequality cannot be improved (this can be verified directly, but it also follows immediately from the fact that if we could replace n^{-2} by $o(n^{-2})$, then (4) would not be valid). Hence, this approach would only give us a rather weak estimate (different by the factor of n^2 from the optimal). The reason why this approach fails is that the inequality (5) is very imprecise, and one needs to work with generalized Ditzian-Totik moduli involving derivatives of functions. In this particular case, we can obtain the needed result taking into account the fact that

$$\omega_6^{\varphi}(f,\delta) \leqslant c\delta^5\omega_{1,5}^{\varphi}(f^{(5)},\delta),$$

and using the estimate

$$E_n^{(2)}(f) \leqslant cn^{-5}\omega_{1,5}^{\varphi}\left(f^{(5)},n^{-1}\right),$$

and the inequality $\omega_{1,5}^{\varphi}\left(h^{(5)},\delta\right) \leqslant c\delta$.

The conclusion that we can reach at this time is that one needs to work with generalized Ditzian-Totik moduli of smoothness $\omega_{k,r}^{\varphi}(f^{(r)},\delta)$ instead of the regular (ordinary or Ditzian-Totik) moduli in order to get exact uniform estimates. Of course, it is now well known that one can also obtain exact estimates of approximation by algebraic polynomials (yielding constructive characterization of classes of functions) in terms of the ordinary moduli of smoothness, but only if pointwise estimates are used. It is well known that for a function $f \in C^r[-1,1]$ and each $n \geqslant k+r$ there is a polynomial $P_n \in \mathbb{P}_n$, such that for every $x \in [-1,1]$,

$$|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)},\rho_n(x)\right),$$

where $\rho_n(x) := n^{-2} + n^{-1}\sqrt{1-x^2}$. (This is a so-called Nikolskii type pointwise estimate.) This immediately implies (2), and is clearly more precise than (2). Hence, the same questions that we discussed above can also be asked for pointwise estimates of rates of approximation by convex polynomials.

So far, we only discussed functions which are convex on the entire interval $[-1,1]$. What if a function has one or more inflection points? Can we approximate it by polynomials having the same "shape"? Again, all of the above questions naturally arise. For example, consider the above mentioned function $g_{2,1}(x) = (1-x)^2\ln(1-x)$. This function is convex on $[-1,y_1]$ and concave on $[y_1,1]$, where $y_1 = 1 - \mathrm{e}^{-3/2}$. We already know (see (4)) that $E_n(g_{2,1}) \leqslant cn^{-4}$. Do we have the same rate of coconvex approximation, *i.e.,* is $E_n^{(2)}(g_{2,1},Y_1) \leqslant cn^{-4}$ valid? (The answer is "yes".) Is it always the case that if a function f changes its convexity at one point of $[-1,1]$ and has the rate of its unconstrained approximation bounded by n^{-4}, then its rate of coconvex approximation is bounded by cn^{-4}? The answer is "yes, but the constant c has to depend on y_1, and may become large as y_1 gets closer to ± 1". This seems like a very obvious and intuitively clear conclusion (the closer the inflection point is to the endpoints, the harder it is to control the approximating polynomial preserving the "shape" of the function). However, this is where intuition fails. We now know that given a function f that has one inflection point in $[-1,1]$ and whose rate of unconstrained approximation is bounded by $n^{-\alpha}$ with $\alpha \neq 4$, its rate of coconvex approximation is bounded by $c(\alpha)n^{-\alpha}$. In other words, in this sense, the case $\alpha = 4$ is totally different from all other cases.

The main purpose of this survey is to summarize the state of the art of this area of research as of the summer of 2011.

4 Definitions, Notations and Glossary of Symbols

Most symbols used throughout this paper are listed in the table below (however, for the reader's convenience, we also define some of them the first time they are used in this survey).

$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$C(S)$	space of continuous functions on S
$C^r(S)$	space of r-times continuously differentiable functions on S, $r \in \mathbb{N}$
$\|f\|_{C(S)}$	$\max_{x\in S} \|f(x)\|$
$\|\cdot\|$	$\|\cdot\|_{C[-1,1]}$
$\|f\|_{\mathbb{L}_\infty(S)}$	ess $\sup_{x\in S} \|f(x)\|$
$\Delta^{(1)}$	$\{f \in C[-1,1] \ : \ f$ is nondecreasing on $[-1,1]\}$
$\Delta^{(2)}$	$\{f \in C[-1,1] \ : \ f$ is convex on $[-1,1]\}$
$\Delta^{(q)}$	$\big\{f \in C[-1,1] \cap C^{q-2}(-1,1) \ : \ f^{(q-2)}$ is convex on $(-1,1)\big\}$, $q \geqslant 3$; f is q-monotone
$\mathbb{P}_n$	space of algebraic polynomials of degree $\leqslant n-1$
$E_n(f)$	$\inf_{P_n\in\mathbb{P}_n} \|f - P_n\|$ (degree of best unconstrained approximation)
$E_n^{(q)}(f)$	$\inf_{P_n\in\mathbb{P}_n\cap\Delta^{(q)}} \|f - P_n\|$ (degree of q-monotone approximation)
Y_s	collection $\{y_i\}_{i=1}^s$ of $s \in \mathbb{N}$ points $-1 =: y_{s+1} < y_s < \cdots < y_1 < y_0 := 1$
Y_0	$\varnothing$
$\Delta^{(1)}(Y_s)$	set of all functions $f \in C[-1,1]$ that change monotonicity at the points Y_s, and are non-decreasing on $[y_1, 1]$
$\Delta^{(2)}(Y_s)$	set of all functions $f \in C[-1,1]$ that change convexity at the points Y_s, and are convex on $[y_1, 1]$
$\Delta^{(q)}(Y_s)$	set of all functions $f \in C[-1,1] \cap C^{q-2}(-1,1)$ such that $f^{(q-2)}$ changes convexity at the points Y_s, and is convex on $[y_1, 1]$, $q \geqslant 3$
$\Delta^{(q)}(Y_0)$	$\Delta^{(q)}$, $q \geqslant 1$
$E_n^{(q)}(f, Y_s)$	$\inf_{P_n\in\Delta^{(q)}(Y_s)\cap\mathbb{P}_n} \|f - P_n\|$ (degree of co-q-monotone approximation)

$\Delta_h^k(f,x)$	$\sum_{i=0}^k \binom{k}{i}(-1)^{k-i} f(x-kh/2+ih)$, if $\lvert x \pm kh/2\rvert < 1$, and 0, otherwise ($k$th symmetric difference)
$\omega_k(f,t)$	$\sup_{0\leqslant h\leqslant t} \lVert\Delta_h^k(f,\cdot)\rVert$ (kth modulus of smoothness)
$\omega(f,t)$	same as $\omega_1(f,t)$ (ordinary modulus of continuity of f)
$\omega_0(f,t)$	$\lVert f\rVert_{\mathbb{L}_\infty[-1,1]}$
$\varphi(x)$	$\sqrt{1-x^2}$
$\rho_n(x)$	$n^{-1}\sqrt{1-x^2}+n^{-2}$
$\omega_k^\varphi(f,t)$	$\sup_{0\leqslant h\leqslant t} \lVert\Delta_{h\varphi(\cdot)}^k(f,\cdot)\rVert$ (Ditzian-Totik (D-T) kth modulus of smoothness)
$\mathbb{W}^r$	space of functions f defined on $[-1,1]$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in \mathbb{L}_\infty[-1,1]$, $r \geqslant 1$
$\mathbb{B}^r$	space of functions f defined on $[-1,1]$ such that $f^{(r-1)}$ is locally absolutely continuous in $(-1,1)$ and $\varphi^r f^{(r)} \in \mathbb{L}_\infty[-1,1]$, $r \geqslant 1$
C_φ^0	$C[-1,1]$
C_φ^r	$\left\{f \in C^r(-1,1) \ : \ \lim_{x\to\pm1} \varphi^r(x) f^{(r)}(x) = 0\right\}$, $r \geqslant 1$
$K(x,\mu)$	$\varphi(\lvert x\rvert + \mu\varphi(x))$
$\omega_{k,r}^\varphi(f^{(r)},t)$	$\sup_{0\leqslant h\leqslant t} \sup_{x:\lvert x\rvert+kh\varphi(x)/2<1} K^r(x,kh/2)\lvert\Delta_{h\varphi(x)}^k(f^{(r)},x)\rvert$ (D-T generalized modulus of smoothness)
$\omega_{0,r}^\varphi(f^{(r)},t)$	$\lVert\varphi^r f^{(r)}\rVert_{\mathbb{L}_\infty[-1,1]}$
c	absolute positive constants that can be different even if they appear on the same line
$c(\cdot)$	positive constants that depend on the parameters appearing inside the parentheses and nothing else
$a_n \sim b_n$	there exists an absolute positive constant c such that $c^{-1}a_n \leqslant b_n \leqslant ca_n$ for all $n \in \mathbb{N}$
$a_n \overset{\alpha_1,\alpha_2,\dots}{\sim} b_n$	there exists a positive constant $c = c(\alpha_1,\alpha_2,\dots)$ such that $c^{-1}a_n \leqslant b_n \leqslant ca_n$ for all $n \in \mathbb{N}$
$a_n(f) = O(b_n)$	there exist $c(f) \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $\lvert a_n(f)\rvert \leqslant c(f)\lvert b_n\rvert$, for $n \geqslant m$

Remark. Throughout this survey, the letters "q", "s", "r" and "k" always stand for nonnegative integers. The letter "q" is always used to describe the shape of a function (*e.g.,* $\Delta^{(q)}$), the letter "s" always stands for the number of changes of monotonicity, convexity or q-monotonicity (*e.g.,* $\mathbb{Y}_s$, Y_s, $\Delta^{(1)}(Y_s)$), the letter "r" always refers to the r-th derivative (*e.g.,* $\mathbb{W}^r$, $\mathbb{B}^r$, $f^{(r)}$, $\omega_{k,r}^\varphi(f^{(r)},\delta)$), and the letter "$k$" is used to describe the order of the appropriate moduli of smoothness or similar quantities (*e.g.,* ω_k, ω_k^φ, Φ^k).

The following are the types of estimates of errors of SPA and unconstrained approximation that we discuss in this survey.

Type of estimate	Estimate is given in terms of	Notation
Jackson-Stechkin (uniform)	$n^{-r}\omega_k\left(f^{(r)},1/n\right)$	$\delta_n(x):=1/n$, $\mathrm{w}_k:=\omega_k$
Ditzian-Totik (uniform)	$n^{-r}\omega^{\varphi}_{k,r}\left(f^{(r)},1/n\right)$	$\delta_n(x):=1/n$, $\mathrm{w}_k:=\omega^{\varphi}_{k,r}$
Nikolskii (pointwise)	$\underbrace{\rho^r_n(x)\omega_k\left(f^{(r)},\rho_n(x)\right)}$	$\delta_n(x):=\rho_n(x)$, $\mathrm{w}_k:=\omega_k$
	$\delta^r_n(x)\mathrm{w}_k\left(f^{(r)},\delta_n(x)\right)$	

With the above notation, it is convenient to refer to all of the above types of estimates at once. Namely, given q, s, r and k as above, for collections Y_s and functions $f\in\Delta^{(q)}(Y_s)$ (assumed to have appropriate smoothness so that $\mathrm{w}_k(f^{(r)},t)$ is defined and finite), we formally write an estimate for the error of (co)-q-monotone polynomial approximation of f (which may or may not be valid for $P_n\in\mathbb{P}_n\cap\Delta^{(q)}(Y_s)$):

$$|f(x)-P_n(x)|\leqslant c(k,r,q)\delta^r_n(x)\mathrm{w}_k\left(f^{(r)},\delta_n(x)\right),\quad n\geqslant\mathcal{N},\tag{6}$$

and distinguish the following cases for the quadruple (k,r,q,s).

Case "+" ("strongly positive case"): Inequality (6) holds with $\mathcal{N}$ depending only on k, r, q and s.

Case "$\oplus$" ("weakly positive case"): Inequality (6) holds with a constant $\mathcal{N}$ that, in addition, depends on the set Y_s (*i.e.,* location of points where f changes its monotonicity, convexity or q-monotonicity), and does not hold in general with $\mathcal{N}$ independent of Y_s. (Note that this case is only applicable if $s\geqslant1$ since, in the case $s=0$, $Y_0=\emptyset$.)

Case "$\ominus$" ("weakly negative case"): Inequality (6) holds with a constant $\mathcal{N}$ that depends on the function f, and does not hold in general with $\mathcal{N}$ independent of f.

Case "$-$" ("strongly negative case"): Inequality (6) does not hold in general even with a constant $\mathcal{N}$ that depends on the function f. This means that there exists a function $f\in\Delta^{(q)}(Y_s)\cap C^r[-1,1]$ such that, for every sequence of polynomials $P_n\in\mathbb{P}_n\cap\Delta^{(q)}(Y_s)$,

$$\limsup_{n\to\infty}\left\|\frac{f-P_n}{\delta^r_n\mathrm{w}_k\left(f^{(r)},\delta_n\right)}\right\|=\infty.$$

We will also have the case "$\varnothing$", which we do not discuss here because, so far, it appears only when pointwise estimates for coconvex approximation are considered (see Section 14.3, page 65).

Remark. In all cases "+" discussed in this survey, inequality (6) holds with $\mathcal{N}=k+r$. (This is the best possible case since (6) certainly cannot hold if $\mathcal{N}<k+r$ because $\mathrm{w}_k(f^{(r)},t)$ vanishes if $f\in\mathbb{P}_{k+r}$.)

Along with different direct estimates on the error of SPA, we will also consider relations between degrees of best unconstrained approximation and SPA, in particular, the so-called α-relations. Namely, given q and s as above, and $\alpha>0$, for collections Y_s and functions $f\in\Delta^{(q)}(Y_s)$, we investigate the validity of the implication

$$n^\alpha E_n(f)\leqslant1,\ \ n\geqslant\mathcal{N}\quad\Longrightarrow\quad n^\alpha E^{(q)}_n(f,Y_s)\leqslant c(s,\alpha),\ \ n\geqslant\mathcal{N}^*,\tag{7}$$

distinguishing the following cases. (Note that, while we use the same symbols as for direct estimates, their meaning for α-relations is different.)

Case "+" ("strongly positive case"): Implication (7) is valid with $\mathcal{N}^*$ depending on $\mathcal{N}$, α, s.

Case "⊕" ("weakly positive case"): Implication (7) is valid with $\mathcal{N}^*$ depending on $\mathcal{N}$, α, s and Y_s, and is not valid in general with $\mathcal{N}$ independent of Y_s.

Case "⊖" ("weakly negative case"): Implication (7) is valid with $\mathcal{N}^*$ that depends on the function f (as well as on $\mathcal{N}$, α, s and Y_s), and is invalid in general with $\mathcal{N}$ independent of f.

Case "−" ("strongly negative case"): Implication (7) is invalid in general even with a constant $\mathcal{N}^*$ that depends on the function f.

Remark. We emphasize that, in all cases "+" of this type discussed in this survey, inequality (7) holds with $\mathcal{N}^* = \mathcal{N}$.

We finally mention that all "Statements" throughout this survey are valid for some values of parameters and invalid for some other values. The above cases/symbols "+", "⊕", "⊖" and "−" will be used to describe various cases of their validity (expressions of type "the quadruple (k, r, q, s) is strongly positive in Statement X", "Statement X is strongly positive for the quadruple (k, r, q, s)", etc. all have the same meaning and will be used interchangeably).

We use the symbol "?" to indicate that we do not know at all which of the symbols "+", "⊕" (if applicable), "⊖" or "−" should be put in its place (*i.e.,* if a problem is completely open). The symbol "?*" is used if we know something about this case, but the problem is still not completely resolved.

5 Historical background

Perhaps the first investigation of SPA was done in 1873 by Chebyshev, who constructed an algebraic polynomial having the minimum uniform norm on $[-1, 1]$ among all nondecreasing polynomials of the form $\epsilon x^n + a_1x^{n-1} + \cdots + a_n$ with $\epsilon = 1$ or -1 (see [Che1873] or [Che55]). Namely, Chebyshev [Che1873] showed that

$$\inf\left\{\|P_n\| \ : \ P_n(x) = x^n + Q_n(x)\,,\ Q_n \in \mathbb{P}_n \text{ and } P_n \in \Delta^{(1)}\right\}$$
$$= \begin{cases} 2\left(\dfrac{m!}{(2m-1)!!}\right)^2, & \text{if } n = 2m, \\ \left(\dfrac{m!}{(2m-1)!!}\right)^2, & \text{if } n = 2m+1, \end{cases}$$

and

$$\inf\left\{\|P_n\| \ : \ P_n(x) = x^n + Q_n(x)\,,\ Q_n \in \mathbb{P}_n \text{ and } (-P_n) \in \Delta^{(1)}\right\}$$
$$= \begin{cases} 2\left(\dfrac{m!}{(2m-1)!!}\right)^2, & \text{if } n = 2m, \\ \left(1+\dfrac{1}{m}\right)\left(\dfrac{m!}{(2m-1)!!}\right)^2, & \text{if } n = 2m+1. \end{cases}$$

In 1927, Bernstein [Ber27] (see also [Ber52, pp. 339–349]) obtained several analogous results for multiply monotone polynomials (smooth functions are called "multiply monotone of order μ" if their first μ derivatives are nonnegative; sometimes these functions are referred to as "absolutely monotone of order μ").

It is rather well known by now that, if $f \in \Delta^{(q)}$, then its Bernstein polynomial (introduced by Bernstein in 1912, see [Ber12] and [Ber52, pp. 105-106])

$$B_n(f,x) = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j} f\left(\frac{n-2j}{n}\right)(1+x)^{n-j}(1-x)^j$$

is also in $\Delta^{(q)}$. It is not clear who was the first to notice this shape preserving property of Bernstein polynomials. Popoviciu knew about it as early as in 1934 (see [Pop34]), but it is not clear if Bernstein himself was aware of this property before then. Since Bernstein polynomials associated with $f \in C[-1,1]$ uniformly approximate f, it follows that the Weierstrass approximation theorem is valid for SPA, that is, for every $f \in \Delta^{(q)}$,

$$E_n^{(q)}(f) \to 0, \quad n \to \infty.$$

In 1965, Shisha [Shi65] proved that, for $f \in C^r \cap \Delta^q$, $1 \leqslant q \leqslant r$,

$$E_n^{(q)}(f) \leqslant c(q,r)\frac{1}{n^{r-q}}\omega(f^{(r)}, 1/n). \tag{8}$$

The proof was based on the rather obvious observation that, for $f \in C^q[-1,1] \cap \Delta^q$, we have

$$E_n^{(q)}(f) \leqslant c(q)E_{n-q}(f^{(q)}). \tag{9}$$

Indeed, let $Q_{n-q} \in \mathbb{P}_{n-q}$ be such that $E_{n-q}(f^{(q)}) = \|f^{(q)} - Q_{n-q}\|$, and $P_n \in \mathbb{P}_n$ be such that $P_n^{(\nu)}(0) = f^{(\nu)}(0)$, $0 \leqslant \nu \leqslant q-1$, and $P_n^{(q)}(x) := Q_{n-q}(x) + E_{n-q}(f^{(q)})$. Hence,

$$\left\|f^{(q)} - P_n^{(q)}\right\| \leqslant 2E_{n-q}(f^{(q)}),$$

and $P_n^{(q)}(x) \geqslant f^{(q)}(x) \geqslant 0$, *i.e.*, $P_n \in \Delta^q$. Finally,

$$\begin{aligned} E_n^{(q)}(f) &\leqslant \|f - P_n\| = \left\|\int_0^x \int_0^{t_1} \cdots \int_0^{t_{q-1}} \left(f^{(q)}(t_q) - P_n^{(q)}(t_q)\right) \mathrm{d}t_q \cdots \mathrm{d}t_1\right\| \\ &\leqslant \frac{1}{q!}\left\|f^{(q)} - P_n^{(q)}\right\| \leqslant \frac{2}{q!}E_{n-q}(f^{(q)}). \end{aligned}$$

While inequality (8) differs from "the optimal estimate of this type" by the factor of n^q, it was perhaps the first attempt to obtain a nontrivial direct estimate for SPA and brought attention to this area.

The first major developments in this area appeared in papers by Lorentz and Zeller and by DeVore. Lorentz and Zeller [LZ69] constructed, for each $q \geqslant 1$, a function $f \in \Delta^{(q)} \cap C^q[-1,1]$, such that

$$\limsup_{n\to\infty} \frac{E_n^{(q)}(f)}{E_n(f)} = \infty.$$

This means that questions on SPA do not trivially reduce to those on unconstrained approximation. Yet, Lorentz and Zeller [LZ68] (for $r = 0$), Lorentz [Lor72] ($r = 1$) and DeVore [DeV77], proved the exact analogue of the Jackson type estimate, namely that, for each function $f \in \Delta^{(1)} \cap C^r[-1,1]$,

$$E_n^{(1)}(f) \leqslant \frac{c(r)}{n^r}\omega\big(f^{(r)}, n^{-1}\big), \quad n \geqslant r.$$

Furthermore, for each function $f \in \Delta^{(1)}$, DeVore [DeV76] proved the estimate for the second modulus of smoothness,

$$E_n^{(1)}(f) \leqslant c\omega_2\left(f, n^{-1}\right), \quad n \geqslant 2.$$

It is impossible to replace ω_2 by ω_3 in the above estimate due to a negative result by Shvedov (see [Shv81a]), who proved that, for each $A > 0$ and $n \in \mathbb{N}$, there is a function $f = f_{n,A} \in \Delta^{(q)}$ such that

$$E_n^{(q)}(f) \geqslant A\omega_{q+2}\left(f, 1\right). \tag{10}$$

Newman [New79] obtained the first "optimal" estimate for comonotone approximation (earlier results on comonotone approximation are due to Newman, Passow and Raymon [NPR72], Passow, Raymon and Roulier [PRR74], Passow and Raymon [PR74], and Iliev [Ili78a]). He showed that, if $f \in \Delta^{(1)}(Y_s)$, then

$$E_n^{(1)}(f, Y_s) \leqslant c(s)\omega\left(f, n^{-1}\right), \quad n \geqslant 1.$$

Shvedov [Shv81b] proved that, if $f \in \Delta^{(1)}(Y_s)$, then

$$E_n^{(1)}(f, Y_s) \leqslant c(s)\omega_2\left(f, n^{-1}\right), \quad n \geqslant \mathcal{N},$$

where $\mathcal{N} = \mathcal{N}(Y_s)$, and that this estimate is no longer valid with $\mathcal{N}$ independent of Y_s.

We also mention the following pre-1980 papers which are somewhat related to the topics discussed in this survey: Roulier [Rou68,Rou71,Rou73,Rou75,Rou76], Lorentz and Zeller [LZ70], Lim [Lim71], R. Lorentz [Lor71], Zeller [Zel73], DeVore [DeV74], Popov and Sendov [PS74], Gehner [Geh75], Kimchi and Leviatan [KL76], Passow and Roulier [PR76], Ishisaki [Ish77], Iliev [Ili78b], and Myers and Raymon [MR78].

6 Unconstrained polynomial approximation

In this section, we remind the reader of the direct and inverse theorems for polynomial approximation. In particular, we emphasize again that, in order to get matching direct and inverse estimates, one should use (i) pointwise estimates involving the usual moduli of smoothness $\omega_k\left(f^{(r)}, \rho_n(x)\right)$, or (ii) uniform estimates in terms of generalized Ditzian-Totik moduli of smoothness $\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})$. Jackson-Stechkin type estimates (see Corollary 2 below) are NOT optimal in this sense.

6.1 Nikolskii type pointwise estimates

In 1946, Nikolskii [Nik46] showed that, for any function f such that $\omega(f,\delta) \leqslant \delta$, it is possible to construct a sequence of polynomials $P_n \in \mathbb{P}_n$ such that, for all $x \in [-1,1]$,

$$|f(x) - P_n(x)| \leqslant \frac{\pi}{2} \cdot \frac{\sqrt{1-x^2}}{n} + |x| O\left(\frac{\ln n}{n^2}\right).$$

Towards the end of the 1960's, the constructive theory of approximation of functions by algebraic polynomials was completed.

Timan [Tim51] (for $k = 1$), Dzjadyk [Dzy58] and, independently, Freud [Fre59] (for $k = 2$), and Brudnyi [Bru63] (for $k \geqslant 3$) proved the following direct theorem for approximation by algebraic polynomials involving Nikolskii type pointwise estimates.

Theorem 1 (Direct theorem: pointwise estimates)**.** *Let $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$. If $f \in C^r[-1,1]$, then for each $n \geqslant k+r$ there is a polynomial $P_n \in \mathbb{P}_n$, such that for every $x \in [-1,1]$,*

$$|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\big(f^{(r)}, \rho_n(x)\big).$$

This pointwise estimate readily implies the classical Jackson-Stechkin type estimate.

Corollary 2 (Jackson-Stechkin type estimates)**.** *If $f \in C^r[-1,1]$, then*

$$E_n(f) \leqslant \frac{c(k,r)}{n^r}\omega_k\big(f^{(r)}, n^{-1}\big), \quad n \geqslant k+r.$$

Trigub [Tri62] ($k = 1$) and Gopengauz [Gop67] ($k \geqslant 1$) proved the generalization of Theorem 1 for the simultaneous approximation of a function and its derivatives by a polynomial and its corresponding derivatives. (For $k = 1$ and $1/n$ instead of $\rho_n(x)$, this result was proved by Gelfond [Gel55].)

Theorem 3 (Simultaneous approximation of a function and its derivatives: pointwise estimates)**.** *Let $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$. If $f \in C^r[-1,1]$, then for each $n \geqslant k+r$ there is a polynomial $P_n \in \mathbb{P}_n$ satisfying, for $0 \leqslant \nu \leqslant r$ and $x \in [-1,1]$,*

$$|f^{(\nu)}(x) - P_n^{(\nu)}(x)| \leqslant c(k,r)\rho_n^{r-\nu}(x)\omega_k\big(f^{(r)}, \rho_n(x)\big).$$

The following stronger result on simultaneous polynomial approximation, Theorem 4, first appeared in [Kop96]. We also note that, while Theorem 4 was not stated in [She92b], it can be proved similarly to [She92b, Theorem 15.3] using [She92b, Lemmas 15.3 and 4.2′].

Theorem 4 (Simultaneous approximation of a function and its derivatives: pointwise estimates)**.** *Let $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$. If $f \in C^r[-1,1]$, then for each $n \geqslant k+r$ there is a polynomial $P_n \in \mathbb{P}_n$ satisfying, for $0 \leqslant \nu \leqslant r$ and $x \in [-1,1]$,*

$$|f^{(\nu)}(x) - P_n^{(\nu)}(x)| \leqslant c(k,r)\omega_{k+r-\nu}\big(f^{(\nu)}, \rho_n(x)\big).$$

Remark. One of the consequences of Theorem 4 is that if $k, q \in \mathbb{N}$, $f \in C^q[-1,1]$ and $f^{(q)}$ is *strictly* positive on $[-1,1]$, then, for sufficiently large n (depending on k, q and f), there exists a polynomial $P_n \in \mathbb{P}_n$ with a positive q-th derivative on $[-1,1]$ (*i.e.*, $P_n \in \Delta^{(q)}$) such that

$$|f(x) - P_n(x)| \leqslant c(k,q)\omega_k\big(f, \rho_n(x)\big).$$

We call a function ϕ a k-majorant (and write $\phi \in \Phi^k$) if it satisfies the following conditions:

(i) $\phi \in C[0,\infty)$, $\phi(0) = 0$,

(ii) ϕ is nondecreasing on $(0,\infty)$,

(iii) $x^{-k}\phi(x)$ is nonincreasing on $(0,\infty)$.

Theorem 5 (Inverse theorem for pointwise estimates: Dzjadyk [Dzy56], Timan [Tim57], Lebed' [Leb57], Brudnyi [Bru59]). *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $\phi \in \Phi^k$, and let f be a given function. If for every $n \geqslant k+r$ there exists $P_n \in \mathbb{P}_n$ such that*

$$|f(x) - P_n(x)| \leqslant \rho_n^r(x)\phi\left(\rho_n(x)\right), \quad x \in [-1,1],$$

then

$$\omega_k(f^{(r)},\delta) \leqslant c(k,r)\left(\int_0^\delta r u^{-1}\phi(u)\,\mathrm{d}u + \delta^k \int_\delta^1 u^{-k-1}\phi(u)\,\mathrm{d}u\right), \quad 0 \leqslant \delta \leqslant 1/2.$$

In particular, if $\int_0^1 r u^{-1}\phi(u)\,\mathrm{d}u < \infty$, then $f \in C^r[-1,1]$.

Theorems 1 and 5 (with $\phi(u) := u^\alpha$) imply the following result.

Corollary 6 (Constructive characterization: pointwise estimates). *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $0 < \alpha < k$, and let f be a given function. Then, for every $n \geqslant k+r$ there exist $P_n \in \mathbb{P}_n$ such that*

$$|f(x) - P_n(x)| \leqslant c(k,r,\alpha)\rho_n^{r+\alpha}(x), \quad x \in [-1,1],$$

if and only if $f \in C^r[-1,1]$ and

$$\omega_k(f^{(r)},\delta) \leqslant c(k,r,\alpha)\delta^\alpha, \quad 0 \leqslant \delta \leqslant 1/2.$$

6.2 Ditzian-Totik type estimates

As discussed above, Jackson-Stechkin type estimates are rather weak in the sense that they do not provide a constructive characterization of classes of functions having prescribed order of approximation by algebraic polynomials. One has to measure smoothness taking into account the distance from the endpoints of $[-1,1]$ in order to get this characterization.

Theorem 7 (Direct theorem: Ditzian and Totik [DT87]). *If $f \in C[-1,1]$ and $k \in \mathbb{N}$, then for each $n \geqslant k$,*

$$E_n(f) \leqslant c(k)\omega_k^\varphi\left(f, n^{-1}\right),$$

where $\omega_k^\varphi(f,\delta) = \sup_{0\leqslant h\leqslant\delta}\|\Delta^k_{h\varphi(\cdot)}(f,\cdot)\|$ and $\varphi(x) := \sqrt{1-x^2}$.

The generalized Ditzian-Totik moduli $\omega^\varphi_{k,r}(f^{(r)},\delta)$ discussed below in this section were introduced in [She92b].

For $r \geqslant 1$, we say that $f \in \mathbb{B}^r$ if $f^{(r-1)}$ is locally absolutely continuous in $(-1,1)$ and $\varphi^r f^{(r)} \in \mathbb{L}_\infty[-1,1]$. Also, let $C^0_\varphi := C[-1,1]$ and

$$C^r_\varphi = \left\{ f \in C^r(-1,1) \ : \ \lim_{x\to\pm1}\varphi^r(x) f^{(r)}(x) = 0 \right\}, \quad r \geqslant 1.$$

For $f \in C^r_\varphi$, $r \geqslant 0$, we denote

$$\omega^\varphi_{k,r}(f^{(r)},\delta) := \sup_{0\leqslant h\leqslant\delta}\ \sup_{x:|x|+\frac{kh}{2}\varphi(x)<1} K^r\left(x,\frac{kh}{2}\right)|\Delta^k_{h\varphi(x)}(f^{(r)},x)|,$$

where $K(x,\mu) := \varphi(|x| + \mu\varphi(x))$.

Note that if $r = 0$, then

$$\omega^{\varphi}_{k,0}(f,\delta) \equiv \omega^{\varphi}_{k}(f,\delta),$$

where $\omega^{\varphi}_{k}(f,\delta)$ is the kth Ditzian-Totik modulus of smoothness defined above.

Clearly $C^r_\varphi \subset \mathbb{B}^r$, while it is known (see, *e.g.*, [DS08, Chapter 3.10]) that if $f \in \mathbb{B}^r$, then $f \in C^l_\varphi$ for all $0 \leqslant l < r$, and

$$\omega^{\varphi}_{r-l,l}(f^{(l)},\delta) \leqslant c\delta^{r-l} \left\| \varphi^r f^{(r)} \right\|_{\mathbb{L}_\infty[-1,1]}, \quad \delta > 0. \tag{11}$$

Note that if $f \in C^r_\varphi$, then the following inequality holds for all $0 \leqslant l \leqslant r$ and $k \geqslant 1$ (see [DS08, Chapter 3.10]):

$$\omega^{\varphi}_{k+r-l,l}(f^{(l)},\delta) \; \leqslant \; c\,\delta^{r-l}\omega^{\varphi}_{k,r}(f^{(r)},\delta), \quad \delta > 0. \tag{12}$$

By virtue of (12), Theorem 7 immediately implies the following.

Corollary 8 (Direct theorem)**.** *If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in C^r_\varphi$, then for each $n \geqslant k + r$,*

$$E_n(f) \leqslant c(k,r) n^{-r} \omega^{\varphi}_{k,r}\big(f^{(r)}, n^{-1}\big).$$

A matching inverse theorem is the following generalization of [DT87, Theorem 7.2.4] in the case $p = \infty$ (see [She92b] or [KLS10, Theorem 3.2]).

Theorem 9 (Inverse theorem for uniform estimates)**.** *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $\mathcal{N} \in \mathbb{N}$, and let $\phi : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\phi(0+) = 0$ and*

$$\int_0^1 \frac{r\phi(u)}{u^{r+1}}\,\mathrm{d}u < +\infty.$$

If

$$E_n(f) \leqslant \phi\left(n^{-1}\right), \quad \textit{for all} \quad n \geqslant \mathcal{N},$$

then $f \in C^r_\varphi$, and, for any $0 \leqslant \delta \leqslant 1/2$,

$$\omega^{\varphi}_{k,r}(f^{(r)},\delta) \leqslant c(k,r) \int_0^\delta \frac{r\phi(u)}{u^{r+1}}\,\mathrm{d}u + c(k,r)\delta^k \int_\delta^1 \frac{\phi(u)}{u^{k+r+1}}\,\mathrm{d}u + c(k,r,\mathcal{N})\delta^k E_{k+r}(f).$$

If, in addition, $\mathcal{N} \leqslant k + r$, then the following Bari–Stechkin type estimate holds:

$$\omega^{\varphi}_{k,r}(f^{(r)},\delta) \leqslant c(k,r) \int_0^\delta \frac{r\phi(u)}{u^{r+1}}\,\mathrm{d}u + c(k,r)\delta^k \int_\delta^1 \frac{\phi(u)}{u^{k+r+1}}\,\mathrm{d}u, \quad \delta \in [0,1/2]\,.$$

If $\phi(u) := u^\alpha$ we get the following corollary (see [KLS10, Theorem 3.3]).

Corollary 10. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $\alpha > 0$, be such that $r < \alpha < k + r$. If*

$$n^\alpha E_n(f) \leqslant 1, \quad \textit{for all} \quad n \geqslant \mathcal{N},$$

where $\mathcal{N} \geqslant k + r$, then $f \in C^r_\varphi$ and

$$\omega^{\varphi}_{k,r}(f^{(r)},\delta) \leqslant c(\alpha,k,r)\delta^{\alpha-r} + c(\mathcal{N},k,r)\delta^k E_{k+r}(f)\,.$$

In particular, if $\mathcal{N} = k + r$, then

$$\omega^{\varphi}_{k,r}(f^{(r)},\delta) \leqslant c(\alpha,k,r)\delta^{\alpha-r}\,.$$

Corollaries 8 and 10 imply the following result.

Corollary 11 (Constructive characterization: uniform estimates). *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $\alpha > 0$, be such that $r < \alpha < k + r$. Then,*

$$E_n(f) \leqslant c(k,r,\alpha)n^{-\alpha}, \quad \textit{for all} \quad n \geqslant k+r,$$

if and only if $f \in C^r_\varphi$ and

$$\omega^\varphi_{k,r}(f^{(r)},\delta) \leqslant c(k,r,\alpha)\delta^{\alpha-r}\,.$$

7 Jackson-Stechkin type estimates for q-monotone approximation, $q \geqslant 1$

7.1 q-monotone approximation of functions from $C[-1,1]$ $(r=0)$

In this section, we discuss the validity of the following statement (note that this is the case $r = 0$ of the more general Statement 2 discussed in Section 7.2).

Statement 1. Let $q \in \mathbb{N}$, $k \in \mathbb{N}$ and $\mathcal{N} \in \mathbb{N}$. If $f \in \Delta^{(q)} \cap C[-1,1]$, then

$$E^{(q)}_n(f) \leqslant c(k,q)\omega_k\left(f,n^{-1}\right), \quad n \geqslant \mathcal{N}. \tag{13}$$

7.1.1 Case "+"

Case "+" is the case when (13) holds with $\mathcal{N} = k + r$.

For $(k = 1, q = 1)$, the estimate (13) with $\mathcal{N} = 1$ was proved by Lorentz and Zeller [LZ68]. Beatson [Bea78] proved (13) for $k = 1$ and all q. For $(k = 2, q = 1)$, (13) with $\mathcal{N} = 2$ was established by DeVore [DeV76]. Shevdov [Shv80] later extended it to $k = 2$ and all $q \geqslant 1$.

So, for the first and second moduli of smoothness, we have exactly the same estimate as in the unconstrained approximation. Thus the cases $(k = 1, q \in \mathbb{N})$ and $(k = 2, q \in \mathbb{N})$ are of type "+". Only two other cases of type "+" are known. Namely, Hu, Leviatan and Yu [HLY94] and, independently, Kopotun [Kop94] proved (13) in the case $(k = 3, q = 2)$, with $\mathcal{N} = 3$, and Bondarenko [Bon02] proved (13) for $(k = 3, q = 3)$, with $\mathcal{N} = 3$.

7.1.2 Case "−"

Wu and Zhou [WZ92] (for $k \geqslant q + 3$ and $q \geqslant 1$), and Bondarenko and Prymak [BP04] (for $k \geqslant 3$ and $q \geqslant 4$), proved that there is a function $f \in \Delta^{(q)}$, such that

$$\limsup_{n\to\infty} \frac{E^{(q)}_n(f)}{\omega_k\left(f, 1/n\right)} = \infty. \tag{14}$$

In other words, in the cases $(k \geqslant q + 3, q \geqslant 1)$ and $(k \geqslant 3, q \geqslant 4)$, estimate (13) is not valid in general even if $\mathcal{N}$ is allowed to depend on f.

7.1.3 Case "$\ominus$"

Shvedov's negative result (see [Shv81a]) implies that, in the cases $(k = 3, q = 1)$, $(k = 4, q = 2)$, and $(k = 5, q = 3)$, for each $A > 0$ and $n \in \mathbb{N}$ there is a function $f = f_{n,A} \in \Delta^{(q)}$ such that

$$E_n^{(q)}(f) \geqslant A\omega_k(f, 1)\,. \tag{15}$$

At the same time, positive results for the first two of these cases were proved in [LS98] and [LS03]. Namely, it was shown that, in the cases $(k = 3, q = 1)$ and $(k = 4, q = 2)$, if $f \in \Delta^{(q)}$, then (13) holds with a constant $\mathcal{N}$ that depends on the function f. We emphasize that (15) implies that, in these cases, (13) cannot be valid with the constant $\mathcal{N}$ independent of f.

We refer to all such cases as "$\ominus$". Namely, we denote by "$\ominus$" the cases when (13) holds with $\mathcal{N}$ that depends on f (this means that there is no f for which (14) is valid) and does not hold in general with $\mathcal{N}$ independent of f.

All of the above cases are conveniently summarized in Table 1.

q	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	+	+	−	−	−	−	−	$\cdots$
4	+	+	−	−	−	−	−	$\cdots$
3	+	+	+	?	?*	−	−	$\cdots$
2	+	+	+	$\ominus$	−	−	−	$\cdots$
1	+	+	$\ominus$	−	−	−	−	$\cdots$
	1	2	3	4	5	6	7	k

Table 1: q-monotone approx. of functions from $C[-1,1]$ $(r = 0)$, validity of $E_n^{(q)}(f) \leqslant c(k,q)\omega_k\left(f, n^{-1}\right)$, $n \geqslant \mathcal{N}$

Remark. It follows from (15) that "?*" in Table 1 cannot be replaced by "+".

We therefore have two problems related to (13).

Open Problem 1. *Does there exist a function $f \in C[-1,1]$ with a convex derivative on $(-1,1)$, that is, $f \in \Delta^{(3)}$, such that, for each sequence $\{P_n\}_{n=1}^{\infty} \subset \mathbb{P}_n$ of algebraic polynomials satisfying*

$$P_n^{(3)}(x) \geqslant 0,$$

we have

$$\limsup_{n\to\infty} \frac{\|f - P_n\|_{C[-1,1]}}{\omega_5(f, 1/n)} = \infty?$$

In other words, is the case $(k = 5, q = 3)$ strongly negative ("−")?

Open Problem 2. *What can be said if ω_5 in* Open Problem 1 *is replaced by ω_4?*

7.2 q-monotone approximation of functions from $C^r[-1,1]$ and $\mathbb{W}^r$, $r \geqslant 0$

In this section, we discuss the validity of the following Statement 2, and ask for which triples (k,r,q) this statement is valid and for which it is invalid (note that the case $r = 0$ was already considered in Statement 1 in Section 7.1).

Statement 2. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $q \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$ and $f \in \Delta^{(q)} \cap C^r[-1,1]$, then

$$E_n^{(q)}(f) \leqslant \frac{c(k,r,q)}{n^r}\omega_k\big(f^{(r)}, n^{-1}\big), \quad n \geqslant \mathcal{N}. \tag{16}$$

Recall that an estimate similar to (16) is valid with $\mathcal{N} = k + r$ in the unconstrained case. Also we remind the reader that we say that, for the triple (k,r,q), Statement 2 is

- "strongly positive" ("+") if (16) holds with $\mathcal{N} = k + r$,
- "weakly negative" ("$\ominus$") if (16) holds with $\mathcal{N} = \mathcal{N}(f)$ and is not valid with $\mathcal{N}$ independent of f,
- "strongly negative" ("$-$") if (16) is not valid at all, that is, there is a function $f \in \Delta^{(q)} \cap C^r[-1,1]$ such that
$$\limsup_{n\to\infty} \frac{n^r E_n^{(q)}(f)}{\omega_k\left(f^{(r)}, n^{-1}\right)} = \infty. \tag{17}$$

For completeness, we also consider the case $k = 0$ in (16) requiring that f belong to $\mathbb{W}^r$, the space of $(r-1)$ times continuously differentiable functions f in $[-1,1]$ such that $f^{(r-1)}$ is absolutely continuous in $(-1,1)$ and $\|f^{(r)}\|_{\mathbb{L}_\infty[-1,1]} < \infty$.

Statement 3 ($k = 0$)**.** If $r \in \mathbb{N}$, $q \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(q)} \cap \mathbb{W}^r$, then

$$E_n^{(q)}(f) \leqslant \frac{c(r,q)}{n^r}\|f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}, \quad n \geqslant \mathcal{N}.$$

The following are the "truth tables" for Statements 2 and 3.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	+	$\cdots$
1	+	+	+	+	+	+	$\cdots$
0		+	+	$\ominus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Table 2: Monotone approx. ($q = 1$), validity of $E_n^{(1)}(f) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

These results appeared in the papers by Lorentz and Zeller [LZ68], Lorentz [Lor72], DeVore [DeV76] and [DeV77], Shvedov [Shv79], Wu and Zhou [WZ92], Shevchuk [She92a] (see also [She89]), and Leviatan and Shevchuk [LS98].

It is convenient to summarize the above references in another table. Note that, in the case "$\ominus$", we first put a reference to the negative result, then to the positive one.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
3	[DeV77]	[DeV77]	[She92a]	[She92a]	[She92a]	[She92a]	$\cdots$
2	[Lor72]	[DeV77]	[She92a]	[She92a]	[She92a]	[She92a]	$\cdots$
1	[LZ68]	[Lor72]	[She92a]	[She92a]	[She92a]	[She92a]	$\cdots$
0		[LZ68]	[DeV76]	[Shv79], [LS98]	[WZ92]	[WZ92]	$\cdots$
	0	1	2	3	4	5	k

Table 2a: References for Table 2

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
3	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
1	$+$	$+$	$+$	$\ominus$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$+$	$\ominus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	6	k

Table 3: Convex approx. $(q = 2)$, validity of $E_n^{(2)}(f) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

These results appeared in the papers by Beatson [Bea78], Shvedov [Shv81a], Wu and Zhou [WZ92], Mania (see in [She92b, Theorems 17.2 and 16.1]), Hu, Leviatan and Yu [HLY94], Kopotun [Kop94], Nissim and Yushchenko [NY03], and Leviatan and Shevchuk [LS03].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
3	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	$\cdots$
2	[Shv81a]	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	$\cdots$
1	[Bea78]	[Shv81a]	[HLY94, Kop94]	[She92b], [LS03]	[NY03]	[NY03]	[NY03]	$\cdots$
0		[Bea78]	[Shv81a]	[HLY94, Kop94]	[Shv81a], [LS03]	[WZ92]	[WZ92]	$\cdots$
	0	1	2	3	4	5	6	k

Table 3a: References for Table 3

Remark. The cases $(k = 3, r = 0, q = 2)$ and $(k = 2, r = 1, q = 2)$ (both of type "+"), were proved in [HLY94] and [Kop94] simultaneously and independently.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$-$	$-$	$-$	$-$	$-$	$\cdots$
3	$-$	$-$	$-$	$-$	$-$	$\cdots$
2	$+$	$-$	$-$	$-$	$-$	$\cdots$
1	$+$	$+$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$-$	$-$	$\cdots$
	0	1	2	3	4	k

Table 4: q-monotone approx. ($q \geqslant 4$), validity of $E_n^{(q)}(f) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

The results appeared in the papers by Beatson [Bea78], Shvedov [Shv80], and Bondarenko and Prymak [BP04].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	[BP04]	[BP04]	[BP04]	[BP04]	[BP04]	$\cdots$
3	[BP04]	[BP04]	[BP04]	[BP04]	[BP04]	$\cdots$
2	[Bea78, Shv80]	[BP04]	[BP04]	[BP04]	[BP04]	$\cdots$
1	[Bea78]	[Bea78, Shv80]	[BP04]	[BP04]	[BP04]	$\cdots$
0		[Bea78]	[Shv80]	[BP04]	[BP04]	$\cdots$
	0	1	2	3	4	k

Table 4a: References for Table 4

Remark. It is worth mentioning that the breakthrough in the surprising negative results is due to Konovalov and Leviatan [KL03] who proved that the negative assertion of "$\ominus$" is valid for all "$-$" entries in Table 4, breaking the Shvedov [Shv81a] pattern (compare with (10)). In fact, the negative results in [KL03] are of width type and are valid for any increasing sequence of n-dimensional linear subspaces of $C[-1, 1]$.

Finally, in the case of 3-monotone approximation, the following is known.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	?	?	?	?	?	?	?	$\cdots$
4	?	?	?	?	?	?	?	$\cdots$
3	+	?	?	?	?	?	?	$\cdots$
2	+	+	?	?*	−	−	−	$\cdots$
1	+	+	+	?	?*	−	−	$\cdots$
0		+	+	+	?	?*	−	$\cdots$
	0	1	2	3	4	5	6	k

Table 5: 3-monotone approx. $(q = 3)$, validity of $E_n^{(3)}(f) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

Remark. It follows from Shvedov [Shv81a] and Mania (see [She92b]) that "?*" in the cases $(k = 5 - r, 0 \leqslant r \leqslant 2)$ in Table 5 cannot be replaced by "+".

These results appeared in the papers by Beatson [Bea78], Shvedov [Shv80, Shv81a], Bondarenko [Bon02], Mania (see [She92b, Theorem 16.1]), Nissim and Yushchenko [NY03], and Wu and Zhou [WZ92].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	?	?	?	?	?	?	?	$\cdots$
4	?	?	?	?	?	?	?	$\cdots$
3	[Bon02]	?	?	?	?	?	?	$\cdots$
2	[Bea78, Shv80]	[Bon02]	?	[She92b]	[NY03]	[NY03]	[NY03]	$\cdots$
1	[Bea78]	[Bea78, Shv80]	[Bon02]	?	[She92b]	[NY03]	[NY03]	$\cdots$
0		[Bea78]	[Shv80]	[Bon02]	?	[Shv81a]	[WZ92]	$\cdots$
	0	1	2	3	4	5	6	k

Table 5a: References for Table 5

Obviously, we would like to replace the question marks in Table 5 with definitive answers. We emphasize that it seems to be a hard problem to replace the question mark in *any* place with a definitive symbol.

Open Problem 3. *Determine which symbol* (*among "+", "$\ominus$" and "−"*) *replaces "?" or "?*" in* Table 5 *in any of the places.*

8 Nikolskii type pointwise estimates for q-monotone approximation, $q \geqslant 1$

In this section, we discuss the validity of the following statement on pointwise estimates for q-monotone polynomial approximation.

Statement 4. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $q \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(q)} \cap C^r[-1,1]$, then there is a sequence $\{P_n\}_{n=1}^\infty$ of polynomials $P_n \in \Delta^{(q)} \cap \mathbb{P}_n$ such that, for every $n \geqslant \mathcal{N}$ and each $x \in [-1,1]$, we have

$$|f(x) - P_n(x)| \leqslant c(k,r,q)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right). \tag{18}$$

Recall that the case "+" means that Statement 4 is valid with $\mathcal{N} = k + r$. On the other hand, note that the case "−" here means (compare with (17)) that there exists a function $f \in \Delta^{(q)} \cap C^r[-1,1]$ such that, for every sequence of polynomials $P_n \in \mathbb{P}_n \cap \Delta^{(q)}$,

$$\limsup_{n\to\infty} E_{n,r,k}^{(q)}(f) := \limsup_{n\to\infty} \left\| \frac{f - P_n}{\rho_n^r \omega_k\left(f^{(r)}, \rho_n\right)} \right\| = \infty. \tag{19}$$

For $k = 0$, similarly to what was done in Section 7.2, we have the following modification of Statement 4.

Statement 5 ($k = 0$)**.** If $r \in \mathbb{N}$, $q \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(q)} \cap \mathbb{W}^r$, then there is a sequence $\{P_n\}_{n=1}^\infty$ of polynomials $P_n \in \Delta^{(q)} \cap \mathbb{P}_n$ such that, for every $n \geqslant \mathcal{N}$ and each $x \in [-1,1]$, we have

$$|f(x) - P_n(x)| \leqslant c(r,q)\rho_n^r(x)\|f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}. \tag{20}$$

Note that "+" now means that (20) is true with $\mathcal{N} = r$, and "−" means that, for every sequence of polynomials $P_n \in \mathbb{P}_n \cap \Delta^{(q)}$,

$$\limsup_{n\to\infty} E_{n,r,0}^{(q)}(f) := \limsup_{n\to\infty} \left\| \frac{f - P_n}{\rho_n^r} \right\| = \infty. \tag{21}$$

Since

$$\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right), \tag{22}$$

all positive estimates in all tables in this section imply positive estimates in the corresponding places (in the corresponding tables) in Section 7.2 (and all negative results in tables in Section 7.2 imply negative results in the corresponding places in the corresponding tables in this section).

The following are the "truth tables" for Statements 4 and 5 for $q = 1$, $q = 2$, $q \geqslant 4$ and $q = 3$ (the most difficult case).

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	$\cdots$
1	+	+	+	+	+	$\cdots$
0		+	+	−	−	$\cdots$
	0	1	2	3	4	k

Table 6: Monotone approx. ($q = 1$), validity of $|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(1)}$

These results appeared in the papers by Lorentz and Zeller [LZ68], DeVore and Yu [DY85], Shevchuk [She92a] (see also [She89]), Wu and Zhou [WZ92], and Leviatan and Shevchuk [LS98].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	[She92a]	[She92a]	[She92a]	[She92a]	[She92a]	$\cdots$
2	[DY85]	[She92a]	[She92a]	[She92a]	[She92a]	$\cdots$
1	[LZ68]	[DY85]	[She92a]	[She92a]	[She92a]	$\cdots$
0		[LZ68]	[DY85]	[LS98]	[WZ92]	$\cdots$
	0	1	2	3	4	k

Table 6a: References for Table 6

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	+	$\cdots$
1	+	+	+	−	−	−	$\cdots$
0		+	+	+	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 7: Convex approx. ($q = 2$), validity of $|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(2)}$

The results appear in the papers by Beatson [Bea80], Leviatan [Lev86], Mania and Shevchuk (see in [She92b, Theorem 17.2]), Kopotun [Kop94], Wu and Zhou [WZ92], and Yushchenko [Yus00].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	$\cdots$
2	[Lev86]	[She92b]	[She92b]	[She92b]	[She92b]	[She92b]	$\cdots$
1	[Bea80]	[Lev86]	[Kop94]	[Yus00]	[Yus00]	[Yus00]	$\cdots$
0		[Bea80]	[Lev86]	[Kop94]	[Yus00]	[WZ92]	$\cdots$
	0	1	2	3	4	5	k

Table 7a: References for Table 7

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	$-$	$-$	$-$	$-$	$-$	$\cdots$
2	$+$	$-$	$-$	$-$	$-$	$\cdots$
1	$+$	$+$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$-$	$-$	$\cdots$
	0	1	2	3	4	k

Table 8: q-monotone approx. ($q \geqslant 4$), validity of $|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(q)}$

These results appeared in the papers by Beatson [Bea80], Cao and Gonska [CG94], and Bondarenko and Prymak [BP04].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	[BP04]	[BP04]	[BP04]	[BP04]	[BP04]	$\cdots$
2	[CG94]	[BP04]	[BP04]	[BP04]	[BP04]	$\cdots$
1	[Bea80]	[CG94]	[BP04]	[BP04]	[BP04]	$\cdots$
0		[Bea80]	[CG94]	[BP04]	[BP04]	$\cdots$
	0	1	2	3	4	k

Table 8a: References for Table 8

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	?*	?*	?*	?*	?*	?*	?*	$\cdots$
4	?	?*	?*	?*	?*	?*	?*	$\cdots$
3	?	?	?*	?*	?*	?*	?*	$\cdots$
2	$+$	?	?	$-$	$-$	$-$	$-$	$\cdots$
1	$+$	$+$	?	?	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	?	?	$-$	$-$	$\cdots$
	0	1	2	3	4	5	6	k

Table 9: 3-monotone approx. ($q = 3$), validity of $|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(3)}$

Remark. It follows from Bondarenko and Gilewicz [BG09] that "?*" in the cases ($k \geqslant 0, r \geqslant 5$), ($k \geqslant 1, r = 4$) and ($k \geqslant 2, r = 3$) in Table 9 cannot be replaced by "+".

These results appeared in the papers by Beatson [Bea80], Cao and Gonska [CG94], Wu and Zhou [WZ92], and Yushchenko [Yus00].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	[BG09]	[BG09]	[BG09]	[BG09]	[BG09]	[BG09]	[BG09]	$\cdots$
4	?	[BG09]	[BG09]	[BG09]	[BG09]	[BG09]	[BG09]	$\cdots$
3	?	?	[BG09]	[BG09]	[BG09]	[BG09]	[BG09]	$\cdots$
2	[CG94]	?	?	[Yus00]	[Yus00]	[Yus00]	[Yus00]	$\cdots$
1	[Bea80]	[CG94]	?	?	[Yus00]	[Yus00]	[Yus00]	$\cdots$
0		[Bea80]	[CG94]	?	?	[Yus00]	[WZ92]	$\cdots$
	0	1	2	3	4	5	6	k

Table 9a: References for Table 9

Again, we would like to replace the question marks with definitive answers.

Open Problem 4. *Determine which symbol (among "+", "⊖" and "−") replaces "?" or "?*" in* Table 9 *in any of the places.*

Remark. For the triples ($k \leqslant 2, r \leqslant 2-k, q \geqslant 1$), DeVore and Yu [DY85] (for $q=1$), Leviatan [Lev86] (for $q=2$) and Cao and Gonska [CG94] (for $q \in \mathbb{N}$), proved that (18) and (20) are satisfied with $\rho_n(x)$ replaced by the smaller quantity $n^{-1}\varphi(x)$, so that, in particular, the polynomials interpolate the function at the endpoints. One can also achieve interpolation at the endpoints if $k \geqslant 3$ (see [Kop94, (8)], for example), but it is known that, in general, the quantity $\rho_n(x)$ cannot be replaced by $n^{-1}\varphi(x)$ (see, *e.g.,* [Yu85, Li86, Dah89, Kop96, GLSW00] for details).

9 Ditzian-Totik type estimates for q-monotone approximation, $q \geqslant 1$

In this section, we discuss the validity of the following statements.

Statement 6. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $q \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(q)} \cap C_\varphi^r$, then

$$E_n^{(q)}(f) \leqslant \frac{c(k,r,q)}{n^r}\omega_{k,r}^{\varphi}\big(f^{(r)}, n^{-1}\big), \quad n \geqslant \mathcal{N}.$$

If $k=0$, Statement 6 is modified as follows.

Statement 7. If $r \in \mathbb{N}$, $q \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(q)} \cap \mathbb{B}^r$, then

$$E_n^{(q)}(f) \leqslant \frac{c(r,q)}{n^r}\|\varphi^r f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}, \quad n \geqslant \mathcal{N}.$$

The following are the "truth tables" for Statements 6 and 7.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	+	+	+	+	+	+	$\cdots$
3	+	+	+	+	+	+	$\cdots$
2	+	$\ominus$	−	−	−	−	$\cdots$
1	+	+	$\ominus$	−	−	−	$\cdots$
0		+	+	$\ominus$	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 10: Monotone approx. ($q = 1$), validity of $E_n^{(1)}(f) \leqslant c(k,r)n^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

The results appear in the papers by Shvedov [Shv79], Leviatan [Lev86], Wu and Zhou [WZ92], Dzyubenko, Listopad and Shevchuk [DLS93], Kopotun and Listopad [KL94], Kopotun [Kop95], Leviatan and Shevchuk [LS98, LS00], and Nesterenko and Petrova [NP05].

Recalling that, in the case "$\ominus$", a reference to the negative result is followed by a reference to the positive one, we summarize all references for Table 10 in the following table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	[DLS93]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	$\cdots$
3	[DLS93]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	$\cdots$
2	[Lev86]	[KL94], [LS98]	[NP05]	[LS00]	[LS00]	[LS00]	$\cdots$
1	[Lev86]	[Lev86]	[KL94], [LS98]	[NP05]	[LS00]	[LS00]	$\cdots$
0		[Lev86]	[Lev86]	[Shv79], [LS98]	[WZ92]	[WZ92]	$\cdots$
	0	1	2	3	4	5	k

Table 10a: References for Table 10

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
6	+	+	+	+	+	+	+	$\cdots$
5	+	+	+	+	+	+	+	$\cdots$
4	$\ominus$	$\ominus$	−	−	−	−	−	$\cdots$
3	+	$\ominus$	$\ominus$	−	−	−	−	$\cdots$
2	+	+	$\ominus$	$\ominus$	−	−	−	$\cdots$
1	+	+	+	$\ominus$	−	−	−	$\cdots$
0		+	+	+	$\ominus$	−	−	$\cdots$
	0	1	2	3	4	5	6	k

Table 11: Convex approx. ($q = 2$), validity of $E_n^{(2)}(f) \leqslant c(k,r)n^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

We emphasize that Table 11 shows that we now have not only cases when Statement 6 is invalid, but also the case $(r = 4, k = 0)$ when Statement 7 is valid only with $\mathcal{N}$ that depends on the function f. This is in contrast with Table 10 for monotone approximation that did not contain "$\ominus$" in the column corresponding to $k = 0$.

These results appeared in the papers by Shvedov [Shv79], Leviatan [Lev86], Wu and Zhou [WZ92], Mania (see [She92b]), Kopotun [Kop92], [Kop94, Kop95], Leviatan and Shevchuk [LS03], Kopotun, Leviatan and Shevchuk [KLS05], and Nissim and Yushchenko [NY03].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
6	[Kop92]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	$\cdots$
5	[Kop92]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	[Kop95]	$\cdots$
4	[Kop92], [LS03]	[Kop92], [KLS05]	[KLS05]	[KLS05]	[KLS05]	[KLS05]	[KLS05]	$\cdots$
3	[Kop92]	[Kop92], [LS03]	[Kop92], [KLS05]	[KLS05]	[KLS05]	[KLS05]	[KLS05]	$\cdots$
2	[Lev86]	[Kop94]	[Kop92], [LS03]	[Kop92], [KLS05]	[KLS05]	[KLS05]	[KLS05]	$\cdots$
1	[Lev86]	[Lev86]	[Kop94]	[She92b], [LS03]	[NY03]	[NY03]	[NY03]	$\cdots$
0		[Lev86]	[Lev86]	[Kop94]	[Shv79], [LS03]	[WZ92]	[WZ92]	$\cdots$
	0	1	2	3	4	5	6	k

Table 11a: References for Table 11

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	?	?	?	?	?	?	?	$\cdots$
4	?	?	?	?	?	?	?	$\cdots$
3	+	?	?	?	?	?	?	$\cdots$
2	+	+	?	?*	−	−	−	$\cdots$
1	+	+	+	?	?*	−	−	$\cdots$
0		+	+	+	?	?*	−	$\cdots$
	0	1	2	3	4	5	6	k

Table 12: 3-monotone approx. $(q = 3)$, validity of $E_n^{(3)}(f) \leqslant c(k,r)n^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

Since

$$\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right) \leqslant \omega_k\left(f^{(r)}, n^{-1}\right), \tag{23}$$

all positive estimates in Table 12 imply positive estimates in corresponding places in Table 5 (and all negative results in Table 5 imply negative results in corresponding places in Table 12). At this time, Table 12 is identical with Table 5 (all cases "+" follow from the article by Bondarenko [Bon02], and all other cases are determined by corresponding entries in Table 5). However, we emphasize that there is no guarantee that once all question marks are replaced by definitive symbols these tables will remain identical.

Open Problem 5. *Determine which symbol (among "+", "⊖" and "−") replaces "?" or "?*" in* Table 12 *in any of the places.*

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$-$	$-$	$-$	$-$	$-$	$\cdots$
3	$-$	$-$	$-$	$-$	$-$	$\cdots$
2	$+$	$-$	$-$	$-$	$-$	$\cdots$
1	$+$	$+$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$-$	$-$	$\cdots$
	0	1	2	3	4	k

Table 13: q-monotone approx. ($q \geqslant 4$), validity of $E_n^{(q)}(f) \leqslant c(k,r)n^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

Because of (23), all cases "−" in Table 13 follow from corresponding "−"'s in Table 4. The cases "+" can be derived from results in the article by Gavrea, Gonska, Păltănea and Tachev [GGPT03], combined with the q-monotonicity preservation properties of the Gavrea operators (see Gavrea [Gav96]), appearing in the paper of Cottin, Gavrea, Gonska, Kacsó and Zhou [CGG+99].

10 Relations between degrees of best unconstrained and q-monotone approximation

Clearly, for each $f \in C[-1,1]$ and $q \geqslant 1$,

$$E_n(f) \leqslant E_n^{(q)}(f).$$

Moreover, Lorentz and Zeller [LZ68], proved that there is a function $f \in \Delta^{(q)} \cap C^q[-1,1]$, such that

$$\limsup_{n\to\infty} \frac{E_n^{(q)}(f)}{E_n(f)} = \infty. \tag{24}$$

It is well known that in the unconstrained case, for all $f \in C^r[-1,1]$, we have

$$E_n(f) \leqslant \frac{c(r)}{n^r} E_{n-r}(f^{(r)}), \quad n > r.$$

At the same time, it was shown in [LS95] and [She96] that, for each $n > q$, there is a function $f_n \in \Delta^{(q)} \cap C^q[-1,1]$, such that

$$E_n^{(q)}(f_n) > c(q) E_{n-q}(f_n^{(q)}), \quad c(q) > 0, \tag{25}$$

and so the (almost obvious) estimate (9) may not in general be improved.

Remark. It was proved in [BP04] that, if $q \geqslant 4$ and $r \geqslant q-1$, then for any nonnegative sequence $\{\alpha_n\}$ such that $\lim_{n\to\infty} \alpha_n = \infty$, there exists a function $f = f_{r,q} \in \Delta^{(q)} \cap C^r[-1,1]$ for which

$$\limsup_{n\to\infty} \alpha_n E_n^{(q)}(f) n^{r-q+3} = \infty. \tag{26}$$

Open Problem 6. *Given $q, r \in \mathbb{N}$ such that $r \geqslant q+1$, is it true that, for each $n > r$, there is a function $f_n \in \Delta^{(q)} \cap C^r[-1,1]$, such that*

$$E_n^{(q)}(f_n) > c(q,r)E_{n-r}(f_n^{(r)}), \quad c(q,r) > 0\,?$$

Open Problem 7. *Given $q, r \in \mathbb{N}$ such that $r \geqslant q+1$ and $1 \leqslant q \leqslant 3$, is it true that, for each $f \in \Delta^{(q)} \cap C^r[-1,1]$,*

$$E_n^{(q)}(f) \leqslant c(q,r)n^{-r}E_{n-r}(f^{(r)}), \quad n > r\,?$$

Note that, in the case $q \geqslant 4$, estimate (26) implies that the answer to the question posed in Open Problem 7 is "no".

Open Problem 8. *Given $q, r \in \mathbb{N}$, does there exist $f \in \Delta^{(q)} \cap C^r[-1,1]$ such that*

$$\limsup_{n\to\infty} \frac{E_n^{(q)}(f)}{E_{n-r}(f^{(r)})} > 0\,?$$

The following is a weaker version of Open Problem 8.

Open Problem 9. *Given $1 \leqslant q \leqslant 3$ and $r \geqslant q$, does there exist $f \in \Delta^{(q)} \cap C^r[-1,1]$ such that*

$$\limsup_{n\to\infty} \frac{n^r E_n^{(q)}(f)}{E_{n-r}(f^{(r)})} = \infty\,?$$

Note that, for $q \geqslant 4$, "$-$"'s in Table 4 imply that the answer to the question posed in Open Problem 9 is "yes". The answer is also "yes" if $0 \leqslant r < q \leqslant 3$ (this follows from the cases "$-$" in Tables 2, 3 and 5).

11 α-relations for q-monotone approximation

Notwithstanding (24), it was shown in [KLS09] for $q = 2$ (the case $q = 1$ is similar) that, for each $\alpha > 0$ and $f \in \Delta^{(q)}$, $1 \leqslant q \leqslant 2$,

$$n^\alpha E_n(f) \leqslant 1, \quad n \geqslant 1 \quad \Longrightarrow \quad n^\alpha E_n^{(q)}(f) \leqslant c(\alpha), \quad n \geqslant 1.$$

What happens if we only have the information on the left-hand side beginning from some fixed $\mathcal{N} \geqslant 2$? In other words, what can be said about the validity of the following statement?

Statement 8. Let $\alpha > 0$ and $f \in \Delta^{(q)}$, $1 \leqslant q \leqslant 2$. Then

$$n^\alpha E_n(f) \leqslant 1, \quad n \geqslant \mathcal{N} \quad \Longrightarrow \quad n^\alpha E_n^{(q)}(f) \leqslant c(\alpha, \mathcal{N}), \quad n \geqslant \mathcal{N}^*. \tag{27}$$

We are interested in determining all cases for which Statement 8 is valid and the exact dependence of $\mathcal{N}^*$ on the various parameters involved in (27), namely, α, $\mathcal{N}$, and perhaps f itself.

It turns out that the parameter that has an influence on the behavior of $\mathcal{N}^*$ is $\lceil \alpha/2 \rceil$, where $\lceil \mu \rceil$ is the ceiling function (*i.e.,* the smallest integer not smaller than μ).

$\lceil\alpha/2\rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	$\cdots$
2	+	+	+	+	$\cdots$
1	+	+	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	$\mathcal{N}$

Table 14: α-relations for monotone approx. ($q = 1$), validity of "$n^\alpha E_n(f) \leqslant 1,\ n \geqslant \mathcal{N} \implies n^\alpha E_n^{(1)}(f) \leqslant c(\alpha, \mathcal{N}),\ n \geqslant \mathcal{N}^*$"

$\lceil\alpha/2\rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	+	+	+	+	+	$\cdots$
3	+	+	+	+	+	$\cdots$
2	+	+	+	$\ominus$	$\ominus$	$\cdots$
1	+	+	+	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	$\mathcal{N}$

Table 15: α-relations for convex approx. ($q = 2$), validity of "$n^\alpha E_n(f) \leqslant 1,\ n \geqslant \mathcal{N} \implies n^\alpha E_n^{(2)}(f) \leqslant c(\alpha, \mathcal{N}),\ n \geqslant \mathcal{N}^*$"

We recall that "+" in Tables 14 and 15 means that $\mathcal{N}^* = \mathcal{N}$, and "$\ominus$" means that $\mathcal{N}^*$ depends on α, $\mathcal{N}$ and, in addition, must depend on f, that is, it cannot be independent of f. Note that Tables 14 and 15 show that (27) is always true, that is, Statement 8 is always valid (perhaps with $\mathcal{N}^*$ depending on f).

For $q = 2$, Table 15 was completed in [KLS09, KLS10]. For $q = 1$, one can use the same arguments to complete Table 14.

Open Problem 10. *Determine α-relations for 3-monotone approximation,* i.e., *construct a table analogous to Tables 14 and 15 in the case $q = 3$.*

In the case $q \geqslant 4$ and $\alpha > 2$ it is known that the implication (27) is invalid for any $\mathcal{N} \in \mathbb{N}$, since there exists a function $f = f_{\alpha,q} \in \Delta^{(q)}$ such that $E_n(f) \leqslant n^{-\alpha}$, $n \in \mathbb{N}$, but for some $\beta < \alpha$, $E_n^{(q)}(f) \geqslant c(\alpha, \beta, q) n^{-\beta}$, for infinitely many n.

Indeed, if $\alpha > q - 2 \geqslant 2$ then (26) (with $r := \lceil\alpha\rceil$ and $\alpha_n := n^\epsilon$) implies that there exists $f \in \Delta^{(q)} \cap C^r[-1, 1]$ such that $\limsup_{n\to\infty} E_n^{(q)}(f) n^{r-q+3+\epsilon} = \infty$, and so (if $\epsilon > 0$ is sufficiently small)

$$E_n^{(q)}(f) \geqslant n^{-r+q-3-\epsilon} \geqslant n^{-\alpha+\epsilon},$$

for infinitely many n. At the same time, $E_n(f) \leqslant cn^{-r} \leqslant cn^{-\alpha}$.

It was also shown in [BP04] that $E_n^{(q)}(x_+^{q-1}) \geqslant c(q) n^{-2}$ if $q \geqslant 4$. Therefore, for $2 < \alpha \leqslant q - 1$, $E_n(x_+^{q-1}) \leqslant cn^{-q+1} \leqslant cn^{-\alpha}$ and $E_n^{(q)}(x_+^{q-1}) \geqslant c(q) n^{-\alpha+\epsilon}$.

Open Problem 11. *Let $\alpha > 0$, $q \geqslant 4$, $f \in \Delta^{(q)}$ and $\mathcal{N} \in \mathbb{N}$. Determine the largest $\beta > 0$ such that*

$$n^\alpha E_n(f) \leqslant 1, \quad n \geqslant \mathcal{N} \quad \Longrightarrow \quad n^\beta E_n^{(q)}(f) \leqslant c(\alpha, \beta, q, \mathcal{N}), \quad n \geqslant \mathcal{N}^*,$$

and investigate the dependence of $\mathcal{N}^*$ on α, β, q, $\mathcal{N}$ and f.

12 Comonotone and coconvex approximation: introducing the case "$\oplus$"

For $s \in \mathbb{N}$, let $Y_s = \{y_i\}_{i=1}^s$ be a collection of s points $-1 < y_s < \cdots < y_1 < 1$, and denote $y_0 := 1$ and $y_{s+1} := -1$.

We say that $f \in \Delta^{(1)}(Y_s)$, if f is continuous on $[-1, 1]$, changes monotonicity at the points Y_s, and is nondecreasing on $[y_1, 1]$. In other words, $f \in C[-1, 1]$ is in $\Delta^{(1)}(Y_s)$ if it is nondecreasing on $[y_{2i+1}, y_{2i}]$ and nonincreasing on $[y_{2i}, y_{2i-1}]$.

We say that $f \in \Delta^{(2)}(Y_s)$, if $f \in C[-1, 1]$ and Y_s is its set of inflection points with f being convex in $[y_1, 1]$ (*i.e.*, f is convex on $[y_{2i+1}, y_{2i}]$ and concave on $[y_{2i}, y_{2i-1}]$).

Note that a function f may belong to more than one class $\Delta^{(q)}(Y_s)$, for $q = 1$ or $q = 2$, both for different sets Y_s (with the same number of changes s), and for different sets Y_s with a different number of changes. For example, this may happen when f is constant on a subinterval of $[-1, 1]$. Also f may belong to $\Delta^{(1)}(Y_{s_1})$ and to $\Delta^{(2)}(Y_{s_2})$.

In particular, if $f \in C^q(-1, 1) \cap C[-1, 1]$, then $f \in \Delta^{(q)}(Y_s)$ if and only if

$$f^{(q)}(x) \prod_{i=1}^s (x - y_i) \geqslant 0, \quad x \in (-1, 1).$$

For $f \in \Delta^{(q)}(Y_s)$, $q = 1, 2$, we define

$$E_n^{(q)}(f, Y_s) := \inf_{P_n \in \Delta^{(q)}(Y_s) \cap \mathbb{P}_n} \|f - P_n\|.$$

Newman [New79] proved the first strongly positive result (the case "+") showing that, if $f \in \Delta^{(1)}(Y_s)$, then

$$E_n^{(1)}(f, Y_s) \leqslant c(s)\omega\left(f, n^{-1}\right), \quad n \geqslant 1.$$

Furthermore, Shvedov [Shv81b] proved that, if $f \in \Delta^{(1)}(Y_s)$, then

$$E_n^{(1)}(f, Y_s) \leqslant c(s)\omega_2\left(f, n^{-1}\right), \quad n \geqslant \mathcal{N},$$

where $\mathcal{N} = \mathcal{N}(Y_s)$, and that this estimate is no longer valid with $\mathcal{N}$ independent of Y_s. Namely, for each $A > 0$, $n \in \mathbb{N}$ and $s \geqslant 1$, there are a collection Y_s and a function $f \in \Delta^{(1)}(Y_s)$ such that, for every polynomial $P_n \in \Delta^{(1)}(Y_s) \cap \mathbb{P}_n$,

$$\|f - P_n\|_{C[-1,1]} > A\omega_2\left(f, n^{-1}\right).$$

Thus, we arrive at a phenomenon we have not seen before: the constant $\mathcal{N}$ cannot depend only on k, r, s and q, and at the same time, it does not have to (fully) depend on the function f itself. Rather, it depends only on where f changes its monotonicity (the set Y_s). We refer to the cases where $\mathcal{N}$ depends on the location of the changes in monotonicity or convexity as "weakly positive" cases and denote them by "$\oplus$".

13 Comonotone approximation: uniform and pointwise estimates

13.1 Jackson-Stechkin type estimates for comonotone approximation

We begin by discussing the validity of the following statements.

Statement 9. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(1)}(Y_s) \cap C^r[-1,1]$, then

$$E_n^{(1)}(f, Y_s) \leqslant \frac{c(k,r,s)}{n^r}\omega_k\big(f^{(r)}, n^{-1}\big), \quad n \geqslant \mathcal{N}. \tag{28}$$

Statement 10. If $r \in \mathbb{N}$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(1)}(Y_s) \cap \mathbb{W}^r$, then

$$E_n^{(1)}(f, Y_s) \leqslant \frac{c(r,s)}{n^r}\|f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}, \quad n \geqslant \mathcal{N}.$$

We have the following truth tables.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	+	$\cdots$
1	+	+	+	$\oplus$	−	−	$\cdots$
0		+	$\oplus$	−	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 16: Comonotone approx. with $s = 1$, validity of $E_n^{(1)}(f, Y_1) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	+	+	+	+	+	+	$\cdots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	$\oplus$	$\oplus$	$\oplus$	$\cdots$
1	+	+	$\oplus$	$\oplus$	−	−	$\cdots$
0		+	$\oplus$	−	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 17: Comonotone approx. with $s = 2$, validity of $E_n^{(1)}(f, Y_2) \leqslant c(k,r)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$s+2$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$+$	$\cdots$
$s+1$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$+$	$\cdots$
s	$+$	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\cdots$
$s-1$	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
2	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\cdots$
1	$+$	$+$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$	$-$	$\cdots$
0		$+$	$\oplus$	$-$	$-$	$-$	$\cdots$	$-$	$\cdots$
	0	1	2	3	4	5	$\cdots$	$\cdots$	k

Table 18: Comonotone approx. with $s \geqslant 3$, validity of $E_n^{(1)}(f, Y_s) \leqslant c(k,r,s)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

The results for $r = 0$ in Tables 16, 17 and 18 are by Newman [New79] ($k = 1$), Shvedov [Shv81a] ($k = 2$), and Zhou [Zho93] ($k \geqslant 3$). The "+" result for $(k = 1, r = 1)$ (and so for $(k = 0, r = 2)$) is due to Beatson and Leviatan [BL83]. All other cases are due to Gilewicz and Shevchuk [GS96], and Dzyubenko, Gilewicz and Shevchuk [DGS98]. We will not go into details. Instead, we advise the interested reader to consult these two papers for their exact results.

We now summarize all references for the three previous tables in one table. We refer to [GS96] and [DGS98] together, so we present them in the table as [GS96]&[DGS98].

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	$\cdots$
2	[BL83]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	$\cdots$
1	[New79]	[BL83]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	[GS96]&[DGS98]	$\cdots$
0		[New79]	[Shv81a]	[Zho93]	[Zho93]	[Zho93]	$\cdots$
	0	1	2	3	4	5	k

Table 18a: References for Tables 16, 17 and 18

Remark. Estimates for comonotone approximation were first discussed, independently, by Iliev [Ili78a] and Newman [New79]. Iliev claimed the same estimates as Newman's for $(k = 1, r = 0)$ in Tables 16, 17 and 18. However, his proof was somewhat incomplete except in the case $s = 1$.

13.2 Ditzian-Totik type estimates for comonotone approximation

In this section, we investigate the validity of the following statements.

Statement 11. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(1)}(Y_s) \cap C^r_\varphi$, then

$$E_n^{(1)}(f, Y_s) \leqslant \frac{c(k,r,s)}{n^r} \omega^\varphi_{k,r}\big(f^{(r)}, n^{-1}\big), \quad n \geqslant \mathcal{N}.$$

Statement 12. If $r \in \mathbb{N}$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(1)}(Y_s) \cap \mathbb{B}^r$, then

$$E_n^{(1)}(f, Y_s) \leqslant \frac{c(r,s)}{n^r} \|\varphi^r f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}, \quad n \geqslant \mathcal{N}.$$

We have the following truth tables.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
6	$+$	$+$	$+$	$+$	$+$	$\cdots$
5	$+$	$+$	$+$	$+$	$+$	$\cdots$
4	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$+$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$-$	$-$	$-$	$\cdots$
1	$+$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
0		$+$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	k

Table 19: Comonotone approx. with $s = 1$, validity of $E_n^{(1)}(f, Y_1) \leqslant c(k,r) n^{-r} \omega^\varphi_{k,r}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$2s+4$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$2s+3$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$2s+2$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
$2s+1$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$-$	$-$	$-$	$\cdots$
1	$+$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
0		$+$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	k

Table 20: Comonotone approx. with $s \geqslant 2$, validity of $E_n^{(1)}(f, Y_s) \leqslant c(k,r,s) n^{-r} \omega^\varphi_{k,r}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

Remark. Note the unique phenomenon of the case "+" for $(k = 0, r = 3)$ when the function has just one point of inflection $(s = 1)$; see Leviatan and Shevchuk [LS99a].

The results for $r = 0, 1, 2$ in Tables 19 and 20 are by Shvedov [Shv81a], Zhou [Zho93], Leviatan and Shevchuk [LS97, LS99a, LS00], Kopotun and Leviatan [KL97], and Nesterenko and Petrova [NP05].

Again, we summarize all references for both tables in one table of references.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$2s+4$	[LS99a]	[LS99a]	[LS99a]	[LS99a]	[LS99a]	$\cdots$
$2s+3$	[LS99a]	[LS99a]	[LS99a]	[LS99a]	[LS99a]	$\cdots$
$2s+2$	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
4	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	$\cdots$
3	[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	[LS97],[LS99a]	$\cdots$
2	[LS97], [KL97]	[LS97], [LS00]	[NP05]	[LS00]	[LS00]	$\cdots$
1	[LS97]	[LS97], [KL97]	[LS97], [LS00]	[NP05]	[LS00]	$\cdots$
0		[LS97]	[Shv81a], [KL97]	[Zho93]	[Zho93]	$\cdots$
	0	1	2	3	4	k

Table 20a: References for Tables 19 and 20

13.3 Pointwise estimates for comonotone approximation

We now discuss the validity of analogous pointwise estimates.

Statement 13. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(1)}(Y_s) \cap C^r[-1,1]$, then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \Delta^{(1)}(Y_s) \cap \mathbb{P}_n$, such that for every $n \geqslant \mathcal{N}$ and each $x \in [-1,1]$, we have

$$|f(x) - P_n(x)| \leqslant c(k,r,s)\rho_n^r(x)\omega_k\big(f^{(r)}, \rho_n(x)\big). \tag{29}$$

Statement 14. If $r \in \mathbb{N}$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(1)}(Y_s) \cap \mathbb{W}^r$, then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \Delta^{(1)}(Y_s) \cap \mathbb{P}_n$, such that every $n \geqslant \mathcal{N}$, and each $x \in [-1,1]$, we have

$$|f(x) - P_n(x)| \leqslant c(r,s)\rho_n^r(x)\|f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}. \tag{30}$$

The truth tables for Statements 13 and 14 are the following.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	+	+	+	$\cdots$
1	+	+	+	$\oplus$	−	−	$\cdots$
0		+	$\oplus$	−	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 21: Comonotone approx. with $s = 1$, validity of $|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(1)}(Y_1)$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$\oplus$	−	−	$\cdots$
0		$\oplus$	$\oplus$	−	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 22: Comonotone approx. with $s \geqslant 2$, validity of $|f(x) - P_n(x)| \leqslant c(k,r,s)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(1)}(Y_s)$

The results for $r = 0$ are by Shvedov [Shv81a], Zhou [Zho93], and Dzyubenko [Dzy94]. All other results are due to Dzyubenko, Gilewicz and Shevchuk [DGS98].

We summarize all references for both tables in one table of references.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
2	[DGS98]	[DGS98]	[DGS98]	[DGS98]	[DGS98]	[DGS98]	$\cdots$
1	[DGS98]	[DGS98]	[DGS98]	[DGS98]	[DGS98]	[DGS98]	$\cdots$
0		[DGS98]	[Shv81a], [Dzy94]	[Zho93]	[Zho93]	[Zho93]	$\cdots$
	0	1	2	3	4	5	k

Table 22a: References for Tables 21 and 22

14 Coconvex approximation: uniform and pointwise estimates

14.1 Jackson-Stechkin type estimates for coconvex approximation

We use the structure of section 13 and begin with the following two statements.

Statement 15. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(2)}(Y_s) \cap C^r[-1,1]$, then

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c(k,r,s)}{n^r} \omega_k\big(f^{(r)}, n^{-1}\big), \quad n \geqslant \mathcal{N}.$$

Statement 16. If $r \in \mathbb{N}$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(2)}(Y_s) \cap \mathbb{W}^r$, then

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c(r,s)}{n^r} \|f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}, \quad n \geqslant \mathcal{N}.$$

We have the following truth tables.

r							
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	+	+	+	+	+	+	$\cdots$
3	+	+	+	+	+	+	$\cdots$
2	+	+	+	$\oplus$	−	−	$\cdots$
1	+	+	$\oplus$	−	−	−	$\cdots$
0		+	+	$\oplus$	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 23: Coconvex approx. with $s = 1$, validity of $E_n^{(2)}(f, Y_1) \leqslant c(k,r) n^{-r} \omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

The results are due to Zhou [Zho93], Kopotun, Leviatan and Shevchuk [KLS99], Pleshakov and Shatalina [PS00], Gilewicz and Yushchenko [GY02], and Leviatan and Shevchuk [LS02, LS03].

r							
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	[LS03]	[LS03]	[LS03]	[LS03]	[LS03]	[LS03]	$\cdots$
2	[LS02]	[LS03]	[LS03]	[PS00], [LS03]	[GY02]	[GY02]	$\cdots$
1	[LS02]	[LS02]	[PS00], [KLS99]	[Zho93]	[Zho93]	[Zho93]	$\cdots$
0		[LS02]	[LS02]	[PS00], [KLS99]	[Zho93]	[Zho93]	$\cdots$
	0	1	2	3	4	5	k

Table 23a: References for Table 23

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$-$	$\cdots$
0		$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Table 24: Coconvex approx. with $s \geqslant 2$, validity of $E_n^{(2)}(f, Y_s) \leqslant c(k,r,s)n^{-r}\omega_k\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

The results are due to Zhou [Zho93], Kopotun, Leviatan and Shevchuk [KLS99], Gilewicz and Yushchenko [GY02], Leviatan and Shevchuk [LS02, LS03], and Kopotun, Leviatan and Shevchuk [KLS06] (some earlier negative results are due to Pleshakov and Shatalina [PS00]).

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	[KLS06], [LS03]	[KLS06], [LS03]	[KLS06], [LS03]	[KLS06], [LS03]	[KLS06], [LS03]	[KLS06], [LS03]	$\cdots$
3	[LS02], [KLS99]	[LS02], [LS03]	[LS02], [LS03]	[LS02], [LS03]	[LS02], [LS03]	[LS02], [LS03]	$\cdots$
2	[LS02]	[LS02], [KLS99]	[LS02], [LS03]	[LS02], [LS03]	[GY02]	[GY02]	$\cdots$
1	[LS02]	[LS02]	[LS02], [KLS99]	[Zho93]	[Zho93]	[Zho93]	$\cdots$
0		[LS02]	[LS02]	[LS02], [KLS99]	[Zho93]	[Zho93]	$\cdots$
	0	1	2	3	4	5	k

Table 24a: References for Table 24

14.2 Ditzian-Totik type estimates for coconvex approximation

For the generalized D-T moduli, we have the following two statements.

Statement 17. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(2)}(Y_s) \cap C_r^{\varphi}$, then

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c(k,r,s)}{n^r}\omega_{k,r}^{\varphi}\big(f^{(r)}, n^{-1}\big), \quad n \geqslant \mathcal{N}.$$

Statement 18. If $r \in \mathbb{N}$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(2)}(Y_s) \cap \mathbb{B}^r$, then

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c(r,s)}{n^r}\|\varphi^r f^{(r)}\|_{\mathbb{L}_\infty[-1,1]}, \quad n \geqslant \mathcal{N}.$$

The truth tables in this case are the following.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
8	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
7	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
6	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
5	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
4	$\oplus$	$\ominus$	$-$	$-$	$-$	$-$	$\cdots$
3	$\oplus$	$\oplus$	$\ominus$	$-$	$-$	$-$	$\cdots$
2	$+$	$\oplus$	$\oplus$	$\ominus$	$-$	$-$	$\cdots$
1	$+$	$+$	$\oplus$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Table 25: Coconvex approx. with $s = 1$, validity of $E_n^{(2)}(f, Y_1) \leqslant c(k,r)n^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
6	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
5	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
4	$\oplus$	$\oplus$	$-$	$-$	$-$	$-$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$-$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$-$	$\cdots$
0		$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Table 26: Coconvex approx. with $s \geqslant 2$, validity of $E_n^{(2)}(f, Y_s) \leqslant c(k,r,s)n^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)$, $n \geqslant \mathcal{N}$

Remark. It is interesting to note that, in some cases, estimates for coconvex approximation with $s = 1$ are "better" than those with $s \geqslant 2$ (see all cases "+" in Table 25), and, in some other cases, they are "worse" (see cases "$\ominus$").

The results are due to Zhou [Zho93], Kopotun, Leviatan and Shevchuk [KLS99, KLS06], Pleshakov and Shatalina [PS00], Gilewicz and Yuschenko [GY02], and Leviatan and Shevchuk [LS02].

We summarize the references for both tables in one table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	[KLS06]	[KLS06]	[KLS06]	[KLS06]	[KLS06]	[KLS06]	$\cdots$
4	[KLS06]	[KLS06]	[KLS06]	[KLS06]	[KLS06]	[KLS06]	$\cdots$
3	[LS02], [KLS99]	[KLS06]	[KLS06]	[KLS06]	[KLS06]	[KLS06]	$\cdots$
2	[LS02], [KLS99]	[LS02], [KLS99]	[LS02], [KLS06]	[LS02], [KLS06]	[GY02]	[GY02]	$\cdots$
1	[LS02], [KLS99]	[LS02], [KLS99]	[PS00], [KLS99]	[Zho93]	[Zho93]	[Zho93]	$\cdots$
0		[LS02], [KLS99]	[LS02], [KLS99]	[PS00], [KLS99]	[Zho93]	[Zho93]	$\cdots$
	0	1	2	3	4	5	k

Table 26a: References for Tables 25 and 26

14.3 Pointwise estimates for coconvex approximation

Finally, we discuss the pointwise estimates for coconvex approximation.

Statement 19. If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(2)}(Y_s) \cap C^r[-1,1]$, then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \Delta^{(2)}(Y_s) \cap \mathbb{P}_n$ such that (29) holds for every $n \geqslant \mathcal{N}$ and each $x \in [-1,1]$.

Statement 20. If $r \in \mathbb{N}$, $s \in \mathbb{N}$, $\mathcal{N} \in \mathbb{N}$, and $f \in \Delta^{(2)}(Y_s) \cap \mathbb{W}^r$, then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \Delta^{(2)}(Y_s) \cap \mathbb{P}_n$ such that (30) holds for every $n \geqslant \mathcal{N}$ and each $x \in [-1,1]$.

The cases $s = 1$ and $s \geqslant 2$ are significantly different. We begin with $s = 1$, where we have the following truth table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
3	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
2	+	$\ominus$	$\ominus$	$\ominus$	−	−	$\cdots$
1	+	+	$\ominus$	−	−	−	$\cdots$
0		+	+	$\ominus$	−	−	$\cdots$
	0	1	2	3	4	5	k

Table 27: Coconvex approx. with $s = 1$, validity of $|f(x) - P_n(x)| \leqslant c(k,r)\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1,1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(2)}(Y_1)$

The results are due to Zhou [Zho93], Gilewicz and Yushchenko [GY02], Dzyubenko, Gilewicz and Shevchuk [DGS02], and Dzyubenko, Leviatan and Shevchuk [DLS10a], [DLS10b]. We summarize the references in the following table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	$\cdots$
3	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	$\cdots$
2	[DGS02]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[DGS02], [DLS10a]	[GY02]	$\cdots$
1	[DGS02]	[DGS02]	[DGS02], [DLS10b]	[Zho93]	[Zho93]	$\cdots$
0		[DGS02]	[DGS02]	[DGS02], [DLS10b]	[Zho93]	$\cdots$
	0	1	2	3	4	k

Table 27a: References for Table 27

Now we will discuss the case $s \geqslant 2$.

Recall that the case "$\oplus$" means that we have an estimate of the form

$$|f(x) - P_n(x)| \leqslant c(k,r,s,q)\rho_n^r(x)\omega_k\big(f^{(r)}, \rho_n(x)\big), \quad n \geqslant \mathcal{N} = \mathcal{N}(k,r,q,Y_s), \tag{31}$$

which in general is not true if the constant $\mathcal{N}$ is independent of Y_s. For comonotone approximation, a Whitney type estimate has been proved in [She95] (in the case "$\oplus$"):

$$E_{k+r}^{(1)}(f, Y_s) \leqslant c(k,r,Y_s)\omega_k(f^{(r)}, 1).$$

This inequality, combined with (31), implies for all comonotone cases "$\oplus$" above that, together with the estimate (31), we also have the estimate

$$|f(x) - P_n(x)| \leqslant c(k,r,q,Y_s)\rho_n^r(x)\omega_k\big(f^{(r)}, \rho_n(x)\big), \quad n \geqslant \mathcal{N} = k + r. \tag{32}$$

For the case of coconvex pointwise approximation with $s \geqslant 2$, a new case appears, which is somewhere between "$\oplus$" and "$\ominus$". Namely, inequality (32) is valid (see [PS00] for the corresponding Whitney inequality), but the inequality (31) is not valid.

We refer to this case "$\oslash$" and formally define it as follows.

Case "$\oslash$": Inequality

$$|f(x) - P_n(x)| \leqslant c\rho_n^r(x)\omega_k\big(f^{(r)}, \rho_n(x)\big), \quad n \geqslant \mathcal{N},$$

holds with $c = c(k,r,Y_s)$ and $\mathcal{N} = k + r$, as well as with $c = c(k,r,s)$ and $\mathcal{N} = \mathcal{N}(k,r,Y_s,f)$, but this inequality does not hold with $c = c(k,r,s)$ and $\mathcal{N}$ which may depend on k, r, and Y_s, but is independent of f.

We have the following truth table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
4	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\cdots$
3	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\cdots$
2	$\oplus$	$\oslash$	$\oslash$	$\oslash$	$-$	$-$	$\cdots$
1	$\oplus$	$\oplus$	$\oslash$	$-$	$-$	$-$	$\cdots$
0		$\oplus$	$\oplus$	$\oslash$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Table 28: Coconvex approx. with $s \geqslant 2$, validity of $|f(x) - P_n(x)| \leqslant c\rho_n^r(x)\omega_k\left(f^{(r)}, \rho_n(x)\right)$ for $x \in [-1, 1]$ and $n \geqslant \mathcal{N}$ with $P_n \in \Delta^{(2)}(Y_s)$

The results are due to Zhou [Zho93], Gilewicz and Yushchenko [GY02], Leviatan and Shevchuk [LS02], Dzyubenko, Gilewicz and Shevchuk [DGS02], [DGS06], Dzyubenko and Zalizko [DZ04], [DZ05], and Dzyubenko, Leviatan and Shevchuk [DLS10a], [DLS10b]. We summarize the references in the following table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
3	[DGS06], [DZ04], [DLS10a]	[DGS06], [DZ05], [DLS10a]	[DGS06], [DZ05], [DLS10a]	[DGS06], [DZ05], [DLS10a]	[DGS06], [DZ05],[DLS10a]	$\cdots$
2	[LS02], [DGS02]	[DGS06], [DZ04], [DLS10a]	[DGS06], [DZ05], [DLS10a]	[DGS06], [DZ05], [DLS10a]	[GY02]	$\cdots$
1	[LS02], [DGS02]	[LS02], [DGS02]	[DGS06], [DZ04], [DLS10b]	[Zho93]	[Zho93]	$\cdots$
0		[LS02], [DGS02]	[LS02], [DGS02]	[DGS06], [DZ04], [DLS10b]	[Zho93]	$\cdots$
	0	1	2	3	4	k

Table 28a: References for Table 28

15 α-relations for comonotone and coconvex approximation

It follows from Leviatan and Shevchuk [LS97, LS99b] and from Kopotun, Leviatan and Shevchuk [KLS99] that, if Y_1 and $\alpha > 0$, $\alpha \neq 2$, are given, and if a function $f \in \Delta^{(1)}(Y_1)$ satisfies

$$n^\alpha E_n(f) \leqslant 1, \quad n \geqslant 1, \tag{33}$$

then

$$n^\alpha E_n^{(1)}(f, Y_1) \leqslant c(\alpha), \quad n \geqslant 1.$$

For coconvex approximation, Kopotun, Leviatan and Shevchuk [KLS09] recently proved that if Y_1 and $\alpha > 0$, $\alpha \neq 4$, are given, and if a function $f \in \Delta^{(2)}(Y_1)$ satisfies (33), then

$$n^\alpha E_n^{(2)}(f, Y_1) \leqslant c(\alpha), \quad n \geqslant 1.$$

For $\alpha = 4$ this statement is false (see [KLS09] or [KLS10]) since, for every Y_1, there is a function $f \in \Delta^{(2)}(Y_1)$, satisfying (33) with $\alpha = 4$, and

$$\sup_{n \geqslant 1} n^4 E_n^{(2)}(f, Y_1) > c\left|\ln \varphi(y_1)\right|.$$

Similar arguments yield that, for every Y_1, there is a function $f \in \Delta^{(1)}(Y_1)$, satisfying (33) with $\alpha = 2$, and

$$\sup_{n \geqslant 1} n^2 E_n^{(1)}(f, Y_1) > c \left|\ln \varphi(y_1)\right|.$$

However, it is still true that, if a function $f \in \Delta^{(2)}(Y_1)$ satisfies (33) with $\alpha = 4$, or if a function $f \in \Delta^{(1)}(Y_1)$ satisfies (33) with $\alpha = 2$, then

$$n^4 E_n^{(2)}(f, Y_1) \leqslant c, \quad n \geqslant \mathcal{N}^*(Y_1)$$

or

$$n^2 E_n^{(1)}(f, Y_1) \leqslant c, \quad n \geqslant \mathcal{N}^*(Y_1),$$

respectively.

Also, for $s \geqslant 2$, it has been shown in [KLS09] that, if Y_s and $\alpha > 0$, are given, and if a function $f \in \Delta^{(2)}(Y_s)$ satisfies (33), then

$$n^\alpha E_n^{(2)}(f, Y_s) \leqslant c(\alpha, s), \quad n \geqslant \mathcal{N}^*(\alpha, Y_s),$$

but we cannot have the above with $\mathcal{N}^*$ that is independent of Y_s.

Surprisingly, the case $s \geqslant 2$ in comonotone approximation is more complicated. Leviatan, Radchenko and Shevchuk [LRS], recently obtained the following.

Theorem 12. *If Y_s, $s \geqslant 1$, and $\alpha > 0$ are given, and if a function $f \in \Delta^{(1)}(Y_s)$ satisfies* (33)*, then*

$$n^\alpha E_n^{(1)}(f, Y_s) \leqslant c(\alpha, s), \quad n \geqslant \mathcal{N}^*,$$

where $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$, if $\alpha = j$, $j = 1, \ldots, 2\lfloor \frac{s}{2} \rfloor$, or $\alpha = 2j$, $j = 1, \ldots, s$, and $\mathcal{N}^* = 1$ in all other cases.

Moreover, this statement cannot be improved since, for $s \geqslant 1$ and $\alpha = j$, $j = 1, \ldots, 2\lfloor \frac{s}{2} \rfloor$, or $\alpha = 2j$, $j = 1, \ldots, s$, for every m there are a collection Y_s and a function $f \in \Delta^{(1)}(Y_s)$ satisfying (33) *and*

$$m^\alpha E_m^{(1)}(f, Y_s) \geqslant c(s) \ln m,$$

where $c(s) > 0$, depends only on s.

For coconvex approximation, we discuss the validity of the following statement.

Statement 21. If $f \in \Delta^{(2)}(Y_s)$ and

$$n^\alpha E_n(f) \leqslant 1, \quad n \geqslant \mathcal{N}, \tag{34}$$

then

$$n^\alpha E_n^{(2)}(f, Y_s) \leqslant c(s, \alpha), \quad n \geqslant \mathcal{N}^*.$$

Obviously $\mathcal{N}^* \geqslant \mathcal{N}$. Is $\mathcal{N}^* = \mathcal{N}^*(\mathcal{N}, \alpha, s)$ (strongly positive), or $\mathcal{N}^* = \mathcal{N}^*(\mathcal{N}, \alpha, Y_s)$ (weakly positive), or $\mathcal{N}^* = \mathcal{N}^*(\mathcal{N}, \alpha, Y_s, f)$ (weakly negative), or is it possible that the above is invalid (strongly negative)?

We have the following truth tables which show, in particular, that the strongly negative case is impossible. That is, Statement 21 is always valid and there always exists an $\mathcal{N}^*$. We observe that the validity of Statement 21 depends on $\lceil \frac{\alpha}{2} \rceil$ rather than on α itself with only one exception.

Namely, the symbol "$\mathring{+}$" for $\lceil \frac{\alpha}{2} \rceil = 2$ and $\mathcal{N} = 1, 2$ in Table 29 is meant to indicate that $\alpha = 4$ behaves differently than the other α's with $\lceil \frac{\alpha}{2} \rceil = 2$. Namely, we have "$\mathring{+}$":= "$\oplus$" for $\alpha = 4$, and "$\mathring{+}$":="$+$" otherwise.

$\lceil \frac{\alpha}{2} \rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
4	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$\oplus$	$\oplus$	$\cdots$
2	$\mathring{+}$	$\mathring{+}$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
1	$+$	$+$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	$\mathcal{N}$

Table 29: α-relations for coconvex approx. with $s = 1$, validity of "$n^\alpha E_n(f) \leqslant 1,\ n \geqslant \mathcal{N} \implies n^\alpha E_n^{(2)}(f, Y_1) \leqslant c(\alpha),\ n \geqslant \mathcal{N}^*$"

We emphasize once again that, in all cases "$+$" in Table 29, Statement 21 holds with $\mathcal{N}^* = \mathcal{N}$.

$\lceil \frac{\alpha}{2} \rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	7	$\mathcal{N}$

Table 30: α-relations for coconvex approx. with $s = 2$, validity of "$n^\alpha E_n(f) \leqslant 1,\ n \geqslant \mathcal{N} \implies n^\alpha E_n^{(2)}(f, Y_2) \leqslant c(\alpha),\ n \geqslant \mathcal{N}^*$"

$\lceil \frac{\alpha}{2} \rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\cdots$	$\oplus$	$\oplus$	?*	$\ominus$	$\ominus$	$\cdots$
1	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	$\cdots$	$s+1$	$s+2$	$s+3$	$s+4$	$s+5$	$\mathcal{N}$

Table 31: α-relations for coconvex approx. with $s \geqslant 3$, validity of "$n^\alpha E_n(f) \leqslant 1,\ n \geqslant \mathcal{N} \implies n^\alpha E_n^{(2)}(f, Y_s) \leqslant c(s, \alpha),\ n \geqslant \mathcal{N}^*$"

Remark. Note that "?*" in Table 31 can be replaced neither by "+" nor by "−".

All the results in Tables 29, 30 and 31, have appeared in [KLS09] and [KLS10].

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