Logarithmic Potential Theory with Applications to Approximation Theory

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12 October 2010

Abstract

We provide an introduction to logarithmic potential theory in the complex plane that particularly emphasizes its usefulness in the theory of polynomial and rational approximation. The reader is invited to explore the notions of Fekete points, logarithmic capacity, and Chebyshev constant through a variety of examples and exercises. Many of the fundamental theorems of potential theory, such as Frostman's theorem, the Riesz Decomposition Theorem, the Principle of Domination, etc., are given along with essential ideas for their proofs. Equilibrium measures and potentials and their connections with Green functions and conformal mappings are presented. Moreover, we discuss extensions of the classical potential theoretic results to the case when an external field is present.

MSC: Primary: 31Cxx, 41Axx

Keywords: Logarithmic potential, Polynomial approximation, Rational approximation, Transfinite diameter, Capacity, Chebyshev constant, Fekete points, Equilibrium potential, Superharmonic functions, Subharmonic functions, Green functions, Rates of polynomial and rational approximation, Condenser capacity, External fields.

[*]Research was supported, in part, by the U. S. National Science Foundation under grant DMS-0808093.

Surveys in Approximation Theory
Volume 5, 2010. pp. 165–200.

ISSN 1555-578X

0 Introduction

Logarithmic potential theory is an elegant blend of real and complex analysis that has had a profound effect on many recent developments in approximation theory. Since logarithmic potentials have a direct connection with polynomial and rational functions, the tools provided by classical potential theory and its extensions to cases when an external field (or weight) is present, have resolved some long-standing problems concerning orthogonal polynomials, rates of polynomial and rational approximation, convergence behavior of Padé approximants (both classical and multi-point), to name but a few. Here are some problems where potential theory has played a crucial role:

(i) *Rate of Polynomial Approximation*: Let f be analytic on a compact set E of the complex plane $\mathbb{C}$, whose complement $\overline{\mathbb{C}} \setminus E$ is connected. How well can f be uniformly approximated on E by polynomials (in z) of degree at most n?

(ii) *Asymptotic Behavior of Zeros of Polynomials*: Let $p_n^*(x)$ denote the polynomial of degree at most n of best uniform approximation to a continuous function on $[-1,1]$, say $f(x) = |x|$. In the complex plane, p_n^* has n zeros (at most).* Where are these zeros located as $n \to \infty$?

(iii) *Fast Decreasing Polynomials*: Given $\varphi \in C[-1,1]$, does there exist a sequence of "needle-like" polynomials (p_n), $\deg p_n \le n$, such that $p_n(0) = 1$ and $|p_n(x)| \le C\mathrm{e}^{-cn\varphi(x)}$, $x \in [-1,1]$, for some positive constants c, C?

(iv) *Recurrence Coefficients for Orthogonal Polynomials*: Let $\{p_n\}$ denote orthonormal polynomials with respect to the weight $\exp(-|x|^\alpha)$, $\alpha > 0$, on $\mathbb{R}$. That is,

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) \mathrm{e}^{-|x|^\alpha} \, \mathrm{d}x = \delta_{mn}.$$

Then the p_n satisfy a 3-term recurrence of the form

$$x p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x), \qquad n = 1, 2, \ldots,$$

where (a_n) is the sequence of recurrence coefficients. These coefficients go to infinity as n increases, but exactly what is their asymptotic growth rate?

(v) *Generalized Weierstrass Problem*: A famous theorem of Weierstrass states that $f \in C[-1,1]$ if and only if there exists a sequence of polynomials (p_n), $\deg p_n \le n$, such that $p_n \to f$ uniformly on $[-1,1]$. But how would you characterize those $f \in C[-1,1]$ that are uniform limits on $[-1,1]$ of "incomplete" polynomials of the form $q_{2n}(x) = \sum_{k=n}^{2n} a_k x^k$, for which half the coefficients are missing? More generally, what functions f are the uniform limits of *weighted polynomials* of the form $w(x)^n p_n(x)$, where the power of the weight matches the degree of the polynomial?

(vi) *Optimal Point Arrangements on the Sphere*: How well separated are N (≥ 2) points on the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ that maximize the product of their pairwise distances:

$$\prod_{1 \le i < j \le N} |x_i - x_j| \quad ?$$

*By symmetry and uniqueness of the best approximants, $p_{2n+1}^* = p_{2n}^*$ for $f(x) = |x|$.

(vii) *Rational Approximation*: Determine the rate of best uniform approximation to e^{-x} on $[0,+\infty)$ by rational functions of the form $p_n(x)/q_n(x)$, $\deg p_n \le n$, $\deg q_n \le n$.

In this article we provide an introduction to the tools of classical and "weighted" potential theory that are the keys to resolving the above questions. The essential reason for the usefulness of potential theory in obtaining results on polynomials is the fact that for any monic polynomial $p(z) = \prod_{k=1}^{n}(z - z_k)$ the function $\log(1/|p(z)|)$ can be written as a logarithmic potential:

$$\log \frac{1}{|p(z)|} = \int \log \frac{1}{|z-t|} \, d\nu(t),$$

where ν is the discrete measure with mass 1 at each of the zeros of p.

1 Transfinite Diameter, Capacity, and Chebyshev Constant

We begin by introducing three "different" quantities associated with a compact (closed and bounded) set in the plane.

A Geometric Problem. Place n points on a compact set E so that they are "as far apart" as possible in the sense of the *geometric mean* of the pairwise distances between the points. Since the number of different pairs of n points is $n(n-1)/2$, we consider the quantity

$$\delta_n(E) := \max_{z_1,\ldots,z_n \in E} \left(\prod_{1 \le i < j \le n} |z_i - z_j| \right)^{2/n(n-1)} . \tag{1.1}$$

Any system of points $\mathcal{F}_n := \left\{ z_1^{(n)}, \ldots, z_n^{(n)} \right\}$ for which the maximum is attained is called an **n-point Fekete set** for E; the points $z_i^{(n)}$ in $\mathcal{F}_n$ are called **Fekete points**.

For example, if $n = 2$, then $\mathcal{F}_2 = \left\{ z_1^{(2)}, z_2^{(2)} \right\}$, where $\left| z_1^{(2)} - z_2^{(2)} \right| = \operatorname{diam} E$. Obviously, any such points lie on the boundary of E. In general, it follows from the maximum modulus principle for analytic functions that for all n, the Fekete sets lie on the *outer boundary* $\partial_\infty E$, that is, the boundary of the unbounded component of the complement of E.

Exercise. Prove that the determinant of the $n \times n$ Vandermonde matrix $[z_i^j]$, $1 \le i \le n$, $0 \le j \le n-1$, is given by $\prod_{1 \le i < j \le n}(z_j - z_i)$. Consequently, an n-point Fekete set for E maximizes the modulus of this determinant over all n-point subsets of E.

Exercise. Let E be the closed unit disk (or the unit circle). Prove that the set of nth roots of unity is an n-point Fekete set for E (and so is any of its rotations) and that $\delta_n(E) = n^{1/(n-1)}$. [Hint: Use Hadamard's inequality for determinants.]

If $E = [-1, 1]$, then the set $\mathcal{F}_n$ turns out to be unique and it coincides with the zeros of $(1 - x^2)P'_{n-1}(x)$, where P_{n-1} is the Legendre polynomial of degree $n-1$ (cf. [Sz]).

Fekete points are "good points" for polynomial interpolation. We denote by $\mathcal{P}_n$ the linear space of all algebraic polynomials with complex coefficients of degree at most n. Recall that if $z_1, \ldots, z_{n+1}$

are any $n+1$ distinct points, then the unique polynomial in $\mathcal{P}_n$ that interpolates a function f in these points is given by

$$p_n(z) = \sum_{k=1}^{n+1} f(z_k)L_k(z),$$

where $L_k(z)$ is the **fundamental Lagrange polynomial** that satisfies $L_k(z_j) = \delta_{jk}$.

Exercise. Prove that if $\{z_1, \dots, z_{n+1}\}$ is an $(n+1)$-point Fekete set for a compact set E, then the associated fundamental Lagrange polynomials satisfy $|L_k(z)| \le 1$ for all $z \in E$. Furthermore, show that if $P_n \in \mathcal{P}_n$, then

$$\|P_n\|_E \le (n+1)\|P_n\|_{\mathcal{F}_{n+1}},$$

where $\|\cdot\|_A$ denotes the sup norm on A.

Exercise. Prove that if $P_n(f;z)$ denotes the polynomial of degree at most n that interpolates a continuous function f in an $(n+1)$-point Fekete set for E, then

$$\|f - P_n(f;\cdot)\|_E \le (n+2)\|f - p_n^*\|_E,$$

where p_n^* is the best uniform approximation to f on E out of $\mathcal{P}_n$.

On taking the logarithm in (1.1), we see that the max problem in (1.1) is equivalent to the *minimization problem*

$$\mathcal{E}_n(E) := \min_{z_1,\dots,z_n \in E} \sum_{1\le i<j\le n} \log \frac{1}{|z_i - z_j|}. \tag{1.2}$$

The summation in (1.2) can be interpreted as the energy of a system of n like-charged particles located at the points $\{z_i\}_{i=1}^n$, where the repelling force between two particles is proportional to the reciprocal of the distance between them. Thus

$$\mathcal{E}_n(E) = \frac{n(n-1)}{2} \log \frac{1}{\delta_n(E)} \tag{1.3}$$

denotes the minimal logarithmic energy that can be attained by n particles that are constrained to lie on E. Any set of n points that attains this minimal energy is called an **equilibrium configuration** for E; that is, a Fekete set $\mathcal{F}_n$ represents an n-point equilibrium configuration for E.

Essential questions are:

(i) What is the asymptotic behavior of the minimal energy $\mathcal{E}_n(E)$ (or, equivalently, of $\delta_n(E)$) as $n \to \infty$?

(ii) How are optimal configurations (Fekete points) distributed on E as $n \to \infty$?

As a first step we establish

Lemma 1.1. *The sequence* $\left(\frac{\mathcal{E}_n(E)}{n(n-1)}\right)_{n=2}^{\infty}$ *is increasing (i.e., nondecreasing) or, equivalently, the sequence* $(\delta_n(E))_{n=2}^{\infty}$ *is decreasing (i.e., nonincreasing).*

Proof. With $\mathcal{F}_n = \left\{ z_k^{(n)} \right\}_{k=1}^n$ we have, for each $k = 1, \dots, n$,

$$\begin{aligned}\mathcal{E}_n(E) &= \sum_{i \neq k} \log \frac{1}{\left| z_i^{(n)} - z_k^{(n)} \right|} + \sum_{\substack{1 \leq i < j \leq n \\ i \neq k \\ j \neq k}} \log \frac{1}{\left| z_i^{(n)} - z_j^{(n)} \right|} \\ &\geq \sum_{i \neq k} \log \frac{1}{\left| z_i^{(n)} - z_k^{(n)} \right|} + \mathcal{E}_{n-1}(E).\end{aligned}$$

Now add these n inequalities together and divide by $n(n-1)(n-2)$ to get result. □

The sequence $(\delta_n(E))$ therefore has a limit†

$$\boxed{\tau(E) := \lim_{n \to \infty} \delta_n(E),} \tag{1.4}$$

which is called the **transfinite diameter** of E. For example, the transfinite diameter of the disk $E = \{z \in \mathbb{C} : |z| \leq R\}$ is R since $\delta_n(E) = Rn^{1/(n-1)} \to R$ as $n \to \infty$.

Note that $0 \leq \tau(E) \leq$ diam E and that $E_1 \subset E_2$ implies $\tau(E_1) \leq \tau(E_2)$.

Exercise. Let $aE + b := \{az + b : z \in E\}$, with a, b fixed complex constants. Prove that $\tau(aE + b) = |a|\tau(E)$ for any compact set $E \subset \mathbb{C}$.

Exercise. Show that the closed set $E = \{0\} \cup \{1/k : k = 1, 2, \dots\}$ has transfinite diameter zero.

Remark. The transfinite diameter τ (considered as a set function) has some of the properties of Lebesgue measure on compact subsets of $\mathbb{C}$; in fact, if E is the closed interval $[a, b]$, then $\tau([a, b]) = (b-a)/4$. However, τ *fails to be subadditive*; $\tau(E_1 \cup E_2)$ may exceed the sum $\tau(E_1) + \tau(E_2)$.

To investigate the asymptotic behavior of a sequence of Fekete sets $\mathcal{F}_n$, $n = 2, 3, \dots$, we utilize *weak-star convergence* of measures.

Definition 1.2. Let μ_n be a sequence of finite positive measures with supports‡ $\operatorname{supp}(\mu_n) \subset K$ for all n, where K is some compact set. We write $\mu_n \xrightarrow{*} \mu$ if

$$\lim_{n \to \infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu \quad \forall f \in C(K). \tag{1.5}$$

(If $\mu_n(K) \leq M$ for some constant M and all n (which clearly holds when μ is a finite measure), this is equivalent to pointwise convergence in the dual space of $C(K)$.) The same definition applies to signed measures and complex measures. In (1.5), we can always take K to be the extended complex plane $\overline{\mathbb{C}}$; however, knowing a specific compact set K that contains all the supports of the μ_n's serves to remind us that the limit measure will also be supported on K.

†If E consists of only finitely many points, then $\tau(E) = 0$. Why?

‡Recall that a point z_0 belongs to $\operatorname{supp}(\mu)$ if and only if every open set containing z_0 has positive μ-measure.

For a discrete set consisting of n points of $\mathbb{C}$, say $A_n = \{z_1, \ldots, z_n\}$, we associate the **normalized counting measure**

$$\nu(A_n) := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k},$$

where δ_z is the unit point mass at z.

Example 1.3. If A_{n+1} consists of the $n+1$ **Chebyshev nodes** for $[-1,1]$; that is $A_{n+1} = \{\cos(k\pi/n) : k = 0, 1, \ldots, n\}$, then

$$\nu(A_n) \xrightarrow{*} \frac{\mathrm{d}x}{\pi\sqrt{1-x^2}}, \tag{1.6}$$

which is the **arcsine distribution** on $[-1,1]$. The nodes A_{n+1} are the extreme points of the **Chebyshev polynomials** $T_n(x) = \cos(n \arccos x)$, which are orthogonal on $[-1,1]$ with respect to the arcsine distribution. Verify (1.6)!

Example 1.4. If A_n consists of the nth roots of unity, then $\nu(A_n) \xrightarrow{*} \frac{1}{2\pi}\,\mathrm{d}\theta$, where $\mathrm{d}\theta$ is arclength on the unit circle $|z| = 1$.

Exercise. Let λ be a real *irrational* number and let $A_n := \{\exp(\lambda k\pi\mathrm{i}) : k = 1, \ldots, n\}$. Prove that $\nu(A_n) \xrightarrow{*} \frac{1}{2\pi}\,\mathrm{d}\theta$. What happens if λ is rational?

As we shall see, many of the results of potential theory are formulated for semi-continuous functions.

Definition 1.5. A function $f : D \to (-\infty, \infty]$ (f omits the value $-\infty$) is **lower semi-continuous (l.s.c.)** on the set $D \subset \mathbb{C}$ if it satisfies any of the following equivalent conditions:

(i) $\{z \in D : f(z) > \alpha\}$ is open relative to D for every $\alpha \in \mathbb{R}$;

(ii) For every $z_0 \in D$,

$$f(z_0) \le \liminf_{z \to z_0} f(z);$$

(iii) For every compact subset $K \subset D$, there exists an increasing sequence of continuous functions on K with pointwise limit f.

Exercise. Prove that if f is l.s.c. on a compact set K, then f attains its minimum on K.

Important for us is the fact, which follows from property (iii) and the Monotone Convergence Theorem, that if f is l.s.c. on a compact set K, then

$$\int_K f\,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_K f\,\mathrm{d}\mu_n \tag{1.7}$$

wherever $\mu_n \xrightarrow{*} \mu$ and $\operatorname{supp}(\mu_n) \subset K$ for all n.

Exercise. Prove that if $\mu_n \xrightarrow{*} \mu$, then for any bounded Borel set E,

$$\mu(\mathring{E}) \le \liminf_{n\to\infty} \mu_n(E) \le \limsup_{n\to\infty} \mu_n(E) \le \mu(\overline{E}),$$

where $\mathring{E}$ and $\overline{E}$ denote, respectively, the interior of E and the closure of E. [Hint: First show that the characteristic function of an open set is l.s.c.]

Our goal now is to determine the weak-star limit (if it exists) for the sequence of normalized counting measures $\nu(\mathcal{F}_n)$ in the Fekete points for a given compact set E. For this purpose we study the continuous analogue of the discrete minimum energy problem (1.2).

Electrostatics Problem for a Capacitor. Place a unit positive charge on a compact set E so that equilibrium is attained in the sense that energy is minimized. Again it is assumed that the repulsive force between like-charged particles located at points z and t is proportional to $1/|z-t|$.

To create a mathematical framework for this problem, we let $\mathcal{M}(E)$ denote the collection of all positive unit Borel measures μ supported on E (so that $\mathcal{M}(E)$ contains all possible distributions of charges placed on E). The **logarithmic potential** associated with μ is

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, \mathrm{d}\mu(t), \tag{1.8}$$

which is harmonic outside the support $\mathrm{supp}(\mu)$ of μ and is l.s.c. in $\mathbb{C}$ since

$$U^{\mu}(z) = \lim_{M\to\infty} \int \min\left(M,\ \log\frac{1}{|z-t|}\right) \mathrm{d}\mu(t).$$

The **energy** of such a potential is defined by

$$I(\mu) := \int U^{\mu}\, \mathrm{d}\mu = \int\int \log\frac{1}{|z-t|}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(z). \tag{1.9}$$

Thus, the electrostatics problem involves the determination of

$$V_E := \inf\{I(\mu):\ \mu \in \mathcal{M}(E)\}, \tag{1.10}$$

which is called the **Robin constant** for E. Note that since E is bounded, we have

$$-\infty < V_E \leq +\infty.$$

First we establish the existence of a measure $\mu_E \in \mathcal{M}(E)$ for which the "inf" is attained. For this purpose we use

Lemma 1.6 (**Principle of Descent**). *Let μ_n be a sequence of measures in $\mathcal{M}(E)$ that converges weak-star to some $\mu \in \mathcal{M}(E)$. Then, for all $z \in \mathbb{C}$,*

$$U^{\mu}(z) \leq \liminf_{n\to\infty} U^{\mu_n}(z), \tag{1.11}$$

and, furthermore,

$$I(\mu) \leq \liminf_{n\to\infty} I(\mu_n). \tag{1.12}$$

Proof. Inequality (1.11) follows from (1.7) on observing that $\log 1/|z-t|$ is l.s.c. in t. Inequality (1.12) follows similarly, on observing that $\mu_n \times \mu_n$ converges weak-star to $\mu \times \mu$. $\square$

Lemma 1.7. *There is some $\mu_E \in \mathcal{M}(E)$ such that $I(\mu_E) = V_E$.*

Proof. By the Banach-Alaoglu Theorem, $\mathcal{M}(E)$ is compact in the weak-star topology (this fact is also known as **Helly's Selection Theorem**). Let μ_n be a sequence in $\mathcal{M}(E)$ satisfying $\lim_{n\to\infty} I(\mu_n) = V_E$ and let ν denote some weak-star cluster point of the μ_n. Then, by the Principle of Descent and the definition of V_E, we obtain $V_E = I(\nu)$. □

When $V_E = +\infty$ (for example this is the case if E is countable), then every measure $\mu \in \mathcal{M}(E)$ is a minimizing measure. However, if V_E is finite, it follows from the strict convexity of $I(\mu)$ on $\mathcal{M}(E)$ (cf. [ST]) that there exists a *unique* measure μ_E such that $V_E = I(\mu_E)$. In this case, we call μ_E the **equilibrium measure** for E, and U^{μ_E} the **equilibrium** or **conductor potential** for E.

Definition 1.8. The **logarithmic capacity** of E, denoted by $\operatorname{cap}(E)$, is defined by

$$\boxed{\operatorname{cap}(E) := \mathrm{e}^{-V_E}.} \tag{1.13}$$

If $V_E = +\infty$, we set $\operatorname{cap}(E) = 0$; such sets E are called **polar sets** because they correspond to sets where potentials can equal $+\infty$. More generally, an arbitrary set $E \subset \mathbb{C}$ is said to be **polar** if every closed subset of E is polar. In electrostatic terms, polar sets are "too small" to hold a charge.[§]

Exercise. Prove that any countable set E is a polar set.

Next we establish the connection with the transfinite diameter.

Theorem 1.9. *For any compact set $E \subset \mathbb{C}$,*

$$\tau(E) = \operatorname{cap}(E). \tag{1.14}$$

Moreover, if E has positive capacity, then

$$\nu(\mathcal{F}_n) \xrightarrow{*} \mu_E \text{ as } n \to \infty. \tag{1.15}$$

Proof. First we show that

$$V_E = \log \frac{1}{\operatorname{cap}(E)} \geq \log \frac{1}{\tau(E)}. \tag{1.16}$$

Let $F(z_1, z_2, \ldots, z_n) := \sum_{1 \leq i < j \leq n} \log(1/|z_i - z_j|)$. Then the expected value of F with respect to the product of equilibrium measures $\mathrm{d}\mu_E(z_1)\,\mathrm{d}\mu_E(z_2)\cdots\,\mathrm{d}\mu_E(z_n)$ cannot be less than its minimum value defined in (1.2); i.e.,

$$\begin{aligned}\int\!\!\int\cdots\int F(z_1, z_2, \ldots, z_n)\,\mathrm{d}\mu_E(z_1)\cdots\,\mathrm{d}\mu_E(z_n) &= \frac{n(n-1)}{2} V_E \\ &\geq \mathcal{E}_n(E) = \frac{n(n-1)}{2} \log \frac{1}{\delta_n(E)}.\end{aligned}$$

[§]Somewhat surprising is the fact that the classical "1/3 Cantor set," which has 1-dimensional Lebesgue measure zero, has *positive* capacity. The precise value of this capacity is as yet still unknown (see [R2] for some numerical approximation methods).

Dividing by $n(n-1)/2$ and letting $n \to \infty$ gives (1.16).

For the reverse direction, let $\hat{\mu}$ be a weak-star limit point of the measures $\nu_n := \nu(\mathcal{F}_n)$, say $\nu_n \xrightarrow{*} \hat{\mu}$ as $n \to \infty$, $n \in \mathcal{N}$. Set $\log_M x := \min\{\log x, M\}$. By the Monotone Convergence Theorem and the weak-star convergence of $\nu_n \times \nu_n$ to $\hat{\mu} \times \hat{\mu}$ for $n \in \mathcal{N}$, we get

$$
\begin{aligned}
I(\hat{\mu}) = \int\int \log \frac{1}{|z-t|} \,\mathrm{d}\hat{\mu}(z)\,\mathrm{d}\hat{\mu}(t) &= \lim_{M\to\infty} \int\int \log_M \frac{1}{|z-t|} \,\mathrm{d}\hat{\mu}(z)\,\mathrm{d}\hat{\mu}(t) \\
&= \lim_{M\to\infty} \lim_{n\to\infty} \int\int \log_M \frac{1}{|z-t|} \,\mathrm{d}\nu_n(z)\,\mathrm{d}\nu_n(t) \\
&\le \lim_{M\to\infty} \lim_{n\to\infty} \left\{ \frac{2}{n^2}\mathcal{E}_n(E) + \frac{nM}{n^2} \right\} \\
&= \lim_{M\to\infty} \lim_{n\to\infty} \log \frac{1}{\delta_n(E)} = \log \frac{1}{\tau(E)} \le V_E.
\end{aligned}
$$

Thus from the minimality property of V_E, we have

$$V_E \le I(\hat{\mu}) \le \log \frac{1}{\tau(E)} \le V_E,$$

which proves (1.14). Furthermore, if $\operatorname{cap}(E) > 0$, then by uniqueness of the equilibrium (minimizing) measure, $\hat{\mu} = \mu_E$. Since $\hat{\mu}$ was an arbitrary limit measure of $\nu(\mathcal{F}_n)$, (1.15) follows. □

As we have earlier observed, Fekete points necessarily lie on the outer boundary of E. Thus from (1.15) we immediately deduce that *the equilibrium measure μ_E is supported on the outer boundary of E*; consequently,

$$\operatorname{cap}(E) = \operatorname{cap}(\partial_\infty E), \qquad \mu_E = \mu_{\partial_\infty E}.$$

If $\partial_\infty E$ is a continuum (not a single point), then $\operatorname{supp}(\mu_E) = \partial_\infty E$. In general, $\partial_\infty E \setminus \operatorname{supp}(\mu_E)$ has capacity zero.

From our knowledge of Fekete points for the disk we deduce the following.

Example 1.10. If E is the closed disk $|z-a| \le r$, then $\operatorname{cap}(E) = r$ and $\mathrm{d}\mu_E = \frac{1}{2\pi r}\,\mathrm{d}s$,¶ where $\mathrm{d}s$ is arclength on the circumference $|z-a| = r$. Furthermore, the equilibrium potential $U^{\mu_E}(z)$ satisfies

$$U^{\mu_E}(z) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{|z-a-r\mathrm{e}^{\mathrm{i}\theta}|}\,\mathrm{d}\theta = \begin{cases} \log \frac{1}{r} & \text{for } |z-a| \le r \\ \log \frac{1}{|z-a|} & \text{for } |z-a| \ge r. \end{cases} \tag{1.17}$$

Exercise. Verify formula (1.17). [Hint: The mean-value property for harmonic functions is useful here; see Theorem 2.1.]

¶This also follows from the fact that the disk E is invariant under rotations about $z = a$ and since the equilibrium measure is unique and supported on the circumference it must also be rotation invariant and hence of the form described.

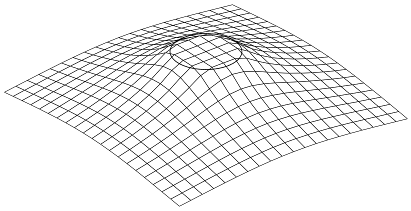

Figure 1: Graph of equilibrium potential for the unit disk

Example 1.11. Let $E = [a, b]$ be a segment on the real line. Then $\mathrm{cap}(E) = (b-a)/4$ and $d\mu_E$ is the arcsine measure; i.e.,

$$d\mu_E = \frac{1}{\pi}\frac{dx}{\sqrt{(x-a)(b-x)}}, \quad x \in [a, b].$$

If $a = -1$, $b = 1$, the conductor potential is given by

$$U^{\mu_E}(z) = \log 2 \text{ if } x \in [-1, 1],$$

$$U^{\mu_E}(z) = \log 2 - \log\left|z + \sqrt{z^2-1}\right| \text{ if } z \notin [-1, 1],$$

where the branch $\sqrt{z^2-1}$ is positive for $z = x > 1$. These facts can be obtained from Example 1.10 by applying the Joukowski conformal map of $\mathbb{C} \setminus [-1, 1]$ onto $|w| > 1$. See Example 3.6.

In the examples for the disk and line segment, observe that the equilibrium potential U^{μ_E} is constant on E, namely it equals $V_E = \log \frac{1}{\mathrm{cap}(E)}$ there. This is certainly consistent with our expectations based on physical grounds, that equilibrium should occur when the potential (voltage) is constant; for otherwise there would be a flow of charge to the points of E at lower potential. From a mathematically rigorous point of view, this assertion is true **quasi-everywhere** (**q.e.**) on E; that is, except for a set of capacity zero.‖ This fact is included in the following result.

Theorem 1.12 (Frostman's Theorem). *Let $E \subset \mathbb{C}$ be compact with $\mathrm{cap}(E) > 0$. Then*

(a) $U^{\mu_E}(z) \le V_E$ for all $z \in \mathbb{C}$;

(b) $U^{\mu_E}(z) = V_E$ q.e. on E.

The proof relies on a maximum principle for potentials which is discussed in Section 2. For full details, see [R1], [ST], [Ts].

The theorem suggests that we visualize the 3-dimensional graph of an equilibrium potential as something like an infinite tent with an "essentially" flat roof consisting of the projection of the set E and tent sides that flow down and outward to $-\infty$; see, e.g., Figure 1.

There are many important consequences of Frostman's result, of which the following will be useful in the proof of the main theorem of this section.

‖An arbitrary Borel set B has capacity zero if $\sup\{\mathrm{cap}(F) : F \subset B \text{ compact}\} = 0$.

Proposition 1.13. *Let $E \subset \mathbb{C}$ be compact with* $\operatorname{cap}(E) > 0$. *If σ is any probability measure with compact support, then*

$$\inf_{z\in E} U^{\sigma}(z) \le V_E = \log \frac{1}{\operatorname{cap}(E)}.$$

Proof. Here we use the **reciprocity law** (a simple consequence of the Fubini-Tonelli Theorem) which asserts that

$$\int U^{\sigma}(z)\,\mathrm{d}\mu_E(z) = \int U^{\mu_E}(z)\,\mathrm{d}\sigma(z).$$

The left-hand side is bounded below by $\inf_{z\in E} U^{\sigma}(z)$ and, from Frostman's theorem, V_E is an upper bound for the right-hand side. □

We now introduce a third quantity associated with a compact set E — the **Chebyshev constant**, $\operatorname{cheb}(E)$ — which arises in a min-max problem.

Polynomial Extremal Problem: Determine the minimal sup norm on E for monic polynomials of degree n. That is, determine**

$$t_n(E) := \min_{p\in\mathcal{P}_{n-1}} \|z^n + p(z)\|_E,$$

where $\mathcal{P}_{n-1}$ denotes the collection of all polynomials of degree $\le n-1$ and $\|\cdot\|_E$ is the sup norm (uniform norm) on E. We assume that E contains infinitely many points (which is always the case if $\operatorname{cap}(E) > 0$). Then for every n there is a unique monic polynomial $T_n(z) = z^n + \cdots$ such that $\|T_n\|_E = t_n(E)$, which is called the nth **Chebyshev polynomial** for E.

Exercise. Prove that all the zeros of T_n lie in the convex hull of E. (This fact is due to Fejér.)

In view of the simple chain of inequalities

$$t_{m+n}(E) = \|T_{m+n}\|_E \le \|T_m T_n\|_E \le \|T_m\|_E \|T_n\|_E = t_m(E)t_n(E),$$

the sequence $\log t_n(E)$ is subadditive, from which it follows that $t_n(E)^{1/n}$ converges and its limit is $\inf_{k\ge1}\{t_k(E)^{1/k}\}$ (cf. [Ts], [ST]). We call this limit the **Chebyshev constant** for E:

$$\operatorname{cheb}(E) := \lim_{n\to\infty} t_n(E)^{1/n} = \inf_{k\ge 1}\{t_k(E)^{1/k}\}. \tag{1.18}$$

From (1.18) and the definition of $t_n(E)$ we deduce the following.

Lemma 1.14. *For any monic polynomial $p_n(z)$ of degree n there holds*

$$\|p_n\|_E \ge [\operatorname{cheb}(E)]^n. \tag{1.19}$$

**This problem is equivalent to finding the polynomial of degree $\le n-1$ of best uniform approximation to the monomial z^n on E.

Example 1.15. Let E be the closed disk of radius R, centered at 0. For any $p \in \mathcal{P}_{n-1}$, the ratio $(z^n + p(z))/z^n$ represents an analytic function in $|z| \geq 1$ that takes the value 1 at ∞. By the maximum principle for analytic functions,

$$\|z^n + p(z)\|_E = \max_{|z|=R} |z^n + p(z)| = R^n \max_{|z|=R} \left| \frac{z^n + p(z)}{z^n} \right| \geq R^n,$$

and strict inequality takes place if $p(z)$ is not identically zero. It follows that $T_n(z) = z^n$. Therefore $t_n(E) = R^n$ and $\operatorname{cheb}(E) = R$.

Example 1.16. Let $E = [-1, 1]$. Then T_n is the classical monic Chebyshev polynomial[††]

$$T_n(x) = 2^{1-n} \cos(n \arccos x), \quad x \in [-1, 1], \quad n \geq 1.$$

Thus, $t_n(E) = \|T_n\|_{[-1,1]} = 2^{1-n}$ from which it follows that $\operatorname{cheb}(E) = 1/2$, which is the same as the capacity of $[-1, 1]$.

Exercise. Verify Example 1.16 by using the fact that $2^{1-n} \cos(n \arccos x)$ equioscillates $n + 1$ times on $[-1, 1]$.

Closely related to Chebyshev polynomials are *Fekete polynomials*. An nth degree Fekete polynomial $F_n(z)$ is a monic polynomial having all its zeros at the n points of a Fekete set $\mathcal{F}_n$.

Example 1.17. If E is the closed unit disk centered at 0, then one can take $F_n(z) = z^n - 1$, so that $\|F_n\|_E = 2$. Comparing this with Example 1.15 we see that the F_n's are asymptotically optimal for the Chebyshev problem:

$$\lim_{n\to\infty} \|F_n\|_E^{1/n} = \lim_{n\to\infty} \|T_n\|_E^{1/n} = 1 = \operatorname{cheb}(E).$$

Moreover, uniformly on compact subsets of $|z| > 1$, we have

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = \lim_{n\to\infty} |T_n(z)|^{1/n} = |z| = \exp(-U^{\mu_E}(z)),$$

(the last equality follows from formula (1.17)).

The above examples illustrate the following fundamental theorem, various parts of which are due to Fekete, Frostman, and Szegő.

Theorem 1.18 (**Fundamental Theorem of Classical Potential Theory**). *For any compact set $E \subset \mathbb{C}$,*
(a) $\operatorname{cap}(E) = \tau(E) = \operatorname{cheb}(E)$;
(b) *Fekete polynomials are asymptotically optimal for the Chebyshev problem:*

$$\lim_{n\to\infty} \|F_n\|_E^{1/n} = \operatorname{cheb}(E) = \operatorname{cap}(E).$$

[††]There is ambiguity in this notation since $T_n(x)$ is traditionally used to denote $\cos(n \arccos x)$.

If $\operatorname{cap}(E) > 0$ *(so that* μ_E *is unique), then we also have*:
(c) *Fekete points (the zeros of* F_n*) have asymptotic distribution* μ_E*, i.e.,* $\nu(\mathcal{F}_n) \xrightarrow{*} \mu_E$ *as* $n \to \infty$*;*
(d) *uniformly on compact subsets of the unbounded component of* $\mathbb{C} \setminus E$*,*

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = \exp(-U^{\mu_E}(z)).$$

Proof. Part (c) was established in (1.15) of Theorem 1.9. Assertion (d) follows from (c) on observing that

$$\frac{1}{n} \log \frac{1}{|F_n(z)|} = U^{\nu(\mathcal{F}_n)}(z),$$

and that all the Fekete points lie on E. Regarding (a) and (b), we already know that $\operatorname{cap}(E) = \tau(E)$. So to establish (a), we prove that $\tau(E) = \operatorname{cheb}(E)$.

Let $\mathcal{F}_n = \{z_k^{(n)}\}_{k=1}^n$ denote an n-point Fekete set for E. Then

$$\delta_{n+1}^{n(n+1)/2} = \max_{\{z_i\}\subset E} \prod_{1\le i<j\le n+1} |z_i - z_j| \ge \left[\prod_{k=1}^n |z - z_k^{(n)}|\right] \delta_n^{n(n-1)/2}$$

for all $z \in E$. Thus

$$\delta_{n+1}^{n(n+1)/2} / \delta_n^{n(n-1)/2} \ge |F_n(z)|, \qquad z \in E,$$

and so on taking nth roots, we get

$$\delta_{n+1}^{(n+1)/2} / \delta_n^{(n-1)/2} \ge \|F_n\|_E^{1/n}. \tag{1.20}$$

The left-hand side of this inequality can be written as

$$\left(\frac{\delta_{n+1}}{\delta_n}\right)^{n/2} \delta_{n+1}^{1/2} \delta_n^{1/2},$$

which is bounded above by $\delta_{n+1}^{1/2}\delta_n^{1/2}$ since the sequence δ_n is decreasing. From (1.19), we have that the right-hand side of (1.20) is bounded below by $\operatorname{cheb}(E)$. Hence

$$\delta_{n+1}^{1/2}\delta_n^{1/2} \ge \|F_n\|_E^{1/n} \ge \operatorname{cheb}(E),$$

and so on letting $n \to \infty$, we get

$$\tau(E) \ge \limsup_{n\to\infty} \|F_n\|_E^{1/n} \ge \liminf_{n\to\infty} \|F_n\|_E^{1/n} \ge \operatorname{cheb}(E).$$

It remains only to prove that $\operatorname{cheb}(E) \ge \tau(E)$. This is obvious if $\tau(E) = \operatorname{cap}(E) = 0$. So assume $\operatorname{cap}(E) > 0$. Let $\nu(T_n)$ denote the normalized counting measure in the zeros of $T_n(z)$. Then by Proposition 1.13,

$$\inf_{z\in E} U^{\nu(T_n)}(z) = \inf_{z\in E} \frac{1}{n} \log \frac{1}{|T_n(z)|} = \frac{1}{n} \log \frac{1}{t_n(E)} \le V_E.$$

Hence

$$t_n(E)^{1/n} \ge \mathrm{e}^{-V_E} = \operatorname{cap}(E) = \tau(E),$$

and on letting $n \to \infty$, we get $\text{cheb}(E) \geq \tau(E)$. $\square$

Exercise. Let $0 < a < b$. Prove that

$$\text{cap}\left([a, b] \cup [-b, -a]\right) = \frac{\sqrt{b^2 - a^2}}{2}.$$

[Hint: What are the Chebyshev polynomials of even degree for this union?]

Exercise. Let $P(z)$ be a monic polynomial of degree n, and consider the *lemniscate set* $L := \{z : |P(z)| \leq R^n\}$. Prove that $\text{cap}(L) = R$. [Hint: Begin by determining the Chebyshev polynomials $T_{kn},\ k = 1, 2, \ldots,$ for L.]

Exercise. Let E be a compact set and $\epsilon > 0$. Show that there exists a lemniscate set L such that $E \subset L$ and $\text{cap}(L) < \text{cap}(E) + \epsilon$.

2 Harmonic, Superharmonic and Subharmonic Functions

Recall that a real-valued function $u(z)$ defined in an open set $D \subset \mathbb{C}$ is **harmonic** in D if u and its 1st and 2nd partial derivatives are continuous in D and u satisfies *Laplace's equation*

$$u_{xx}(z) + u_{yy}(z) = 0, \qquad z \in D. \tag{2.1}$$

(Actually it is enough to merely assume that the 2nd partial derivatives exist and satisfy (2.1).) *Locally*, harmonic functions are the real (or imaginary) parts of an analytic function.

Exercise. Prove that if u is harmonic in D, then $g(z) := u_x(z) - iu_y(z)$ is analytic in D.

Note that the important function $\log|z|$ is harmonic in $\mathbb{C} \setminus \{0\}$, is locally the real part of a branch of $\log z$, but is not globally (in $\mathbb{C} \setminus \{0\}$) the real part of an analytic function.

Exercise. Prove that the logarithmic potential $U^\mu(z)$ is harmonic for z not in the support of μ (assuming this support is compact and μ is a finite measure).

Harmonic functions can also be characterized by the following **Mean-Value Property** (**MVP**).

Theorem 2.1. *A real-valued function $u(z)$ is harmonic in an open set D if and only if u is continuous in D and locally satisfies the* ***mean-value property****; i.e., if the disk $|z - a| \leq r$ is contained in D, then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta. \tag{2.2}$$

(In fact, it is enough that equality holds for $r = r(a)$ sufficiently small.)

Exercise. Prove that if u is harmonic on an open set D containing the disk $|z - z_0| \leq r$, then

$$u(z_0) = \frac{1}{\pi r^2} \int\int_{|z - z_0| \leq r} u(z)\, \mathrm{d}x\, \mathrm{d}y.$$

Exercise. Prove that if $u_n(z)$, $n = 1, 2, \ldots$, is a sequence of functions harmonic in D that converges locally uniformly to a function u in D, then u is harmonic in D.

An important consequence of (2.2) is the **Max-Min Principle for harmonic functions**.

Theorem 2.2. *If u is harmonic in a **domain** D (i.e., an open connected set) and u attains its maximum (or minimum) in D, then u is identically constant in D.*

Furthermore, if u is harmonic in the interior and continuous on the boundary of a compact set, then u attains its max and min on the boundary.

The proof of this principle follows from the observation that if u attains its max at a point $z_0 \in D$ and $\overline{D}_r(z_0)$ is any closed disk centered at z_0, then u must equal $u(z_0)$ for all $z \in \overline{D}_r(z_0)$ since the contrary assumption would lead to a violation of the MVP (simply integrate around a circle centered at z_0 and containing a point where $u(z) < u(z_0)$).

Theorem 2.2 also tells us that harmonic functions are determined by their values on the boundary of a compact set. Indeed, in the case of a disk, we have **Poisson's integral formula**: If u is harmonic in $|z| < R$ and continuous on $|z| \le R$, then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(t,z)u(t)\,\mathrm{d}\theta, \qquad t = R\mathrm{e}^{\mathrm{i}\theta}, \qquad |z| < R, \tag{2.3}$$

where

$$P(t,z) := \frac{|t|^2 - |z|^2}{|t-z|^2} = \Re\left(\frac{t+z}{t-z}\right). \tag{2.4}$$

This formula can be deduced, for example, from the Cauchy integral formula for analytic functions; it includes as a special case the MVP (2.2).

Exercise. Prove that if $U(t)$ is integrable (in the Lebesgue sense) on $|t| = R$, then

$$u(z) := \frac{1}{2\pi}\int_0^{2\pi} P(t,z)U(t)\,\mathrm{d}t \tag{2.5}$$

is harmonic in $|z| < R$. (If $U(t)$ is continuous on the circle $|t| = R$, then Schwarz's theorem asserts that u as given in (2.5) solves the *Dirichlet problem* for the disk; i.e., $\lim_{z\to t} u(z) = U(t)$ for all t on the boundary $|t| = R$.)

If we replace equality in (2.2) by $\ge$, we obtain the class of superharmonic functions.

Definition 2.3. An extended real-valued function f on an open set $D \subset \mathbb{C}$ is called **superharmonic** in D if f is not the constant function $+\infty$ and satisfies

(i) f is lower semi-continuous on D;

(ii) the value of f at any point $z_0 \in D$ is not less than its average over any circle in D centered at z_0; that is

$$f(z_0) \ge \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \tag{2.6}$$

provided the closed disk $\overline{D_r(z_0)} := \{z \in \mathbb{C} : |z - z_0| \le r\}$ is contained in D.

A **subharmonic** function is the negative of a superharmonic function. A real-valued function $u(z)$ that is both superharmonic and subharmonic in D is harmonic in D.

Exercise. Let $f : D \to \mathbb{R}$, D a domain, $f \in C^2(D)$. Prove that f is superharmonic in D if and only if $\Delta f := f_{xx} + f_{yy} \le 0$ at all points of D. [Hint: Begin by showing that a negative Laplacian implies that f is superharmonic.]

As the above exercise illustrates, superharmonic functions in $\mathbb{R}^2$ are analogues of concave functions in $\mathbb{R}$; and subharmonic functions are the analogues of convex functions.

Exercise. Prove that if $F(z)$ is analytic in a domain D and $p > 0$, then $|F(z)|^p$ is subharmonic in D. Furthermore, show $\log|F(z)|$ is subharmonic in D unless F is identically zero.

Essential for us is the fact that *logarithmic potentials are superharmonic in* $\mathbb{C}$. Indeed, as we have seen,

$$U^\mu(z) = \int \log \frac{1}{|z-t|}\, \mathrm{d}\mu(t)$$

is l.s.c. on $\mathbb{C}$ and since $\log(1/|z-t|)$ is superharmonic in $\mathbb{C}$ for fixed t (Why?), it follows from the Fubini-Tonelli theorem that for $a \in \mathbb{C}$

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} U^\mu(a + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta &= \int \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|a + r\mathrm{e}^{\mathrm{i}\theta} - t|}\,\mathrm{d}\theta\,\mathrm{d}\mu(t) \\ &\le \int \log\frac{1}{|a-t|}\,\mathrm{d}\mu(t) = U^\mu(a) \end{aligned}$$

(see also (1.17)).

Exercise. Prove that if U^μ is harmonic in a neighborhood of a point z_0, then $z_0 \notin \operatorname{supp}(\mu)$.

While it may appear that potentials are rather special types of superharmonic functions, their properties are key to the analysis of general superharmonic functions. This is thanks to the following celebrated result.

Theorem 2.4 (**Riesz Decomposition**). *If f is superharmonic in a domain D, then there exists a positive measure λ supported on D such that for every subdomain $D^* \subset D$ for which $\overline{D^*} \subset D$, we have*

$$f(z) = h(z) + \int_D \log\frac{1}{|z-t|}\,\mathrm{d}\lambda(t), \quad z \in D^*, \tag{2.7}$$

where h is harmonic in D^*.

For the case when f is smooth, superharmonicity implies that $\Delta f = f_{xx} + f_{yy} \le 0$ in D and it turns out that the positive measure

$$\boxed{\mathrm{d}\lambda(t) := -\frac{1}{2\pi}\Delta f(t)\,\mathrm{d}m_2(t), \qquad t \in D,} \tag{2.8}$$

where m_2 denotes 2-dimensional Lebesgue measure, yields the appropriate potential U^λ for which (2.7) holds.

Let's verify this for the simple but important case when $f(z) = -|z|^2$, for which we have $\Delta f \equiv -4$. Then we need to show that $\Delta(f - U^\lambda) = 0$ in D^* where $\mathrm{d}\lambda = (2/\pi)\,\mathrm{d}m_2$. For an arbitrary disk $D_r(z_0) := \{z : |z - z_0| < r\}$ contained in D^*, write $U^\lambda = U^{\lambda_1} + U^{\lambda_2}$ where $\lambda_1 = \lambda|_{D_r(z_0)}$ and $\lambda_2 = \lambda - \lambda_1$. Then U^{λ_2} is harmonic in $D_r(z_0)$ and so we need only show that $\Delta(f - U^{\lambda_1}) = 0$ in $D_r(z_0)$. A simple calculation using (1.17) gives that

$$\begin{aligned} U^{\lambda_1}(z) &= \frac{2}{\pi}\int_{D_r(z_0)} \log\frac{1}{|z-t|}\,\mathrm{d}m_2(t) \\ &= 2r^2\log\frac{1}{r} + r^2 - |z - z_0|^2, \end{aligned}$$

from which we get $\Delta U^{\lambda_1}(z) = -4 = \Delta f$ for $z \in D_r(z_0)$, as desired. For more general but smooth f, one can use Green's formula[‡‡] to verify that (2.8) yields the decomposition in (2.7).

For general superharmonic functions f, we interpret the right-hand side of (2.8) in the *distributional sense* (cf. [R1, Sec. 3.7]); more precisely, we identify $-\Delta f\,\mathrm{d}m_2$ as the unique positive measure that satisfies

$$\int_D \phi(-\Delta f\,\mathrm{d}m_2) := -\int_D f\Delta\phi\,\mathrm{d}m_2 \tag{2.9}$$

for all C^∞ functions ϕ whose support is a compact subset of D. This condition is precisely what would be expected from Green's formula. The existence of such a measure $-\Delta f\,\mathrm{d}m_2$ satisfying (2.9) is guaranteed by the Riesz representation theorem for linear functionals. With this interpretation, it follows that for any finite Borel measure μ with compact support there holds

$$\boxed{\mu = -\frac{1}{2\pi}\Delta U^\mu.} \tag{2.10}$$

Just as the MVP (2.2) for harmonic functions implied the Max-Min Principle (Theorem 2.2), the mean-value inequality property (2.6) yields the following.

Theorem 2.5 (**Minimum Principle for Superharmonic Functions**). *Let D be a bounded domain and g a superharmonic function on D such that*

$$\liminf_{z\to\zeta} g(z) \ge m \qquad \forall\,\zeta \in \partial D. \tag{2.11}$$

Then $g(z) > m$ for all z in D, unless g is constant.

Exercise. Use the Min Principle to prove that a l.s.c. function f (not identically $+\infty$) is superharmonic in a domain D if and only if it has the following property: If $D_0 \subset D$ is a bounded domain whose closure is contained in D and u is harmonic in D_0, continuous on $\overline{D_0}$ and satisfies

[‡‡] Recall that Green's formula for smooth functions u, v on a bounded open set D with smooth boundary ∂D asserts that

$$\int_D (v\Delta u - u\Delta v)\,\mathrm{d}m_2 = -\int_{\partial D} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})\,\mathrm{d}s,$$

where $\partial/\partial n$ denotes differentiation in the direction of the inner normal to D.

$u(\zeta) \leq f(\zeta) \ \ \forall \, \zeta \in \partial D_0$, then $u(z) \leq f(z) \ \ \forall \, z \in D_0$.

A more general form of the Min Principle allows us to ignore a set of points on ∂D of capacity zero provided g is lower bounded on D; see [ST].

Theorem 2.6 (**Generalized Min Principle**). *If $D \subset \overline{\mathbb{C}}$ is a domain, $\operatorname{cap}(\partial D) > 0$, g is superharmonic and bounded from below in D and (2.11) holds for q.e. ζ on ∂D, then $g(z) > m \ \ \forall \, z \in D$ unless g is constant.*

Exercise. Give an example to show that the lower boundedness assumption cannot be removed in the above result.

For potentials what is crucial is their behavior on the support of its defining measure.

Theorem 2.7 (**Maximum Principle for Potentials**). *Let μ be a finite positive measure with compact support. If $U^\mu(z) \leq M$ for all $z \in \operatorname{supp}(\mu)$, then $U^\mu(z) \leq M$ for all $z \in \mathbb{C}$.*

The Max Principle is a special case of the following important result.

Theorem 2.8 (**Principle of Domination**). *Let μ, ν be positive finite measures with compact supports, $\nu(\mathbb{C}) \leq \mu(\mathbb{C})$, and μ has finite logarithmic energy ($I(\mu) < \infty$). If for some constant c, the inequality*

$$U^\mu(z) \leq U^\nu(z) + c$$

holds μ-a.e., then it holds for all $z \in \mathbb{C}$.

The idea of the proof of the above theorem is to consider the function

$$U(z) := \min(U^\nu(z) + c, U^\mu(z)),$$

which is superharmonic since the minimum of two superharmonic functions is again superharmonic. Let $\lambda := -\frac{1}{2\pi}\Delta U$ (i.e., λ is the measure guaranteed by the Riesz Decomposition Theorem (RDT) with $D = \mathbb{C}$) and argue that λ must equal μ. See [ST] for details.

As an application of the Principle of Domination, we present

Example 2.9. Let p_n, $n = 1, 2, \ldots$, be a sequence of monic polynomials of respective degrees n that satisfy

$$\limsup_{n\to\infty} \|p_n\|_{[-1,1]}^{1/n} \leq \frac{1}{2} = \operatorname{cap}([-1,1]), \tag{2.12}$$

and let $\nu(p_n)$ denote the normalized counting measure in the zeros of p_n. If all the zeros of the p_n's lie on $[-1,1]$, then

$$\nu(p_n) \xrightarrow{*} \mu_{[-1,1]} = \frac{\mathrm{d}x}{\pi\sqrt{1-x^2}} \quad \text{as} \quad n \to \infty. \tag{2.13}$$

Indeed, by definition of the sup norm,

$$\frac{1}{n}\log\frac{1}{|p_n(z)|} \geq \frac{1}{n}\log\frac{1}{\|p_n\|_{[-1,1]}}, \quad z \in [-1,1].$$

We write this last inequality in the equivalent form

$$U^{\nu(p_n)}(z) + \log 2 \geq \frac{1}{n} \log \frac{1}{\|p_n\|_{[-1,1]}} + U^{\mu_{[-1,1]}}(z), \quad z \in [-1,1], \tag{2.14}$$

where we used the fact that $U^{\mu_{[-1,1]}}(x) = \log 2$ for all $x \in [-1,1]$ (see Examples 1.11 and 3.6). By the Principle of Domination, (2.14) holds for all $z \in \mathbb{C}$. Now let ν be a weak-star limit measure of the sequence $\nu(p_n)$. Then from (2.12) and (2.14), we get

$$U^{\nu}(z) + \log 2 \geq \log 2 + U^{\mu_{[-1,1]}}(z) \quad \forall\, z \notin [-1,1],$$

i.e., $U^{\nu}(z) - U^{\mu_{[-1,1]}}(z) \geq 0$ for all $z \in \Omega := \overline{\mathbb{C}} \setminus [-1,1]$. But since $U^{\nu} - U^{\mu_{[-1,1]}}$ vanishes at ∞ and is harmonic in Ω, the Max-Min Principle asserts that $U^{\nu}(z) = U^{\mu_{[-1,1]}}(z)$ for $z \in \Omega$. By l.s.c., we then deduce that $U^{\nu}(x) \leq U^{\mu_{[-1,1]}}(x) = \log 2$ for all $x \in [-1,1]$ and so $I(\nu) \leq \log 2 = I(\mu_{[-1,1]})$. By uniqueness of the minimizing measure, we get that $\nu = \mu_{[-1,1]}$, which proves (2.13).

Actually, (2.13) *holds without any prior assumptions on the location of zeros of the p_n's.* This follows from the fact that (2.12) implies that the proportion of zeros that lie outside any open set containing $[-1,1]$ is asymptotically negligible (see Section 3).

Another consequence of the Riesz Decomposition Theorem is the following.

Theorem 2.10 (**Unicity Theorem**). *Let μ, ν be positive finite measures having compact support. If, in a region $D \subset \mathbb{C}$, there holds*

$$U^{\mu}(z) = U^{\nu}(z) + h(z) \qquad m_2\text{-a.e.},$$

where h is harmonic in D, then $\mu|_D = \nu|_D$.

In particular, if two potentials U^{μ} and U^{ν} agree except for a set of 2-dimensional Lebesgue measure zero, then $\mu = \nu$.

3 Equilibrium Potentials, Green Functions and Regularity

Throughout this section, $E \subset \mathbb{C}$ denotes a compact set with $\operatorname{cap}(E) > 0$. Here, we discuss properties and characterizations of the equilibrium (conductor) potential U^{μ_E}.

According to Frostman's Theorem 1.12, U^{μ_E} is "essentially" constant on E (more precisely, constant *q.e.* on E). So the following result should come as no surprise.

Theorem 3.1. *If $\nu \in \mathcal{M}(E)$ has finite logarithmic energy* (i.e., $I(\nu) < \infty$) *and $U^{\nu}(z) = c$ q.e. on E, then $c = V_E$ and $\nu = \mu_E$.*

In the proof of this result, a useful fact is the following.

Exercise. If a measure ν has finite logarithmic energy, then any set of capacity zero has ν measure zero.

With this fact, Theorem 3.1 follows by simply integrating the equality $U^{\nu} = c$ with respect to $d\mu_E$ and interchanging order of integration.

Exercise. Give an example to show that if $\nu \in \mathcal{M}(E)$ and U^{ν} is constant on E *except for one point*, then ν need not equal μ_E.

What can be said about the continuity properties of U^{μ_E}? Certainly U^{μ_E} is continuous in $\mathbb{C} \setminus \operatorname{supp}(\mu_E)$ since it is harmonic there. So suppose $z_0 \in \operatorname{supp}(\mu_E)$. Then by l.s.c. and Frostman's Theorem, we have

$$U^{\mu_E}(z_0) \leq \liminf_{z \to z_0} U^{\mu_E}(z) \leq \limsup_{z \to z_0} U^{\mu_E}(z) \leq V_E.$$

Hence if $U^{\mu_E}(z_0) = V_E$, then U^{μ_E} is continuous at z_0. The converse is also true. (Prove it!) To summarize, we have:

Theorem 3.2. *U^{μ_E} is continuous at $z_0 \in \operatorname{supp}(\mu_E)$ if and only if $U^{\mu_E}(z_0) = V_E$. Consequently, U^{μ_E} is continuous q.e. in the plane.*

Definition 3.3. A point $z_0 \in \partial_\infty E$ is said to be a **regular point** of the unbounded component Ω of $\overline{\mathbb{C}} \setminus E$ if $U^{\mu_E}(z_0) = V_E$. Otherwise, z_0 is called an **irregular point**. (From Theorem 3.2, we see that the set of all irregular points has capacity zero.) If every point of $\partial\Omega = \partial_\infty E$ is regular, we say that Ω is regular (with respect to the Dirichlet problem).

Exercise. Prove that every interior point of E satisfies $U^{\mu_E}(z) = V_E$.

The equilibrium potential is related to the Green function associated with the unbounded component of the complement of E; more precisely, we have

Definition 3.4. The **Green function with pole at** ∞ for the unbounded component Ω of $\mathbb{C} \setminus E$ is defined by

$$g_\Omega(z, \infty) := V_E - U^{\mu_E}(z). \tag{3.1}$$

(Some authors write $g_E(z, \infty)$ instead of $g_\Omega(z, \infty)$.)

Three properties uniquely *characterize* this function for $z \in \Omega$; namely

(a) $g_\Omega(z, \infty)$ is harmonic in $\Omega \setminus \{\infty\}$ and bounded from above and below outside each neighborhood of ∞;

(b) $g_\Omega(z, \infty) - \log|z| = O(1)$ as $z \to \infty$;

(c) $g_\Omega(z, \infty) \to 0$ as $z \to \zeta,\ z \in \Omega$, for q.e. $\zeta \in \partial\Omega$.

Regarding property (b), it follows from (3.1) and the fact that μ_E is a unit measure that

$$g_\Omega(z, \infty) - \log|z| \to V_E = \log\frac{1}{\operatorname{cap}(E)} \text{ as } z \to \infty. \tag{3.2}$$

It is also clear from (3.1) that $g_\Omega(z, \infty) \geq 0$, $g_\Omega(z, \infty) > 0$ for $z \in \Omega$, and in view of Theorem 3.2, if $\zeta \in \partial\Omega = \partial_\infty E$, then $g_\Omega(z, \infty) \to 0$ as $z \to \zeta$, $z \in \Omega$, if and only if ζ is a regular point of Ω.

Exercise. Prove that if $\tilde{g}(z)$ is a function that satisfies properties (a), (b), and (c) for $z \in \Omega$, then $\tilde{g}(z) = g_\Omega(z, \infty)$ for $z \in \Omega$.

In the case when Ω is *simply connected*, we can relate the Green function to a Riemann mapping of Ω.

Theorem 3.5. *If the unbounded component Ω of $\mathbb{C} \setminus E$ is simply connected, then $g_\Omega(z, \infty) = \log|\Phi(z)|,\ z \in \Omega$, where $w = \Phi(z)$ is the unique Riemann mapping function from Ω to the exterior $\{w : |w| > 1\}$ of the unit disk such that $\Phi(\infty) = \infty,\ \Phi'(\infty) > 0$.*

Such a function Φ has a Laurent expansion about ∞ of the form

$$\Phi(z) = \frac{z}{c} + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\ , \text{ with } c > 0, \tag{3.3}$$

and using this representation, properties (a), (b), and (c) are easy to establish.

Exercise. Prove Theorem 3.5.

From (3.3), we immediately see that

$$\log|\Phi(z)| - \log|z| \to \log\frac{1}{c} \text{ as } z \to \infty,$$

and comparison with (3.2) shows that

$$\boxed{c = \mathrm{cap}(E).}$$

From this fact, we can determine the capacity of any compact set E providing we know the exterior conformal mapping function Φ. We illustrate this for the line segment.

Example 3.6. Let $E = [-1, 1]$. The well-known Joukowski transformation $z = \psi(w) = \frac{1}{2}(w + w^{-1})$ maps the exterior of the unit circle onto $\Omega := \overline{\mathbb{C}} \setminus [-1, 1]$, with $\psi(\infty) = \infty,\ \psi'(\infty) > 0$. Solving for w, we obtain the desired Riemann mapping:

$$w = \Phi(z) = z + \sqrt{z^2 - 1},$$

where $\sqrt{z^2 - 1}$ behaves like z near infinity. Thus

$$g_\Omega(z, \infty) = \log\left|z + \sqrt{z^2 - 1}\right|,$$

and since $\Phi(z) = 2z + \cdots$ near infinity, we get that $\mathrm{cap}([-1, 1]) = 1/2$. Furthermore, from (3.1), we have

$$U^{\mu_E}(z) = \log 2 - \log\left|z + \sqrt{z^2 - 1}\right|$$

as claimed in Example 1.11.

Exercise. Show that if the unbounded component Ω of $\overline{\mathbb{C}} \setminus E$ is simply connected, then every point of $\partial_\infty E$ is regular, i.e., Ω is a regular domain.

Exercise. By constructing a suitable mapping function, show that the capacity of an ellipse with semi-axis lengths a and b is $(a+b)/2$.

Exercise. Show that if the compact set E has positive capacity and $p(z)$ is a monic polynomial of degree n, then the set $p^{-1}(E)$ has capacity $[\mathrm{cap}(E)]^{1/n}$.

In the case when $\partial_\infty E$ is a smooth closed Jordan curve, there is a simple representation for μ_E in terms of the Green function $g_\Omega = g_\Omega(z,\infty)$. Using Green's formula, one first shows that, for $z \in \Omega$, the equilibrium potential $(= V_E - g_\Omega)$ identically equals the potential of the unit measure $\frac{1}{2\pi}\frac{\partial g_\Omega}{\partial n}\,\mathrm{d}s$, where the derivative is taken in the direction of the *outer* normal on $\partial_\infty E$ and s denotes arclength on $\partial_\infty E$ (cf. [W, Sec. 4.2]). On letting $z \in \Omega$ approach $\partial_\infty E$ and appealing to the lower semi-continuity of potentials, it follows that $\frac{1}{2\pi}\frac{\partial g_\Omega}{\partial n}\,\mathrm{d}s$ has energy at most V_E, and so by uniqueness of the minimizing measure, we deduce that $\mathrm{d}\mu_E = \frac{1}{2\pi}\frac{\partial g_\Omega}{\partial n}\,\mathrm{d}s$. Thus, for any Borel subset γ of $\partial_\infty E$,

$$\mu_E(\gamma) = \frac{1}{2\pi}\int_\gamma \frac{\partial g_\Omega}{\partial n}\,\mathrm{d}s = \frac{1}{2\pi}\int_\gamma |\Phi'|\,\mathrm{d}s.$$

Alternatively, $\mu_E(\gamma)$ is given by the normalized angular measure of the image $\Phi(\gamma)$:

$$\mu_E(\gamma) = \frac{1}{2\pi}\int_{\Phi(\gamma)} \mathrm{d}\theta \tag{3.4}$$

(for this representation, the smoothness of $\partial_\infty E$ is not needed).

The Green function is especially useful for estimating the modulus of a polynomial outside E when its sup norm on E is known.

Lemma 3.7. (**Bernstein-Walsh**). *If $p_n(z)$ is any polynomial of degree $\le n$, then*

$$|p_n(z)| \le \|p_n\|_E\ \mathrm{e}^{n g_\Omega(z,\infty)}, \qquad z \in \Omega, \tag{3.5}$$

where $\|p_n\|_E := \max_{z\in E}|p_n(z)|$ and Ω is the unbounded component of $\mathbb{C}\setminus E$.

Proof. Assume that p_n has exact degree n. Then (3.5) is equivalent to

$$\frac{1}{n}\log\frac{1}{|p_n(z)|} + g_\Omega(z,\infty) \ge \frac{1}{n}\log\frac{1}{\|p_n\|_E}, \qquad z \in \Omega. \tag{3.6}$$

Let $u(z)$ denote the left-hand side of (3.6) and note that u is superharmonic in Ω and harmonic at ∞. Moreover, from property (c) for g_Ω, we deduce that

$$\liminf_{\substack{z\to\zeta\\ z\in\Omega}} u(z) \ge \frac{1}{n}\log\frac{1}{\|p_n\|_E} \text{ for q.e. } \zeta \in \partial\Omega.$$

Thus (3.6) follows from the Minimum Principle for Superharmonic Functions. □

It is useful to consider Green functions with poles at finite points of the plane. If D is a domain with $\mathrm{cap}(\partial D) > 0$, $\partial D \subset \mathbb{C}$ compact, the **Green function** $g_D(z,\zeta)$ **for** D **with pole at** $\zeta \in D$ is the unique real-valued function of z satisfying (a') $g_D(z,\zeta)$ is harmonic in $D\setminus\{\zeta\}$ and bounded outside any neighborhood of ζ.

(b') $g_D(z,\zeta) - \log\frac{1}{|z-\zeta|} = O(1)$ as $z\to\zeta$.

(c') $\lim\limits_{\substack{z\to w\\ z\in D}} g_D(z,\zeta) = 0$ for q.e. $w\in\partial D$.

The relation of this Green function to the one with pole at ∞ is easy to see. Consider the mapping $w = 1/(z-\zeta)$ that takes ζ to ∞ and D to some domain D'. Then

$$g_D(z,\zeta) = g_{D'}\left(\frac{1}{z-\zeta},\infty\right). \tag{3.7}$$

Exercise. Verify that the function of z on the right-hand side of (3.7) satisfies properties (a'), (b'), (c').

Exercise. Verify that for the unit disk $D : |z| < 1$,

$$g_D(z,\zeta) = \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right|, \qquad z,\zeta\in D.$$

A clever application of Green's formula shows that g_D is symmetric: $g_D(z,\zeta) = g_D(\zeta,z)$; see [Ts].

Exercise. Prove that if p_n, $n = 1,2,\ldots$, is a sequence of monic polynomials of respective degrees n that satisfy

$$\limsup_{n\to\infty} \|p_n\|_{[-1,1]}^{1/n} \le \frac{1}{2} = \operatorname{cap}([-1,1]),$$

then the proportion of the number of zeros of the p_n's that lie outside any neighborhood of $[-1,1]$ tends to zero as $n\to\infty$. (Recall Example 2.9 and the remark following it.) [Hint: Let $\{z_{n,k}\}_{k\in J_n}$ denote the zeros of p_n that lie at a distance $\ge\epsilon>0$ from $[-1,1]$ and consider the functions

$$\frac{1}{n}\log\frac{1}{|p_n(z)|} - \frac{1}{n}\sum_{k\in J_n} g(z,z_{n,k}) + g(z,\infty),$$

where $g = g_{\overline{\mathbb{C}}\setminus[-1,1]}$.]

Just as $\log\frac{1}{|z-t|}$ serves as the kernel for logarithmic potential theory, so too does $g_D(z,t)$ serve as the kernel for *Green potential theory.* If ν is a finite positive measure on D with compact support in $\mathbb{C}$, we define

$$U_D^\nu(z) := \int g_D(z,\zeta)\,\mathrm{d}\nu(\zeta), \qquad z\in D, \tag{3.8}$$

and note that $U_D^\nu \ge 0$ in D and U_D^ν is superharmonic in D and harmonic in $D\setminus\operatorname{supp}(\nu)$. Furthermore, if ν has compact support in D, then

$$\lim_{\substack{z\to w\\ z\in D}} U_D^\nu(z) = 0 \text{ for q.e. } w\in\partial D.$$

The **Green energy** of a measure ν is defined by

$$I^D(\nu) := \int\int g_D(z,\zeta)\,\mathrm{d}\nu(z)\,\mathrm{d}\nu(\zeta). \tag{3.9}$$

For a closed subset $E \subset D$ of positive logarithmic capacity, we consider the minimum energy problem

$$V_E^D := \inf\{I^D(\nu) : \nu \in \mathcal{M}(E)\} \tag{3.10}$$

for which there exists a unique measure (the **Green equilibrium measure**) $\mu_E^D \in \mathcal{M}(E)$ such that $I^D(\mu_E^D) = V_E^D$. Analogous to Frostman's Theorem 1.12, there holds

$$U_D^{\mu_E^D}(z) = V_E^D \text{ q.e. on } E \tag{3.11}$$

$$U_D^{\mu_E^D}(z) \le V_E^D \text{ for all } z \in D. \tag{3.12}$$

The constant

$$\mathrm{cap}(E,\partial D) := \frac{1}{V_E^D} \tag{3.13}$$

is called the **capacity of the condenser** $(E,\partial D)$; see Section 5.

Balayage

Let $D \subset \overline{\mathbb{C}}$ be an open set with compact boundary ∂D of positive capacity and let μ be a measure with $\mathrm{supp}(\mu) \subset \overline{D}$. The problem of **balayage** (a French word meaning "sweeping") consists of finding a new measure μ^b *supported on* ∂D such that $\mu^b(\mathbb{C}) = \mu(\mathbb{C})$ and

$$U^{\mu^b}(z) = U^{\mu}(z) \text{ for q.e. } z \notin D. \tag{3.14}$$

For a bounded domain D, the sweeping out of the measure μ to ∂D can always be accomplished, but if the domain $D \subset \overline{\mathbb{C}}$ contains the point at infinity, it is necessary to modify (3.14) so that it reads

$$U^{\mu^b}(z) = U^{\mu}(z) + c \text{ for q.e. } z \notin D, \tag{3.15}$$

for some constant c. Necessarily

$$c = \int g_D(z,\infty)\,\mathrm{d}\mu(z). \tag{3.16}$$

If D is connected and regular, then equality in (3.14) and (3.15) holds for all $z \notin D$. To ensure uniqueness of μ^b, a condition such as boundedness of U^{μ^b} on ∂D suffices.

Exercise. Verify that the constant c in (3.15) is given by (3.16). [Hint: Starting with

$$\int g_D(z,\infty)\,\mathrm{d}\mu(z) = \int [V_E - U^{\mu_{\partial D}}(z)]\,\mathrm{d}\mu(z),$$

use the reciprocity law together with (3.15) on ∂D.]

Balayage measures can also be characterized by the following property: if h is any function that is continuous on $\overline{D}$ and harmonic in D, then

$$\int_D h\,\mathrm{d}\mu = \int_{\partial D} h\,\mathrm{d}\mu^b. \tag{3.17}$$

Example 3.8. If D is the unit disk $|z| < 1$ and δ_ζ is the unit point mass at $\zeta \in D$, then the balayage of δ_ζ to $\partial D : |z| = 1$ is given by

$$\mathrm{d}\delta_\zeta^b(t) = \frac{1}{2\pi} P(t, \zeta)\, \mathrm{d}\theta, \quad t = \mathrm{e}^{\mathrm{i}\theta}, \quad 0 \le \theta \le 2\pi,$$

where P denotes the Poisson kernel (2.4).

If $\overline{D}$ is a compact subset of $\mathbb{C}$ with $\mathrm{cap}(\overline{D}) > 0$, then the balayage of δ_∞ onto the outer boundary of D is the equilibrium measure $\mu_{\overline{D}}$.

The notion of balayage is intimately connected to the Green function of a domain. Indeed, if D is bounded, $\zeta \in D$, and δ_ζ^b denotes the balayage of δ_ζ to $\partial_\infty D$, then

$$g_D(z, \zeta) = \log \frac{1}{|z - \zeta|} - U^{\delta_\zeta^b}(z), \tag{3.18}$$

since, as can be verified, the right-hand satisfies properties (a'), (b'), and (c') that characterize the Green function with pole at ζ. If μ is a finite positive measure on D, then

$$\mu^b = \int \delta_\zeta^b \, \mathrm{d}\mu(\zeta) \tag{3.19}$$

and so, on integrating (3.18) with respect to $\mathrm{d}\mu(\zeta)$, we get

$$U_D^\mu(z) = \int g_D(z, \zeta)\, \mathrm{d}\mu(\zeta) = U^\mu(z) - U^{\mu^b}(z), \quad z \in D. \tag{3.20}$$

As a consequence of (3.20) and the nonnegativity of the Green potential, we get

$$U^{\mu^b}(z) \le U^\mu(z) \qquad \forall z \in \mathbb{C}. \tag{3.21}$$

In case D is an unbounded domain with $\partial D \subset \mathbb{C}$, then (3.18), (3.19), and (3.20) must be modified to include the constant c of (3.16); e.g., (3.21) becomes

$$U^{\mu^b}(z) \le U^\mu(z) + c \qquad \forall z \in \mathbb{C}. \tag{3.22}$$

4 Applications to Polynomial Approximation of Analytic Functions

Let f be a continuous complex-valued function on a compact set $E \subset \mathbb{C}$ and let

$$e_n(f; E) = e_n(f) := \min_{p \in \mathcal{P}_n} \|f - p\|_E \tag{4.1}$$

be the error in best uniform approximation of f by polynomials of degree at most n. We denote by p_n^* the polynomial of best approximation: $\|f - p_n^*\|_E = e_n(f)$.

If $e_n(f) \to 0$ as $n \to \infty$, the series

$$p_1^* + \sum_{n=1}^{\infty} (p_{n+1}^* - p_n^*)$$

converges to f uniformly on E, so that the continuous function f must be analytic at every interior point of E. (The collection of all functions that are continuous on E and analytic in the interior of E is denoted by $\mathcal{A}(E)$.) Furthermore, it follows from the maximum principle for analytic functions, that the above series automatically converges on every bounded component of $\mathbb{C} \setminus E$, so that its sum represents an analytic continuation of f to these components (e.g., if E is the unit circle $|z| = 1$, then the convergence holds in the unit disk $|z| \leq 1$). Such a continuation, however, may be impossible. Therefore, in order to ensure that $e_n(f) \to 0$ for *every function* f in $\mathcal{A}(E)$, it is necessary to assume that the only component of $\mathbb{C} \setminus E$ is the unbounded one; that is, $\mathbb{C} \setminus E$ is connected (so that E does not separate the plane).

A celebrated theorem of S.N. Mergelyan (cf. [Ga]) asserts that this assumption is also sufficient. Here, we prove this result in a special case when E has a connected and regular complement $\Omega := \overline{\mathbb{C}} \setminus E$ and f is analytic in some neighborhood of E. Our aim is to determine the *rate of approximation*.

For any $R > 1$, let Γ_R denote the level curve $\{z : g_\Omega(z, \infty) = \log R\}$, see Fig. 2 (we call such a curve a **level curve with index** R). The assumption that Ω is regular ensures that for any open set V containing E, the level curve Γ_R will lie in V for R sufficiently close to 1.

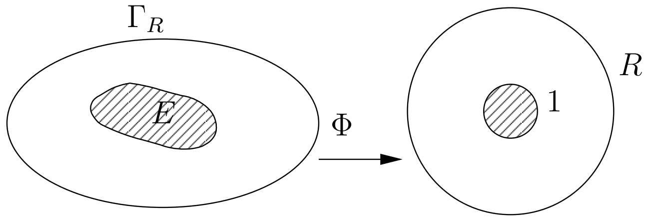

Figure 2: Level curve of $g_\Omega(z, \infty)$

Let F_{n+1} be the $(n+1)$-st degree Fekete polynomial for E and let P_n be the polynomial of degree $\leq n$ that interpolates f at the zeros of F_{n+1}. We are given that f is analytic in a neighborhood of E; hence there exists $R > 1$ such that f is analytic on and inside Γ_R. For any such R, the **Hermite interpolation formula** yields

$$f(z) - P_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_R} \frac{F_{n+1}(z)}{F_{n+1}(t)} \frac{f(t)\,\mathrm{d}t}{t - z}, \quad z \text{ inside } \Gamma_R. \tag{4.2}$$

(The validity of the Hermite formula follows by first observing that the right-hand side vanishes at the zeros of $F_{n+1}(z)$, and then by replacing $f(z)$ by its Cauchy integral representation to deduce that the difference between f and the right-hand side is indeed a polynomial of degree at most n.)

Formula (4.2) leads to a simple estimate:

$$e_n(f) \leq \|f - P_n\|_E \leq K \frac{\|F_{n+1}\|_E}{\min_{\Gamma_R} |F_{n+1}(t)|},$$

where K is some constant independent of n. Applying parts (b), (c) of the Fundamental Theorem 1.18, we obtain that

$$\limsup_{n \to \infty} e_n(f)^{1/n} \leq \frac{\mathrm{cap}(E)}{R\,\mathrm{cap}(E)} = \frac{1}{R} < 1. \tag{4.3}$$

We have proved that indeed $e_n(f) \to 0$ and that the convergence is geometrically fast. Since $R > 1$ was arbitrary (but such that f is analytic on and inside Γ_R), we have actually proved that (4.3) holds with R replaced by $R(f)$, where

$$R(f) := \sup\{R : \ f \text{ admits analytic continuation to the interior of } \Gamma_R\}.$$

Can we improve on this? The answer is — no! In order to show this, we appeal to the Bernstein-Walsh Lemma 3.7.

Assume now that (4.3) holds for some $R > R(f)$ and let $R(f) < \rho < R$. Then for some constant $c > 1$,

$$e_n(f) \le \frac{c}{\rho^n}, \quad n \ge 1.$$

Since, from the triangle inequality,

$$\|p_{n+1}^* - p_n^*\|_E = \|p_{n+1}^* - f + f - p_n^*\|_E \le e_{n+1}(f) + e_n(f) \le 2c\rho^{-n},$$

we obtain from the Bernstein-Walsh Lemma that for any $r > 1$,

$$\|p_{n+1}^* - p_n^*\|_{\Gamma_R} \le 2c\left(\frac{r}{\rho}\right)^n, \quad n \ge 1.$$

If we choose $R(f) < r < \rho$, we obtain that the series $p_1^* + \sum_{n=1}^{\infty}(p_{n+1}^* - p_n^*)$ converges uniformly inside Γ_r. Hence it gives an analytic continuation of f to the interior of Γ_r, which contradicts the definition of $R(f)$.

Let us summarize what we have proved.

Theorem 4.1 (**Walsh** [W, Ch. VII]). *Let E be a compact set with connected and regular complement. Then for any $f \in \mathcal{A}(E)$,*

$$\limsup_{n\to\infty} e_n(f)^{1/n} = \frac{1}{R(f)}.$$

Remark. $R(f)$ is the first value of R for which the level curve Γ_R contains a singularity of f. It may well be possible that f is analytic at some other points of $\Gamma_{R(f)}$, but the geometric rate of best polynomial approximation "does not feel this" — whether every point of $\Gamma_{R(f)}$ is a singularity or merely one point is a singularity, the rate of approximation remains the same as if f was analytic only inside of $\Gamma_{R(f)}$! To take advantage of any extra analyticity, different approximation tools are needed; e.g., rational functions.

Example 4.2. Let $E = [-1, -\alpha] \cup [\alpha, 1]$, $0 < \alpha < 1$, and let $f = 0$ on $[-1, -\alpha]$ and $f = 1$ on $[\alpha, 1]$. Some level curves Γ_R of $g_{\mathbb{C}\setminus E}$ are depicted on Fig. 3. For R small, Γ_R consists of two pieces, while for R large, Γ_R is a single curve. There is a "critical value" $R_0 = g_{\mathbb{C}\setminus E}(0, \infty)$ for which Γ_{R_0} represents a self-intersecting lemniscate-like curve (the bold curve in Fig. 3). Clearly, f can be extended as an analytic function to the interior of Γ_{R_0} (define $f = 0$ inside the left lobe and $f = 1$ inside the right lobe). For $R > R_0$, the interior of Γ_R is a (connected) domain; hence there is no function analytic inside of Γ_R that is equal to 0 on $[-1, -\alpha]$ and to 1 on $[\alpha, 1]$. Therefore

$$R(f) = R_0 = \exp\left\{g_{\mathbb{C}\setminus E}(0, \infty)\right\},$$

and by Theorem 4.1:

$$\limsup_{n\to\infty} e_n(f)^{1/n} = \exp\left\{-g_{\mathbb{C}\setminus E}(0, \infty)\right\}.$$

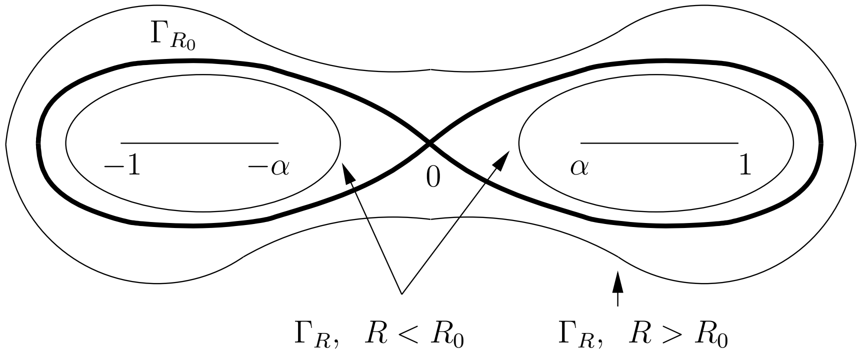

Figure 3: Level curves of $g_{\mathbb{C}\setminus E}$ for Example 4.2

5 Rational Approximation

For a rational function $R(z) = P_1(z)/P_2(z)$, where P_1 and P_2 are monic polynomials of degree n, one can write

$$-\frac{1}{n}\log|R(z)| = U^{\nu_1}(z) - U^{\nu_2}(z),$$

where ν_1, ν_2 are the normalized zero counting measures for P_1, P_2, respectively. The right-hand side represents the logarithmic potential of the *signed measure* $\mu = \nu_1 - \nu_2$:

$$U^{\nu_1}(z) - U^{\nu_2}(z) = U^{\mu}(z) = \int \log\frac{1}{|z-t|}\,\mathrm{d}\mu(t).$$

The theory of such potentials can be developed along the same lines as in the earlier sections. We present below only the very basic notions of this theory that are needed to formulate the approximation results. A more in-depth treatment can be found in the works of Bagby [B], Gonchar [Gon], as well as [ST].

The analogy with electrostatics problems suggests considering the following energy problem. Let $E_1, E_2 \subset \mathbb{C}$ be two closed sets that are a positive distance apart. The pair (E_1, E_2) is called a **condenser** and the sets E_1, E_2 are called the **plates.** Let μ_1 and μ_2 be positive unit measures supported on E_1 and E_2, respectively. Consider the energy integral of the signed measure $\mu = \mu_1 - \mu_2$:

$$I(\mu) = \int\int \log\frac{1}{|z-t|}\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(t).$$

Since $\mu(\mathbb{C}) = 0$, the integral is well-defined, even if one of the sets is unbounded. While not obvious, it turns out that such $I(\mu)$ is always *positive.* We assume that E_1 and E_2 have positive logarithmic capacity. Then the minimal energy (over all signed measures of the above form)

$$V(E_1, E_2) := \inf_{\mu} I(\mu)$$

is finite and positive. We then define the **condenser capacity** $\mathrm{cap}(E_1, E_2)$ by

$$\mathrm{cap}(E_1, E_2) := 1/V(E_1, E_2).$$

One can show, as with the Frostman theorem, that there exists a unique signed measure $\mu^* = \mu_1^* - \mu_2^*$ (the *equilibrium measure* for the condenser) for which $I(\mu^*) = V(E_1, E_2)$. Furthermore, the corresponding potential (called the **condenser potential**) is constant on each plate:

$$U^{\mu^*} = c_1 \text{ on } E_1, \quad U^{\mu^*} = -c_2 \text{ on } E_2 \tag{5.1}$$

(we assume throughout that E_1, E_2 are regular — otherwise the above equalities hold only quasi-everywhere). On integrating against μ^*, we deduce from (5.1) that

$$c_1 + c_2 = V(E_1, E_2) = 1/\mathrm{cap}(E_1, E_2). \tag{5.2}$$

We mention that (similar to the case of the conductor potential) the relations of type (5.1) *characterize* μ^*. Moreover, one can deduce from (5.1) that the measure μ_i^* is supported on the boundary (not necessarily the outer one) of E_i, $i = 1, 2$. Therefore, on replacing each E_i by its boundary, we do not change the condenser capacity or the condenser potential.

Example 5.1. Let E_1, E_2 be, respectively, the circles $|z| = r_1$ $|z| = r_2$, $r_1 < r_2$. These sets are invariant under rotations. Being unique, the measure μ^* is therefore also invariant under rotations and we obtain that

$$\mathrm{d}\mu_1^* = \frac{1}{2\pi r_1}\,\mathrm{d}s, \quad \mathrm{d}\mu_2^* = \frac{1}{2\pi r_2}\,\mathrm{d}s,$$

where $\mathrm{d}s$ denotes the arclength over the respective circles E_1, E_2. Applying the result of Example 1.10, we find that

$$U^{\mu^*}(z) = \begin{cases} 0, & |z| > r_2 \\ \log(r_2/|z|), & r_1 \le |z| \le r_2 \\ \log(r_2/r_1), & |z| < r_1. \end{cases}$$

Therefore (recall (5.2))

$$\mathrm{cap}(E_1, E_2) = 1/\log\frac{r_2}{r_1}. \tag{5.3}$$

Assume now that each plate of a condenser is a single Jordan arc or curve (without self-intersections), and let G be the doubly-connected domain that is bounded by E_1 and E_2, see Fig. 4. We call such a G a **ring domain**. For ring domains one can give an alternative definition

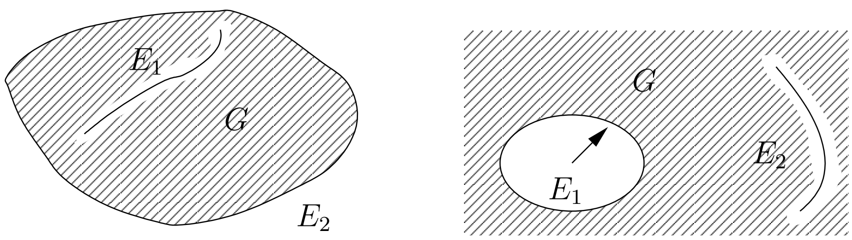

Figure 4: Ring domains

of condenser capacity. Let

$$u(z) := \int \log(z - t)\,\mathrm{d}\mu^*(t) + c_1.$$

The complex function u is locally analytic but not single-valued in G (notice that there is no modulus sign in the integral). Moreover, if we fix t and let z move along a simple closed counterclockwise oriented curve in G that encircles E_1, say, then the imaginary part of $\log(z-t)$ increases by 2π, for $t \in E_1$, while for $t \in E_2$ it returns to the original value. Since μ_1^* and μ_2^* are unit measures, it follows that the function $\phi : z \to w = \exp(u(z))$ is analytic and single-valued. Moreover, it can be shown to be one-to-one in G. By its definition, ϕ satisfies

$$\log|\phi| = -U^{\mu^*} + c_1 = 0 \text{ on } E_1; \quad \log|\phi| = -U^{\mu^*} + c_1 = c_1 + c_2 \text{ on } E_2.$$

Therefore ϕ maps G conformally onto the annulus $1 < |w| < \mathrm{e}^{c_1+c_2}$.

It is known from the theory of conformal mapping that, for a ring domain G, there exists unique $R > 1$, called the **modulus of** G (we denote it by $\mathrm{mod}(G)$), such that G can be mapped conformally onto the annulus $1 < |w| < R$. We have thus shown that

$$\mathrm{cap}(E_1, E_2) = 1/\log(\mathrm{mod}(G)). \tag{5.4}$$

We remark that if $G_1 \supset G_2$ are two ring domains, then $\mathrm{mod}(G_1) \geq \mathrm{mod}(G_2)$.

Example 5.2. Let E_1, E_2 be as above, and assume that E_2 is the R-th level curve for E_1. That is, $|\Phi(z)| = R$ for $z \in E_2$, where Φ maps conformally the unbounded component of $\mathbb{C} \setminus E_1$ onto $|w| > 1$. In particular, Φ maps the corresponding ring domain G onto the annulus $1 < |w| < R$, and we conclude that $\mathrm{mod}(G) = R$ (so that $\mathrm{cap}(E_1, E_2) = 1/\log R$). Applying this to the configuration of Example 5.1, we see that $\Phi(z) = z/r_1$, so that $R = r_2/r_1$, and we obtain again (5.3).

We now turn to rational approximation. Let $E \subset \mathbb{C}$ be compact. We denote by $\mathcal{R}_n$ the collection of all rational functions of the form $R = P/Q$, where P, Q are polynomials of degree at most n, and Q has no zeros in E. For $f \in \mathcal{A}(E)$, let

$$r_n(f; E) = r_n(f) := \inf_{r \in \mathcal{R}_n} \|f - r\|_E$$

be the error in best approximation of f by rational functions from $\mathcal{R}_n$. Clearly, since polynomials are rational functions, we have (cf. (4.1)) $r_n(f) \leq e_n(f)$. A basic theorem regarding the rate of rational approximation was proved by Walsh [W, Ch.IX]. Following is a special case of this theorem.

Theorem 5.3. (Walsh) *Let E be a single Jordan arc or curve and let f be analytic on a simply connected domain $D \supset E$. Then*

$$\limsup_{n \to \infty} r_n(f)^{1/n} \leq \exp(-1/\mathrm{cap}(E, \partial D)). \tag{5.5}$$

The proof of (5.5) follows the same ideas as the proof of inequality (4.3). Let Γ be a contour in $D \setminus E$ that is arbitrarily close to ∂D. Let $\mu^* = \mu_1^* - \mu_2^*$ be the equilibrium measure for the condenser (E, Γ). For any n, let $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ be equally spaced on E (with respect to μ_1^*) and let $\beta_1^{(n)}, \ldots, \beta_n^{(n)}$ be equally spaced on Γ (with respect to μ_2^*). Then one can show that the rational functions $r_n(z)$ with zeros at the $\alpha_i^{(n)}$'s and poles at the $\beta_i^{(n)}$'s satisfy

$$\left(\frac{\max\limits_E |r_n|}{\min\limits_\Gamma |r_n|}\right)^{1/n} \to \mathrm{e}^{-1/\mathrm{cap}(E,\Gamma)}. \tag{5.6}$$

Let $R_n = p_{n-1}/q_n$ be the rational function with poles at the $\beta_i^{(n)}$'s that interpolates f at the points $\alpha_i^{(n)}$'s. Then the Hermite formula (cf. (4.2)) takes the following form:

$$f(z) - R_n(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{r_n(z)}{r_n(t)} \frac{f(t)}{t-z}\, \mathrm{d}t, \quad z \text{ inside } \Gamma,$$

and it follows from (5.6) that

$$\limsup_{n\to\infty} r_n(f)^{1/n} \le \limsup_{n\to\infty} \|f - R_n\|_E^{1/n} \le \mathrm{e}^{-1/\mathrm{cap}(E,\Gamma)}.$$

Letting Γ approach ∂D, we get the result.

Remarks.
(a) Unlike in the polynomial approximation, no rate of convergence of $r_n(f)$ to 0 can ensure that a function $f \in C(E)$ is analytic somewhere beyond E.
(b) One can construct a function for which equality holds in (5.5), so that this bound is sharp. Such a function necessarily has a singularity at every point of ∂D; otherwise f would be analytic in a larger domain, so that the corresponding condenser capacity will become smaller. In view of Theorem 5.3, this would violate the assumed equality in (5.5).

Although sharp, the bound (5.5) is unsatisfactory, in the following sense. Assume, for example, that E is connected and has a connected complement, and let Γ_R, $R > 1$, be a level curve for E. Let f be a function that is analytic in the domain D bounded by Γ_R and such that the equality holds in (5.5). According to Example 5.2, we then obtain that

$$\limsup_{n\to\infty} r_n(f)^{1/n} = \frac{1}{R}. \tag{5.7}$$

By Remark (b) above, such f must have singularities on Γ_R. Hence (recall Remark following Theorem 4.1) the relation (5.7) holds with $r_n(f)$ replaced by $e_n(f)$. But the family $\mathcal{R}_n$ contains $\mathcal{P}_n$ and it is much more rich than $\mathcal{P}_n$ — it depends on $2n+1$ parameters while $\mathcal{P}_n$ depends only on $n+1$ parameters. One would expect, therefore, that at least for a subsequence of n's, $r_n(f)$ behaves asymptotically like $e_{2n}(f)$. This was a motivation for the following conjecture.

Conjecture. (A.A. Gonchar) *Let E be a compact set and f be analytic in an open set D containing E. Then*

$$\liminf_{n\to\infty} r_n(f;E)^{1/n} \le \exp(-2/\mathrm{cap}(E,\partial D)). \tag{5.8}$$

This conjecture was proved by O. Parfenov [Pa] for the case when E is a continuum with connected complement and in the general case by V. Prokhorov [P]; they used a very different method — the so-called "AAK Theory" (cf. [Y]). However this method is not constructive, and it remains a challenging problem to find such a method. Yet, potential theory can be used to obtain bounds like (5.8) in the stronger form

$$\lim_{n\to\infty} r_n(f;E)^{1/n} = \exp(-2/\operatorname{cap}(E,\partial D))$$

for some important *subclasses* of analytic functions, such as Markov functions (cf. [Gon]) and functions with a finite number of algebraic branch-points (cf. [St]).

6 Logarithmic Potentials with External Fields

Let E be a closed (not necessarily compact) subset of $\mathbb{C}$ and let $w(z)$ be a nonnegative weight on E. We define a new "distance function" on E, replacing $|z-t|$ by $|z-t|w(z)w(t)$. This gives rise to weighted versions of logarithmic capacity, transfinite diameter and Chebyshev constant.

Weighted capacity: $\operatorname{cap}(w,E)$.

As before, let $\mathcal{M}(E)$ denote the collection of all unit measures supported on E. We set

$$Q := \log\frac{1}{w}$$

and call it the **external field**. Consider the modified energy integral for $\mu\in\mathcal{M}(E)$:

$$\begin{aligned} I_w(\mu) &:= \int\int \log\frac{1}{|z-t|w(z)w(t)}\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(t) \qquad (6.1)\\ &= \int\int \log\frac{1}{|z-t|}\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(t) + 2\int Q(z)\,\mathrm{d}\mu(z) \end{aligned}$$

and let

$$V_w := \inf_{\mu\in\mathcal{M}(E)} I_w(\mu).$$

The **weighted capacity** is defined by

$$\operatorname{cap}(w,E) := \mathrm{e}^{-V_w}.$$

In the sequel, we assume that w satisfies the following conditions:

(i) $w>0$ on a subset of positive logarithmic capacity;

(ii) w is continuous (or, more generally, upper semi-continuous);

(iii) if E is unbounded, then $|z|w(z)\to 0$ as $|z|\to\infty$, $z\in E$.

Under these restrictions on w, there exists a unique measure $\mu_w\in\mathcal{M}(E)$, called the **weighted equilibrium measure**, such that

$$I(\mu_w) = V_w.$$

The above integral (6.1) can be interpreted as the total energy of the unit charge μ, in the presence of the external field Q (in this electrostatics interpretation, the field is actually $2Q$). Since this field has a strong repelling effect near points where $w=0$ (i.e., $Q=\infty$), assumption (iii) physically means that, for the equilibrium distribution, no charge occurs near ∞. In other words, the support

$\operatorname{supp}(\mu_w)$ of μ_w is necessarily *compact.* However, unlike the unweighted case, the support need not lie entirely on $\partial_\infty E$ and, in fact, it can be quite an arbitrary closed subset of E. Determining this set is one of the most important aspects of weighted potential theory.

Weighted transfinite diameter: $\tau(w, E)$.

Let

$$\delta_n(w) := \max_{z_1,\ldots,z_n \in E} \left(\prod_{1 \le i < j \le n} |z_i - z_j| w(z_i) w(z_j) \right)^{2/n(n-1)}.$$

Points $z_1^{(n)}, \ldots, z_n^{(n)}$ at which the maximum is attained are called **weighted Fekete points**. The corresponding **Fekete polynomial** is the monic polynomial with all its zeros at these points.

As in the unweighted case, the sequence $\delta_n(w)$ is decreasing, so one can define

$$\tau(w, E) := \lim_{n\to\infty} \delta_n(w),$$

which we call the **weighted transfinite diameter** of E.

Weighted Chebyshev constant: $\operatorname{cheb}(w, E)$.

Let

$$t_n(w) := \min_{p \in \mathcal{P}_{n-1}} \|w^n(z)(z^n - p(z))\|_E.$$

Then the **weighted Chebyshev constant** is defined by

$$\operatorname{cheb}(w, E) := \lim_{n\to\infty} t_n(w)^{1/n}.$$

The following theorem generalizes the fundamental results stated in Theorem 1.18.

Theorem 6.1 (**Generalized Fundamental Theorem**). *Let E be a closed set of positive capacity. Assume that w satisfies the conditions* (i)–(iii) *and let $Q = \log(1/w)$. Then*

$$\operatorname{cap}(w, E) = \tau(w, E) = \operatorname{cheb}(w, E) \exp\left(-\int Q \, \mathrm{d}\mu_w\right).$$

Moreover, weighted Fekete points have asymptotic distribution μ_w as $n \to \infty$, and weighted Fekete polynomials are asymptotically optimal for the weighted Chebyshev problem.

How can one find μ_w?

In most applications, the weight w is continuous and the set E is regular. Recall that the latter means that the classical (unweighted) equilibrium potential for E is equal to V_E *everywhere* on E, not just quasi-everywhere. Under these assumptions, the equilibrium measure $\mu = \mu_w$ is characterized by the conditions that $\mu \in \mathcal{M}(E)$, $I(\mu) < \infty$ and, for some constant c_w, the following *variational conditions* hold:

$$\begin{cases} U^\mu + Q = c_w & \text{on } S(\mu) = \operatorname{supp}(\mu) \\ U^\mu + Q \ge c_w & \text{on } E. \end{cases} \tag{6.2}$$

On integrating (against $\mu = \mu_w$) the first condition, we obtain that the constant is given by

$$c_w = I(\mu_w) + \int Q \, \mathrm{d}\mu_w = V_w - \int Q \, \mathrm{d}\mu_w.$$

When trying to find μ_w, an essential step (and a nontrivial problem in its own right!) is to determine the support $S(\mu_w) := \mathrm{supp}(\mu_w)$. There are several methods by which $S(\mu_w)$ can be numerically approximated, but they are complicated from the computational point of view. Therefore, knowing properties of the support can be useful and we list some of them.

Properties of the support $S(\mu_w)$
(a) The sup norm of weighted polynomials "lives" on $S(\mu_w)$. That is, for any n and for any polynomial P_n of degree at most n, there holds

$$\|w^n P_n\|_E = \|w^n P_n\|_{S(\mu_w)}.$$

(b) Let K be a compact subset of E of positive capacity, and define

$$F(K) := \log \mathrm{cap}(K) - \int_K Q \, \mathrm{d}\mu_K,$$

where μ_K is the classical (unweighted) equilibrium measure for K. This so-called **F-functional** of Mhaskar and Saff is often a helpful tool in finding $S(\mu_w)$. Since $\mathrm{cap}(K)$ and μ_K remain the same if we replace K by $\partial_\infty K$, we obtain that $F(K) = F(\partial_\infty K)$. It turns out that the outer boundary of $S(\mu_w)$ maximizes the F-functional:

$$\max_K F(K) = F(\partial_\infty S(\mu_K)).$$

This result is especially useful when E is a real interval and Q is *convex.* It is then easy to derive from (6.2) that $S(\mu_w)$ is an *interval.* Thus, to find the support, one merely needs to maximize $F(K)$ only over intervals $K \subset E$, which amounts to a standard calculus problem for the determination of the endpoints of $S(\mu_w)$.

(c) $S(\mu_w)$ is the set of weighted polynomial **peaking points**; that is, if w is continuous and E is of positive capacity at each of its points, then z belongs to $S(\mu_w)$ iff for every disk $D_r(z)$ there is a weighted polynomial $w^n P_n$ that attains its maximum modulus only in $D_r(z)$ (cf. [ST, Sec. IV.1]).

Example 6.2. Incomplete polynomials
For the study of incomplete polynomials of type θ on the interval $E = [0,1]$; that is, polynomials of the form $p(x) = \sum_{k=s}^{n} a_k x^k$ where $s/n \geq \theta$, the appropriate external field is $Q(x) = \log(1/w(x)) = -\frac{\theta}{1-\theta} \log x$ which is convex. Maximizing the F-functional one gets $S(\mu_w) = [\theta^2, 1]$. (For details, see [ST, Sec. IV.1].)

Example 6.3. Freud Weights
Here $E = \mathbb{R}$ and $w(x) = \exp(-|x|^\alpha)$. Hence $Q(x) = |x|^\alpha$ is convex provided that $\alpha > 1$, and we obtain $S_w = [-a_\alpha, a_\alpha]$, where a_α can be given explicitly in terms of the Gamma function. (Actually, this result also holds for all $\alpha > 0$; see [ST, Sec. IV.1].) For example, when $\alpha = 2$, we get $S_w = [-1, 1]$.

The *Generalized Weierstrass Approximation Problem* mentioned in problem (v) of the introduction states the following: For $E \subset \mathbb{R}$ closed, $w : E \to [0, \infty)$, characterize those functions f

continuous on E that are uniform limits on E of some sequence of weighted polynomials $(w^n P_n)$, $\deg P_n \le n$.

To attack this problem, we begin with a crucial observation. Let E be a closed subset of $\mathbb{R}$ whose complement is regular and $w(x)$ be continuous on E. Then we have the following weighted analogue of the Bernstein-Walsh lemma:

$$|w^n(x)P_n(x)| \le \|w^n P_n\|_{S(\mu_w)} \exp(-n(U^{\mu_w}(x) + Q(x) - c_w)), \quad x \in E \setminus S(\mu_w).$$

With the aid of (6.2) and a variant of the Stone-Weierstrass theorem (cf. [ST]), one can show that if a sequence $(w^n(x)P_n(x))$, $\deg P_n \le n$, converges uniformly on E, then it tends to 0 for every $x \in E \setminus S(\mu_w)$.

Thus, if some $f \in C(E)$ is a uniform limit on E of such a sequence as $n \to \infty$, it must vanish on $E \setminus S(\mu_w)$. The converse is not true, in general, but it is true in many important cases, such as for incomplete polynomials where the weight $w(x) = x^{\theta/(1-\theta)}$ on $[0,1]$ and for Freud weights $w(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, on $\mathbb{R}$. The latter fact provided an essential ingredient in resolving problem (iv) of the introduction (see [LuSa] and [LMS]).

For the case when E is a real interval and $Q = \log(1/w)$ is convex on E, this author conjectured and Totik [To] has proved that, more generally, any $f \in C(E)$ that vanishes on $E \setminus S(\mu_w)$ is the uniform limit on E of some sequence of weighted polynomials $(w^n P_n)$, $\deg P_n \le n$.

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