Korovkin-type Theorems and Approximation by Positive Linear Operators

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To the memory of my parents
Maria Giordano (1915-1989) and Luigi Altomare (1898-1963)

Abstract

This survey paper contains a detailed self-contained introduction to Korovkin-type theorems and to some of their applications concerning the approximation of continuous functions as well as of L^p-functions, by means of positive linear operators.

The paper also contains several new results and applications. Moreover, the organization of the subject follows a simple and direct approach which quickly leads both to the main results of the theory and to some new ones.

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1 Introduction

Korovkin-type theorems furnish simple and useful tools for ascertaining whether a given sequence of positive linear operators, acting on some function space is an approximation process or, equivalently, converges strongly to the identity operator.

Roughly speaking, these theorems exhibit a variety of test subsets of functions which guarantee that the approximation (or the convergence) property holds on the whole space provided it holds on them.

The custom of calling these kinds of results "Korovkin-type theorems" refers to P. P. Korovkin who in 1953 discovered such a property for the functions $\mathbf{1}, x$ and x^2 in the space $C([0,1])$ of all continuous functions on the real interval $[0,1]$ as well as for the functions $\mathbf{1}, \cos$ and $\sin$ in the space of all continuous 2π-periodic functions on the real line ([77-78]).

After this discovery, several mathematicians have undertaken the program of extending Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on. Such developments delineated a theory which is nowadays referred to as Korovkin-type approximation theory.

This theory has fruitful connections with real analysis, functional analysis, harmonic analysis, measure theory and probability theory, summability theory and partial differential equations. But the foremost applications are concerned with constructive approximation theory which uses it as a valuable tool.

Even today, the development of Korovkin-type approximation theory is far from complete, especially for those parts of it that concern limit operators different from the identity operator (see Problems 5.3 and 5.4 and the subsequent remarks).

A quite comprehensive picture of what has been achieved in this field until 1994 is documented in the monographs of Altomare and Campiti ([8], see in particular Appendix D), Donner ([46]), Keimel and Roth ([76]), Lorentz, v. Golitschek and Makovoz ([83]). More recent results can be found, e.g., in [1], [9-15], [22], [47-52], [63], [71-74], [79], [114-116], [117] and the references therein.

The main aim of this survey paper is to give a detailed self-contained introduction to the field as well as a secure entry into a theory that provides useful tools for understanding and unifying several aspects pertaining, among others, to real and functional analysis and which leads to several applications in constructive approximation theory and numerical analysis.

This paper, however, not only presents a survey on Korovkin-type theorems but also contains several new results and applications. Moreover, the organization of the subject follows a simple and direct approach which quickly leads both to the main results of the theory and to some new ones.

In Sections 3 and 4, we discuss the first and the second theorem of Korovkin. We obtain both of them from a simple unifying result which we state in the setting of metric spaces (see Theorem 3.2).

This general result also implies the multidimensional extension of Korovkin's theorem due to Volkov ([118]) (see Theorem 4.1). Moreover, a slight extension of it into the framework of locally compact metric spaces allows to extend the Korovkin's theorems to arbitrary real intervals or, more generally, to locally compact subsets of $\mathbb{R}^d$, $d \geq 1$.

Throughout the two sections, we present some applications concerning several classical approximation processes ranging from Bernstein operators on the unit interval or on the canonical hypercube and the multidimensional simplex, to Kantorovich operators, from Fejér operators to Abel-Poisson operators, from Szász-Mirakjan operators to Gauss-Weierstrass operators.

We also prove that the first and the second theorems of Korovkin are actually equivalent to the algebraic and the trigonometric version, respectively, of the classical Weierstrass approximation theorem.

Starting from Section 5, we enter into the heart of the theory by developing some of the main results in the framework of the space $C_0(X)$ of all real-valued continuous functions vanishing at infinity on a locally compact space X and, in particular, in the space $C(X)$ of all real-valued continuous functions on a compact space X.

We choose these continuous function spaces because they play a central role in the whole theory and are the most useful for applications. Moreover, by means of them it is also possible to easily obtain some Korovkin-type theorems in weighted continuous function spaces and in L^p-spaces, $1 \leq p$. These last aspects are treated at the end of Section 6 and in Section 8.

We point out that we discuss Korovkin-type theorems not only with respect to the identity operator but also with respect to a positive linear operator on $C_0(X)$ opening the door to a variety of problems some of which are still unsolved.

In particular, in Section 10, we present some results concerning positive projections on $C(X), X$ compact, as well as their applications to the approximation of the solutions of Dirichlet problems and of other similar problems.

In Sections 6 and 7, we present several results and applications concerning Korovkin sets for the identity operator. In particular, we show that, if M is a subset of $C_0(X)$ that separates the points of X and if $f_0 \in C_0(X)$ is strictly positive, then $\{f_0\} \cup f_0M \cup f_0M^2$ is a Korovkin set in $C_0(X)$.

This result is very useful because it furnishes a simple way to construct Korovkin sets, but in addition, as we show in Section 9, it turns out that it is equivalent to the Stone generalization to $C_0(X)$-spaces of the Weierstrass theorem. This equivalence was already established in [8, Section 4.4] (see also [12-13]) but here we furnish a different, direct and more transparent proof.

We also mention that, at the end of Sections 7 and 10, we present some applications concerning Bernstein-Schnabl operators associated with a positive linear operator and, in particular, with a positive projection. These operators are useful for the approximation of not just continuous functions but also — and this was the real reason for the increasing interest in them — positive semigroups and hence the solutions of initial-boundary value evolution problems. These aspects are briefly sketched at the end of Section 10.

Following the main aim of "Surveys in Approximation Theory", this paper is directed to the graduate student level and beyond. However, some parts of it as well as some new methods developed here could also be useful to expert readers.

A knowledge of the basic definitions and results concerning locally compact Hausdorff spaces and continuous function spaces on them is required as well as some basic properties of positive linear functionals on these function spaces (Radon measures). However, the reader who is not interested in this level of generality may replace everywhere our locally compact spaces with the space $\mathbb{R}^d, d \geq 1$, or with an open or a closed subset of it or with the intersection of an open subset and a closed subset of $\mathbb{R}^d$. However, this restriction does not produce any simplification of the proofs or of the methods.

For the convenience of the reader and to make the exposition self-contained, we collect all these prerequisites in the Appendix. There, the reader can also find some new simple and direct proofs of the main properties of Radon measures which are required throughout the paper, so that no a priori knowledge of the theory of Radon measures is needed.

This paper contains introductory materials so that many aspects of the theory have been omitted. We refer, e.g., to [8, Appendix D] for further details about some of the main directions developed during the last fifty years.

Furthermore, in the applications shown throughout the paper, we treat only general constructive aspects (convergence of the approximation processes) without any mention of quantitative aspects (estimates of the rate of convergence, direct and converse results and so on) nor to shape preserving properties. For such matters, we refer, e.g., to [8], [26], [38], [41], [42], [44], [45], [64], [65], [81-82], [83], [92], [109].

We also refer to [19, Proposition 3.7], [67], [69] and [84] where other kinds of convergence results for sequences of positive linear operators can be found. The results of these last papers do not properly fall into the Korovkin-type approximation theory but they can be fruitfully used to decide whether a given sequence of positive linear operators is strongly convergent (not necessarily to the identity operator).

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2 Preliminaries and notation

In this section, we assemble the main notation which will be used throughout the paper together with some generalities.

Given a metric space (X, d), for every $x_0 \in X$ and $r > 0$, we denote by $B(x_0, r)$ and $B'(x_0, r)$ the open ball and the closed ball with center x_0 and radius r, respectively, i.e.,

$$B(x_0, r) := \{x \in X \mid d(x_0, x) < r\} \tag{2.1}$$

and

$$B'(x_0, r) := \{x \in X \mid d(x_0, x) \leq r\}. \tag{2.2}$$

The symbol

$$F(X)$$

stands for the linear space of all real-valued functions defined on X. If M is a subset of $F(X)$, then by $\mathcal{L}(M)$ we designate the linear subspace generated by M. We denote by

$$B(X)$$

the linear subspace of all functions $f : X \longrightarrow \mathbb{R}$ that are bounded, endowed with the norm of uniform convergence (briefly, the sup-norm) defined by

$$\|f\|_\infty := \sup_{x\in X}|f(x)| \quad (f \in B(X)), \tag{2.3}$$

with respect to which it is a Banach space.

The symbols

$$C(X) \quad \text{and} \quad C_b(X)$$

denote the linear subspaces of all continuous (resp. continuous and bounded) functions in $F(X)$. Finally, we denote by

$$UC_b(X)$$

the linear subspace of all uniformly continuous and bounded functions in $F(X)$. Both $C_b(X)$ and $UC_b(X)$ are closed in $B(X)$ and hence, endowed with the norm (2.3), they are Banach spaces.

A linear subspace E of $F(X)$ is said to be a **lattice subspace** if

$$|f| \in E \qquad \text{for every} \quad f \in E. \tag{2.4}$$

For instance, the spaces $B(X), C(X), C_b(X)$ and $UC_b(X)$ are lattice subspaces.

Note that from (2.4), it follows that $\sup(f,g), \inf(f,g) \in E$ for every $f,g \in E$ where

$$\sup(f,g)(x) := \sup(f(x),g(x)) \qquad (x \in X) \tag{2.5}$$

and

$$\inf(f,g)(x) := \inf(f(x),g(x)) \qquad (x \in X). \tag{2.6}$$

This follows at once by the elementary identities

$$\sup(f,g) = \frac{f+g+|f-g|}{2} \quad \text{and} \quad \inf(f,g) = \frac{f+g-|f-g|}{2}. \tag{2.7}$$

More generally, if $f_1,\dots,f_n \in E,\ n \geq 3$, then $\sup\limits_{1\leq i\leq n} f_i,\ \inf\limits_{1\leq i\leq n} f_i \in E$.

We say that a linear subspace E of $F(X)$ is a **subalgebra** if

$$f \cdot g \in E \qquad \text{for every} \quad f,g \in E \tag{2.8}$$

or, equivalently, if $f^2 \in E$ for every $f \in E$. In this case, if $f \in E$ and $n \geq 1$, then $f^n \in E$ and hence for every real polynomial $Q(x) := \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ ($x \in \mathbb{R}$) vanishing at 0, the function

$$Q(f) := \alpha_1 f + \alpha_2 f^2 + \cdots + \alpha_n f^n \tag{2.9}$$

belongs to E as well. If E contains the constant functions, then $P(f) \in E$ for every real polynomial P.

Note that a subalgebra is not necessarily a lattice subspace (for instance, $C^1([a,b])$ is such an example). However every closed subalgebra of $C_b(X)$ is a lattice subspace (see Lemma 9.1).

Given a linear subspace E of $F(X)$, a linear functional $\mu : E \longrightarrow \mathbb{R}$ is said to be positive if

$$\mu(f) \geq 0 \qquad \text{for every} \quad f \in E,\ f \geq 0. \tag{2.10}$$

The simplest example of a positive linear functional is the so-called evaluation functional at a point $a \in X$ defined by

$$\delta_a(f) := f(a) \qquad (f \in E). \tag{2.11}$$

If (Y, d') is another metric space, we say that a linear operator $T : E \longrightarrow F(Y)$ is positive if

$$T(f) \geq 0 \qquad \text{for every} \quad f \in E,\ f \geq 0. \tag{2.12}$$

Every positive linear operator $T : E \longrightarrow F(Y)$ gives rise to a family $(\mu_y)_{y \in Y}$ of positive linear functionals on E defined by

$$\mu_y(f) := T(f)(y) \quad (f \in E). \tag{2.13}$$

Below, we state some elementary properties of both positive linear functionals and positive linear operators.

In what follows, the symbol $\mathbb{F}$ stands either for the field $\mathbb{R}$ or for a space $F(Y)$, Y being an arbitrary metric space.

Consider a linear subspace E of $F(X)$ and a positive linear operator $T : E \longrightarrow \mathbb{F}$. Then:

(i) For every $f, g \in E, f \leq g$,

$$T(f) \leq T(g) \tag{2.14}$$

(ii) If E is a lattice subspace, then

$$|T(f)| \leq T(|f|) \quad \text{for every } f \in E. \tag{2.15}$$

(iii) (Cauchy-Schwarz inequality) If E is both a lattice subspace and a subalgebra, then

$$T(|f \cdot g|) \leq \sqrt{T(f^2)T(g^2)} \qquad (f, g \in E). \tag{2.16}$$

In particular, if $\mathbf{1} \in E$, then

$$T(|f|)^2 \leq T(\mathbf{1})T(f^2) \qquad (f \in E). \tag{2.17}$$

(iv) If X is compact, $\mathbf{1} \in E$ and F is either $\mathbb{R}$ or $B(Y)$, then T is continuous and

$$\|T\| = \|T(\mathbf{1})\|. \tag{2.18}$$

Thus, if $\mu : E \longrightarrow \mathbb{R}$ is a positive linear functional, then μ is continuous and $\|\mu\| = \mu(\mathbf{1})$.

3 Korovkin's first theorem

Korovkin's theorem provides a very useful and simple criterion for whether a given sequence $(L_n)_{n\geq 1}$ of positive linear operators on $C([0,1])$ is an **approximation process**, i.e., $L_n(f) \longrightarrow f$ uniformly on $[0,1]$ for every $f \in C([0,1])$.

In order to state it, we need to introduce the functions

$$e_m(t) := t^m \qquad (0 \leq t \leq 1) \tag{3.1}$$

$(m \geq 1)$.

Theorem 3.1. *(Korovkin ([77])) Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from $C([0,1])$ into $F([0,1])$ such that for every $g \in \{\mathbf{1}, e_1, e_2\}$*

$$\lim_{n\to\infty} L_n(g) = g \quad \textit{uniformly on} \quad [0,1].$$

Then, for every $f \in C([0,1])$,

$$\lim_{n\to\infty} L_n(f) = f \quad \textit{uniformly on} \quad [0,1].$$

Below, we present a more general result from which Theorem 3.1 immediately follows.

For every $x \in [0,1]$ consider the auxiliary function

$$d_x(t) := |t-x| \qquad (0 \leq t \leq 1). \tag{3.2}$$

Then

$$d_x^2 = e_2 - 2xe_1 + x^2\mathbf{1}$$

and hence, if $(L_n)_{n\geq 1}$ is a sequence of positive linear operators satisfying the assumptions of Theorem 3.1, we get

$$\lim_{n\to\infty} L_n(d_x^2)(x) = 0 \tag{3.3}$$

uniformly with respect to $x \in [0,1]$, because for $n \geq 1$

$$L_n(d_x^2) = (L_n(e_2) - x^2) + 2x(L_n(e_1) - x) + x^2(L_n(\mathbf{1}) - \mathbf{1}).$$

After these preliminaries, the reader can easily realize that Theorem 3.1 is a particular case of the following more general result which, together with its modification (i.e., Theorem 3.5) as well as the further consequences presented at the beginning of Section 4, should also be compared with the simple but different methods of [79].

Consider a metric space (X,d). Extending (3.2), for any $x \in X$ we denote by $d_x \in C(X)$ the function

$$d_x(y) := d(x,y) \qquad (y \in X). \tag{3.4}$$

Theorem 3.2. *Let (X,d) be a metric space and consider a lattice subspace E of $F(X)$ containing the constant functions and all the functions d_x^2 $(x \in X)$. Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from E into $F(X)$ and let Y be a subset of X such that*

(i) $\lim_{n\to\infty} L_n(\mathbf{1}) = \mathbf{1}$ *uniformly on Y;*

(ii) $\lim_{n\to\infty} L_n(d_x^2)(x) = 0$ *uniformly with respect to* $x \in Y$.

Then for every $f \in E \cap UC_b(X)$

$$\lim_{n\to\infty} L_n(f) = f \quad \textit{uniformly on} \quad Y.$$

Proof. Consider $f \in E \cap UC_b(X)$ and $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon \qquad \text{for every} \quad x, y \in X,\ d(x,y) \le \delta.$$

On the other hand, if $d(x,y) \ge \delta$, then

$$|f(x) - f(y)| \le 2\|f\|_\infty \le \frac{2\|f\|_\infty}{\delta^2} d^2(x,y).$$

Therefore, for $x \in X$ fixed, we obtain

$$|f - f(x)| \le \frac{2\|f\|_\infty}{\delta^2} d_x^2 + \varepsilon \mathbf{1}$$

and hence, for any $n \ge 1$,

$$|L_n(f)(x) - f(x)L_n(\mathbf{1})(x)| \ \le\ L_n(|f - f(x)|)(x) \ \le\ \frac{2\|f\|_\infty}{\delta^2} L_n(d_x^2)(x) + \varepsilon L_n(\mathbf{1})(x).$$

We may now easily conclude that $\lim_{n\to\infty} L_n(f) = f$ uniformly on Y because of the assumptions (i) and (ii). $\square$

Theorem 3.2 has a natural generalization to completely regular spaces (for more details, we refer to [15]). Furthermore, the above proof can be adapted to show the next result.

Theorem 3.3. *Consider* (X, d) *and* $E \subset F(X)$ *as in Theorem 3.2. Consider a sequence* $(L_n)_{n\ge1}$ *of positive linear operators from* E *into* $F(X)$ *and assume that for a given* $x \in X$

(i) $\lim_{n\to\infty} L_n(\mathbf{1})(x) = 1$;

(ii) $\lim_{n\to\infty} L_n(d_x^2)(x) = 0$.

Then, for every bounded function $f \in E$ *that is continuous at* x,

$$\lim_{n\to\infty} L_n(f)(x) = f(x).$$

Adapting the proof of Theorem 3.2, we can show a further result. We first state a preliminary lemma.

Lemma 3.4. *Let* (X, d) *be a locally compact metric space. Then for every compact subset* K *of* X *and for every* $\varepsilon > 0$, *there exist* $0 < \overline{\varepsilon} < \varepsilon$ *and a compact subset* K_ε *of* X *such that*

$$B'(x, \overline{\varepsilon}) \subset K_\varepsilon \quad \textit{for every} \quad x \in K.$$

Proof. Given $x \in K$, there exists $0 < \varepsilon(x) < \varepsilon$ such that $B'(x, \varepsilon(x))$ is compact. Since $K \subset \bigcup_{x \in K} B(x, \varepsilon(x)/2)$, there exist $x_1, \dots, x_p \in K$ such that $K \subset \bigcup_{i=1}^{p} B(x_i, \varepsilon(x_i)/2)$. Set $\overline{\varepsilon} := \min_{1 \le i \le p} \varepsilon(x_i) < \varepsilon$ and $K_\varepsilon := \bigcup_{i=1}^{p} B'(x_i, \varepsilon(x_i))$. Now, if $x \in K$ and $y \in X$ and if $d(x, y) \le \overline{\varepsilon}$, then there exists an $i \in \{1, \dots, p\}$ such that $d(x, x_i) \le \varepsilon(x_i)/2$, and hence $d(y, x_i) \le d(y, x) + d(x, x_i) \le \varepsilon(x_i)$. Therefore $y \in K_\varepsilon$. $\square$

Theorem 3.5. *Let (X, d) be a locally compact metric space and consider a lattice subspace E of $F(X)$ containing the constant function $\mathbf{1}$ and all the functions d_x^2 $(x \in X)$. Let $(L_n)_{n \ge 1}$ be a sequence of positive linear operators from E into $F(X)$ and assume that*

(i) $\lim_{n \to \infty} L_n(\mathbf{1}) = \mathbf{1}$ *uniformly on compact subsets of X;*

(ii) $\lim_{n \to \infty} L_n(d_x^2)(x) = 0$ *uniformly on compact subsets of X.*

Then, for every $f \in E \cap C_b(X)$,

$$\lim_{n \to \infty} L_n(f) = f \quad \textit{uniformly on compact subsets of} \quad X.$$

Proof. Fix $f \in E \cap C_b(X)$ and consider a compact subset K of X. Given $\varepsilon > 0$, consider $0 < \overline{\varepsilon} < \varepsilon$ and a compact subset K_ε of X as in Lemma 3.4.

Since f is uniformly continuous on K_ε, there exists $0 < \delta < \overline{\varepsilon}$ such that

$$|f(x) - f(y)| \le \varepsilon \quad \text{for every} \quad x, y \in K_\varepsilon,\ d(x, y) \le \delta.$$

Given $x \in K$ and $y \in X$, if $d(x, y) \le \delta$, then $y \in B'(x, \overline{\varepsilon}) \subset K_\varepsilon$ and hence $|f(x) - f(y)| \le \varepsilon$.

If $d(x, y) \ge \delta$, then

$$|f(x) - f(y)| \le \frac{2\|f\|_\infty}{\delta^2} d^2(x, y).$$

Therefore, once again,

$$|f - f(x)| \le \frac{2\|f\|_\infty}{\delta^2} d_x^2 + \varepsilon \mathbf{1}$$

so that, arguing as in the final part of the proof of Theorem 3.2, we conclude that

$$\lim_{n \to \infty} L_n(f)(x) = f(x)$$

uniformly with respect to $x \in K$. $\square$

Theorem 3.1 was obtained by P. P. Korovkin in 1953 ([77], see also [78]). However, in [35], H. Bohman showed a result like Theorem 3.1 by considering sequences of positive linear operators on $C([0, 1])$ of the form

$$L(f)(x) = \sum_{i \in I} f(a_i) \varphi_i(x) \quad (0 \le x \le 1),$$

where $(a_i)_{i \in I}$ is a finite family in $[0, 1]$ and $\varphi_i \in C([0, 1])$ $(i \in I)$. Finally, we point out that the germ of the same theorem can be also traced back to a paper by T. Popoviciu ([95]).

Korovkin's theorem 3.1 (often called **Korovkin's first theorem**) has many important applications in the study of positive approximation processes in $C([0, 1])$.

One of them is concerned with the **Bernstein operators** on $C([0,1])$ which are defined by

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \tag{3.5}$$

($n \geq 1, f \in C([0,1]),\ 0 \leq x \leq 1$). Each $B_n(f)$ is a polynomial of degree not greater than n. They were introduced by S. N. Bernstein ([34]) to give the first constructive proof of the Weierstrass approximation theorem (algebraic version) ([119]).

Actually, we have that:

Theorem 3.6. *For every $f \in C([0,1])$,*

$$\lim_{n\to\infty} B_n(f) = f \quad \textit{uniformly on} \quad [0,1].$$

Proof. Each B_n is a positive linear operator on $C([0,1])$. Moreover, it is easy to verify that for any $n \geq 1$

$$B_n(\mathbf{1}) = \mathbf{1}, \quad B_n(e_1) = e_1$$

and

$$B_n(e_2) = \frac{n-1}{n}e_2 + \frac{1}{n}e_1.$$

Therefore, the result follows from Theorem 3.1. □

The original proof of Bernstein's Theorem 3.6 is based on probabilistic considerations (namely, on the weak law of large numbers). For a survey on Bernstein operators, we refer, e.g., to [82] (see also [42] and [8]).

Note that Theorem 3.6 furnishes a constructive proof of the Weierstrass approximation theorem [119] which we state below. (For a survey on many other alternative proofs of Weierstrass' theorem, we refer, e.g., to [93-94].)

Theorem 3.7. *For every $f \in C([0,1])$, there exists a sequence of algebraic polynomials that uniformly converges to f on $[0,1]$.*

Using modern language, Theorem 3.7 can be restated as follows

"The subalgebra of all algebraic polynomials is dense in $C([0,1])$
with respect to the uniform norm".

By means of Theorem 3.6, we have seen that the Weierstrass approximation theorem can be obtained from Korovkin's theorem.

It seems to be not devoid of interest to point out that, from the Weierstrass theorem, it is possible to obtain a special version of Korovkin's theorem which involves only positive linear operators L_n, $n \geq 1$, such that $L_n(C([0,1])) \subset B([0,1])$ for every $n \geq 1$. This special version will be referred to as the **restricted version of Korovkin's theorem.**

Theorem 3.8. *The restricted version of Korovkin's theorem and Weierstrass' Approximation Theorem are equivalent.*

Proof. We have to furnish a proof of the restricted version of Korovkin's theorem based solely on the Weierstrass Theorem.

Consider a sequence of positive linear operators $(L_n)_{n\geq 1}$ from $C([0,1])$ into $B([0,1])$ such that $\lim_{n\to\infty} L_n(g) = g$ uniformly on $[0,1]$ for every $g \in \{\mathbf{1}, e_1, e_2\}$. As in the proof of Theorem 3.1, we then get

$$\lim_{n\to\infty} L_n(d_x^2)(x) = 0$$

uniformly with respect to $x \in [0,1]$.

For $m \geq 1$ and $x, y \in [0,1]$, we have

$$|x^m - y^m| \leq m|y - x|$$

and hence, recalling the function $e_m(x) = x^m \quad (0 \leq x \leq 1)$,

$$|e_m - y^m\mathbf{1}| \leq m|e_1 - y\mathbf{1}| \quad (y \in [0,1]).$$

An application of the Cauchy-Schwarz inequality (2.16) implies, for any $n \geq 1$ and $y \in [0,1]$,

$$\begin{aligned}|L_n(e_m) - y^m L_n(\mathbf{1})| &\leq mL_n(|e_1 - y\mathbf{1}|)\\ &\leq m\sqrt{L_n(\mathbf{1})}\sqrt{L_n((e_1 - y\mathbf{1})^2)} = m\sqrt{L_n(\mathbf{1})}\sqrt{L_n(d_y^2)}.\end{aligned}$$

Therefore, $\lim_{n\to\infty} L_n(e_m) = e_m$ uniformly on $[0,1]$ for any $m \geq 1$ and hence $\lim_{n\to\infty} L_n(P) = P$ uniformly on $[0,1]$ for every algebraic polynomial P on $[0,1]$.

We may now conclude the proof because, setting $M := \sup_{n\geq 1} \|L_n\| = \sup_{n\geq 1} \|L_n(\mathbf{1})\| < +\infty$ and fixing $f \in C([0,1])$ and $\varepsilon > 0$, there exists an algebraic polynomial P on $[0,1]$ such that $\|f - P\| \leq \varepsilon$, and an integer $r \in \mathbb{N}$ such that $\|L_n(P) - P\| \leq \varepsilon$ for every $n \geq r$, so that

$$\begin{aligned}\|L_n(f) - f\| &\leq \|L_n(f) - L_n(P)\| + \|L_n(P) - P\| + \|P - f\|\\ &\leq M\|f - P\| + \|L_n(P) - P\| + \|P - f\| \leq (M + 2)\varepsilon.\end{aligned}$$

□

For another proof of Korovkin's first theorem which involves Weierstrass theorem, see [117].

It is well-known that there are "trigonometric" versions of both Korovkin's theorem and Weierstrass' theorem (see Theorems 4.3 and 4.6). Also, these versions are equivalent (see Theorem 4.7). In the sequel, we shall also prove that the generalizations of these two theorems to compact and to locally compact settings are equivalent as well (see Theorem 9.4).

We proceed now to illustrate another application of Korovkin's theorem that concerns the approximation of functions in $L^p([0,1])$, $1 \leq p < +\infty$, by means of positive linear operators. Note that Bernstein operators are not suitable to approximate Lebesgue integrable functions (see, for instance, [82, Section 1.9]).

The space $C([0,1])$ is dense in $L^p([0,1])$ with respect to the natural norm

$$\|f\|_p := \Big(\int_0^1 |f(t)|^p \,\mathrm{d}t\Big)^{1/p} \quad (f \in L^p([0,1])) \tag{3.6}$$

and

$$\|f\|_p \leq \|f\|_\infty \qquad \text{if} \quad f \in C([0,1]). \tag{3.7}$$

Therefore, the subalgebra of all algebraic polynomials on $[0,1]$ is dense in $L^p([0,1])$.

The **Kantorovich polynomials** introduced by L. V. Kantorovich ([75]) furnish the first constructive proof of the above mentioned density result. They are defined by

$$K_n(f)(x) := \sum_{k=0}^{n} \Big[(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,\mathrm{d}t\Big]\binom{n}{k}x^k(1-x)^{n-k} \tag{3.8}$$

for every $n \geq 1, f \in L^p([0,1]), 0 \leq x \leq 1$. Each $K_n(f)$ is a polynomial of degree not greater than n and every K_n is a positive linear operator from $L^p([0,1])$ (and, in particular, from $C([0,1])$ into $C([0,1])$). For additional information on these operators, see [8, Section 5.3.7], [82], [42, Chapter 10].

Theorem 3.9. *If $f \in C([0,1])$, then*

$$\lim_{n\to\infty} K_n(f) = f \quad \textit{uniformly on} \quad [0,1].$$

Proof. A direct calculation which involves the corresponding formulas for Bernstein operators gives for $n \geq 1$

$$K_n(\mathbf{1}) = \mathbf{1},\ K_n(e_1) = \frac{n}{n+1}e_1 + \frac{1}{2(n+1)}$$

and

$$K_n(e_2) = \frac{n(n-1)}{(n+1)^2}e_2 + \frac{2n}{(n+1)^2}e_1 + \frac{1}{3(n+1)^2}.$$

Therefore, the result follows at once from Korovkin's Theorem 3.1. $\square$

Before showing a result similar to Theorem 3.9 for L^p-functions, we need to recall some properties of convex functions.

Consider a real interval I of $\mathbb{R}$. A function $\varphi : I \longrightarrow \mathbb{R}$ is said to be **convex** if

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y)$$

for every $x, y \in I$ and $0 \leq \alpha \leq 1$. If I is open and φ is convex, then, for every finite family $(x_k)_{1\leq k\leq n}$ in I and $(\alpha_k)_{1\leq k\leq n}$ in $[0,1]$ such that $\sum\limits_{k=1}^{n} \alpha_k = 1$,

$$\varphi\Big(\sum_{k=1}^{n}\alpha_k x_k\Big) \leq \sum_{k=1}^{n}\alpha_k\varphi(x_k)$$

(**Jensen's inequality**).

The function $|t|^p$ $(t \in \mathbb{R})$, $1 \leq p < \infty$, is convex. Given a probability space $(\Omega, \mathcal{F}, \mu)$, an open interval I of $\mathbb{R}$ and a μ-integrable function $f : \Omega \longrightarrow I$, then

$$\int_\Omega f\,\mathrm{d}\mu \in I.$$

Furthermore, if $\varphi : I \longrightarrow \mathbb{R}$ is convex and $\varphi \circ f : \Omega \longrightarrow \mathbb{R}$ is μ-integrable, then

$$\varphi\Big(\int_\Omega f\,\mathrm{d}\mu\Big) \le \int_\Omega \varphi\circ f\,\mathrm{d}\mu$$

(**Integral Jensen inequality**).

In particular, if $f \in \mathcal{L}^p(\Omega,\mu) \subset \mathcal{L}^1(\Omega,\mu)$, then

$$\left|\int f\,\mathrm{d}\mu\right|^p \le \int |f|^p\,\mathrm{d}\mu. \tag{3.9}$$

(For more details see, e.g., [29, pp.18–21].)

After these preliminaries, we now proceed to show the approximation property of $(K_n)_{n\ge1}$ in $L^p([0,1])$.

Theorem 3.10. *If $f \in L^p([0,1])$, $1 \le p < +\infty$, then*

$$\lim_{n\to\infty} K_n(f) = f \quad \textit{in} \quad L^p([0,1]).$$

Proof. For every $n \ge 1$, denote by $\|K_n\|$ the operator norm of K_n considered as an operator from $L^p([0,1])$ into $L^p([0,1])$.

To prove the result, it is sufficient to show that there exists an $M \ge 0$ such that $\|K_n\| \le M$ for every $n \ge 1$. After that, the result will follow immediately because, for a given $\varepsilon > 0$, there exists $g \in C([0,1])$ such that $\|f-g\|_p \le \varepsilon$ and there exists $\nu \in \mathbb{N}$ such that, for $n \ge \nu$,

$$\|K_n(g) - g\|_\infty \le \varepsilon$$

so that

$$\|K_n(f) - f\|_p \le M\|f-g\|_p + \|K_n(g)-g\|_p + \|g-f\|_p \le (2+M)\varepsilon.$$

Now, in order to obtain the desired estimate, we shall use the convexity of the function $|t|^p$ on $\mathbb{R}$ and inequality (3.9).

Given $f \in L^p([0,1])$, for every $n \ge 1$ and $0 \le k \le n$, we have indeed

$$\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \mid f(t) \mid \,\mathrm{d}t\right)^p \le (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \mid f(t) \mid^p \,\mathrm{d}t$$

and hence, for every $x \in [0,1]$,

$$\begin{aligned}|K_n(f)(x)|^p &\le \sum_{k=0}^n \binom{n}{k} x^k(1-x)^{n-k}\Big[(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|\,\mathrm{d}t\Big]^p \\ &\le \sum_{k=0}^n \binom{n}{k} x^k(1-x)^{n-k}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\,\mathrm{d}t.\end{aligned}$$

Therefore,

$$\int_0^1 |K_n(f)(x)|^p\,\mathrm{d}x \le \sum_{k=0}^n \binom{n}{k}\Big(\int_0^1 x^k(1-x)^{n-k}\,\mathrm{d}x\Big)\Big((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\,\mathrm{d}t\Big).$$

On the other hand, by considering the beta function

$$B(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t \qquad (u>0, v>0),$$

it is not difficult to show that, for $0 \le k \le n$,

$$\int_0^1 x^k(1-x)^{n-k}\,\mathrm{d}x = B(k+1, n-k+1) = \frac{1}{(n+1)\binom{n}{k}},$$

and hence

$$\int_0^1 |K_n(f)(x)|^p\,\mathrm{d}x \le \sum_{k=0}^{n} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\,\mathrm{d}t = \int_0^1 |f(t)|^p\,\mathrm{d}t.$$

Thus, $\|K_n(f)\|_p \le \|f\|_p$ for every $f \in L^p([0,1])$, i.e., $\|K_n\| \le 1$. □

Remarks 3.11.

1. For every $f \in L^p([0,1]),\ 1 \le p < +\infty$, it can also be shown that

   $$\lim_{n\to\infty} K_n(f) = f \quad \text{almost everywhere on} \quad [0,1]$$

   (see [82, Theorem 2.2.1]).

2. If $f \in C([0,1])$ is continuously differentiable in [0,1], then, by referring again to Bernstein operators (3.5), it is not difficult to show that, for $n \ge 1$ and $x \in [0,1]$,

   $$B_{n+1}(f)'(x) = \sum_{h=0}^{n} (n+1)\Big[f\Big(\frac{h+1}{n+1}\Big) - f\Big(\frac{h}{n+1}\Big)\Big]\binom{n}{h} x^h(1-x)^{n-h} = K_n(f')(x).$$

   Therefore, by Theorem 3.9, we infer that

   $$\lim_{n\to\infty} B_n(f)' = f' \quad \text{uniformly on} \quad [0,1]. \tag{3.10}$$

   More generally, if $f \in C([0,1])$ possesses continuous derivatives in [0,1] up to the order $m \ge 1$, then for every $1 \le k \le m$,

   $$\lim_{n\to\infty} B_n(f)^{(k)} = f^{(k)} \quad \text{uniformly on} \quad [0,1] \tag{3.11}$$

   ([82, Section 1.8]).

3. Another example of positive approximating operators on $L^p([0,1]),\ 1 \le p < +\infty$, is furnished by the **Bernstein-Durrmeyer operators** defined by

   $$D_n(f)(x) := \sum_{k=0}^{n} \Big(\int_0^1 (n+1)\binom{n}{k} t^k(1-t)^{n-k} f(t)\,\mathrm{d}t\Big)\binom{n}{k} x^k(1-x)^{n-k} \tag{3.12}$$

   $(f \in L^p([0,1]),\ 0 \le x \le 1)$ (see [53], [40], [8, Section 5.3.8]).

We also refer the interested reader to [16] where a generalization of Kantorovich operators is introduced and studied.

4 Korovkin's second theorem and something else

In this section, we shall consider the space $\mathbb{R}^d$, $d \geq 1$, endowed with the Euclidean norm

$$\|x\| = \Big(\sum_{i=1}^{d} x_i^2\Big)^{1/2} \quad (x = (x_i)_{1\leq i\leq d} \in \mathbb{R}^d). \tag{4.1}$$

For every $j = 1, \dots, d$, we shall denote by

$$pr_j : \mathbb{R}^d \longrightarrow \mathbb{R}$$

the j-th coordinate function which is defined by

$$pr_j(x) := x_j \quad (x = (x_i)_{1\leq i\leq d} \in \mathbb{R}^d). \tag{4.2}$$

By a common abuse of notation, if X is a subset of $\mathbb{R}^d$, the restriction of each pr_j to X will be again denoted by pr_j. In this framework, for the functions d_x $(x \in X)$ defined by (3.4), we get

$$d_x^2 = \|x\|^2\mathbf{1} - 2\sum_{i=1}^{d} x_i pr_i + \sum_{i=1}^{d} pr_i^2. \tag{4.3}$$

Therefore, from Theorem 3.5, we then obtain

Theorem 4.1. *Let X be a locally compact subset of $\mathbb{R}^d$, $d \geq 1$, i.e., X is the intersection of an open subset and a closed subset of $\mathbb{R}^d$ (see Appendix). Consider a lattice subspace E of $F(X)$ containing $\{\mathbf{1}, pr_1, \dots, pr_d, \sum_{i=1}^{d} pr_i^2\}$ and let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from E into $F(X)$ such that for every $g \in \{\mathbf{1}, pr_1, \dots, pr_d, \sum_{i=1}^{d} pr_i^2\}$*

$$\lim_{n\to\infty} L_n(g) = g \quad \textit{uniformly on compact subsets of} \quad X.$$

Then, for every $f \in E \cap C_b(X)$

$$\lim_{n\to\infty} L_n(f) = f \quad \textit{uniformly on compact subsets of} \quad X.$$

The special case of Theorem 4.1 when X is compact follows indeed from Theorem 3.2 and is worth being stated separately. It is due to Volkov ([118]).

Theorem 4.2. *Let X be a compact subset of $\mathbb{R}^d$ and consider a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C(X)$ into $F(X)$ such that for every $g \in \{\mathbf{1}, pr_1, \dots, pr_d, \sum_{i=1}^{d} pr_i^2\}$*

$$\lim_{n\to\infty} L_n(g) = g \quad \textit{uniformly on} \quad X.$$

Then for every $f \in C(X)$

$$\lim_{n\to\infty} L_n(f) = f \quad \textit{uniformly on} \quad X.$$

Note that, if X is contained in some sphere of $\mathbb{R}^d$, i.e., $\sum_{i=1}^{d} pr_i^2$ is constant on X, then the test subset in Theorem 4.2 reduces to $\{\mathbf{1}, pr_1, \ldots, pr_d\}$. (In [8, Corollary 4.5.2], the reader can find a complete characterization of those subsets X of $\mathbb{R}^d$ for which $\{\mathbf{1}, pr_1, \ldots, pr_d\}$ satisfies Theorem 4.2.)

This remark applies in particular for the unit circle of $\mathbb{R}^2$

$$\mathbb{T} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}. \tag{4.4}$$

On the other hand, the space $C(\mathbb{T})$ is isometrically (order) isomorphic to the space

$$C_{2\pi}(\mathbb{R}) := \{f \in C(\mathbb{R}) \mid f \quad \text{is} \quad 2\pi\text{-periodic}\} \tag{4.5}$$

(endowed with the sup-norm and pointwise ordering) by means of the isomorphism $\Phi : C(\mathbb{T}) \longrightarrow C_{2\pi}(\mathbb{R})$ defined by

$$\Phi(F)(t) := F(\cos t, \sin t) \quad (t \in \mathbb{R})\,. \tag{4.6}$$

Moreover,

$$\Phi(\mathbf{1}) = \mathbf{1},\ \Phi(pr_1) = \cos,\ \Phi(pr_2) = \sin \tag{4.7}$$

and so we obtain **Korovkin's second theorem**.

Theorem 4.3. *Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $F(\mathbb{R})$ such that*

$$\lim_{n\to\infty} L_n(g) = g \quad \textit{uniformly on} \quad \mathbb{R}$$

for every $g \in \{\mathbf{1}, \cos, \sin\}$. Then

$$\lim_{n\to\infty} L_n(f) = f \quad \textit{uniformly on} \quad \mathbb{R}$$

for every $f \in C_{2\pi}(\mathbb{R})$.

Below, we discuss some applications of Theorem 4.3.

For $1 \leq p < +\infty$, we shall denote by

$$L^p_{2\pi}(\mathbb{R})$$

the Banach space of all (equivalence classes of) functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that are Lebesgue integrable to the p-th power over $[-\pi, \pi]$ and that satisfy $f(x + 2\pi) = f(x)$ for a.e. $x \in \mathbb{R}$. The space $L^p_{2\pi}(\mathbb{R})$ is endowed with the norm

$$\|f\|_p := \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \,\mathrm{d}t\Big)^{1/p} \quad (f \in L^p_{2\pi}(\mathbb{R})). \tag{4.8}$$

A family $(\varphi_n)_{n\geq 1}$ in $L^1_{2\pi}(\mathbb{R})$ is said to be **a positive periodic kernel** if every φ_n is positive, i.e., $\varphi_n \geq 0$ a.e. on $\mathbb{R}$, and

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t = 1. \tag{4.9}$$

Each positive kernel $(\varphi_n)_{n\geq 1}$ generates a sequence of positive linear operators on $L^1_{2\pi}(\mathbb{R})$. For every $n \geq 1, f \in L^1_{2\pi}(\mathbb{R})$ and $x \in \mathbb{R}$, set

$$\begin{aligned} L_n(f)(x) := (f * \varphi_n)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\varphi_n(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\varphi_n(x-t)\,\mathrm{d}t. \end{aligned} \tag{4.10}$$

From Fubini's theorem and Hölder's inequality, it follows that $L_n(f) \in L^p_{2\pi}(\mathbb{R})$ if $f \in L^p_{2\pi}(\mathbb{R}), 1 \leq p < +\infty$.

Moreover, if $f \in C_{2\pi}(\mathbb{R})$, then the Lebesgue dominated convergence theorem implies that $L_n(f) \in C_{2\pi}(\mathbb{R})$. Furthermore,

$$\|L_n(f)\|_p \leq \|\varphi_n\|_1 \|f\|_p \qquad (f \in C_{2\pi}(\mathbb{R})) \tag{4.11}$$

and

$$\|L_n(f)\|_\infty \leq \|\varphi_n\|_1 \|f\|_\infty \qquad (f \in C_{2\pi}(\mathbb{R})). \tag{4.12}$$

A positive kernel $(\varphi_n)_{n\geq 1}$ is called an **approximate identity** if for every $\delta \in]0,\pi[$

$$\lim_{n\to\infty} \int_{-\pi}^{-\delta} \varphi_n(t)\,\mathrm{d}t + \int_{\delta}^{\pi} \varphi_n(t)\,\mathrm{d}t = 0\,. \tag{4.13}$$

Theorem 4.4. *Consider a positive kernel $(\varphi_n)_{n\geq 1}$ in $L^1_{2\pi}(\mathbb{R})$ and the corresponding sequence $(L_n)_{n\geq 1}$ of positive linear operators defined by (4.10). For every $n \geq 1$, set*

$$\beta_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\sin^2\frac{t}{2}\,\mathrm{d}t. \tag{4.14}$$

Then the following properties are equivalent:

a) For every $1 \leq p < +\infty$ and $f \in L^p_{2\pi}(\mathbb{R})$

$$\lim_{n\to\infty} L_n(f) = f \quad \text{in} \quad L^p_{2\pi}(\mathbb{R})$$

as well as

$$\lim_{n\to\infty} L_n(f) = f \quad \text{in} \quad C_{2\pi}(\mathbb{R})$$

provided $f \in C_{2\pi}(\mathbb{R})$.

b) $\lim_{n\to\infty} \beta_n = 0$.

c) $(\varphi_n)_{n\geq 1}$ is an approximate identity.

Proof. To show the implication $(a) \Rightarrow (b)$, it is sufficient to point out that for every $n \geq 1$ and $x \in \mathbb{R}$

$$\begin{aligned} \beta_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(u-x)\sin^2\frac{u-x}{2}\,\mathrm{d}u \\ &= \frac{1}{2}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t - (\cos x)L_n(\cos)(x) - (\sin x)L_n(\sin)(x)\Big) \end{aligned}$$

and hence $\beta_n \to 0$ as $n \to \infty$.

Now assume that (b) holds. Then, for $0 < \delta < \pi$ and $n \geq 1$,

$$\frac{\sin^2(\delta/2)}{2\pi}\Big(\int_{-\pi}^{-\delta} \varphi_n(t)\,\mathrm{d}t + \int_{\delta}^{\pi} \varphi_n(t)\,\mathrm{d}t\Big) \leq \frac{1}{2\pi}\Big(\int_{-\pi}^{-\delta} \varphi_n(t)\sin^2\frac{t}{2}\,\mathrm{d}t + \int_{\delta}^{\pi} \varphi_n(t)\sin^2\frac{t}{2}\,\mathrm{d}t\Big) \leq \beta_n$$

and hence (c) follows.

We now proceed to show the implication $(c) \Rightarrow (a)$. Set $M := \sup_{n\geq 1}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t$. For a given $\varepsilon > 0$ there exists $\delta \in]0,\pi[$ such that $|\cos t - 1| \leq \dfrac{\varepsilon}{6(M+1)}$ and $|\sin t| \leq \dfrac{\varepsilon}{6(M+1)}$ for any $t \in \mathbb{R}, |t| \leq \delta$, and hence, for sufficiently large $n \geq 1$

$$\Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t - 1\Big| \leq \frac{\varepsilon}{3} \quad \text{and} \quad \int_{\delta\leq|t|\leq\pi} \varphi_n(t)\,\mathrm{d}t \leq \pi\frac{\varepsilon}{3}.$$

Therefore, for every $x \in \mathbb{R}$,

$$\begin{aligned}
|L_n(\sin)(x) - \sin x| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(x-t) - \sin x|\varphi_n(t)\,\mathrm{d}t \\
&\qquad + \Big|\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t - 1\Big)\Big||\sin x| \\
&\leq \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} |\sin(x-t) - \sin x|\varphi_n(t)\,\mathrm{d}t \\
&\qquad + \frac{1}{2\pi}\int_{|t|<\delta} |(\cos t - 1)\sin x - \cos x \sin t|\varphi_n(t)\,\mathrm{d}t + \varepsilon/3 \\
&\leq \frac{1}{\pi}\int_{\delta\leq|t|\leq\pi} \varphi_n(t)\,\mathrm{d}t + \frac{\varepsilon}{3(M+1)}\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} L_n(\sin) = \sin$ uniformly on $\mathbb{R}$. The same method can be used to show that $\lim_{n\to\infty} L_n(\cos) = \cos$ uniformly on $\mathbb{R}$ and hence, by Korovkin's second theorem 4.3, we obtain $\lim_{n\to\infty} L_n(f) = f$ in $C_{2\pi}(\mathbb{R})$ for every $f \in C_{2\pi}(\mathbb{R})$.

By reasoning as in the proof of Theorem 3.10, it is a simple matter to get the desired convergence formula in $L^p_{2\pi}(\mathbb{R})$ by using the previous one on $C_{2\pi}(\mathbb{R})$, the denseness of $C_{2\pi}(\mathbb{R})$ in $L^p_{2\pi}(\mathbb{R})$ and formula (4.11) which shows that the operators L_n, $n \geq 1$, are equibounded from $L^p_{2\pi}(\mathbb{R})$ into $L^p_{2\pi}(\mathbb{R})$. □

Two simple applications of Theorem 4.4 are particularly worthy of mention. For other applications, we refer to [8, Section 5.4], [38], [41], [78].

We begin by recalling that a trigonometric polynomial of degree $n \in \mathbb{N}$ is a real function of the form

$$u_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \quad (x \in \mathbb{R}) \tag{4.15}$$

where $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. A series of the form

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \quad (x \in \mathbb{R}) \tag{4.16}$$

$(a_k, b_k \in \mathbb{R})$ is called a **trigonometric series**.

If $f \in L^1_{2\pi}(\mathbb{R})$, the trigonometric series

$$\frac{1}{2}a_0(f) + \sum_{k=1}^{\infty} a_k(f) \cos kx + b_k(f) \sin kx \quad (x \in \mathbb{R}) \tag{4.17}$$

where

$$a_0(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t, \tag{4.18}$$

$$a_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,\mathrm{d}t, \quad k \geq 1, \tag{4.19}$$

$$b_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,\mathrm{d}t, \quad k \geq 1, \tag{4.20}$$

is called the **Fourier series** of f. The a_n's and b_n's are called the **real Fourier coefficients** of f.

For any $n \in \mathbb{N}$, denote by

$$S_n(f)$$

the n-th partial sum of the Fourier series of f, i.e.,

$$S_0(f) = \frac{1}{2}a_0(f) \tag{4.21}$$

and, for $n \geq 1$,

$$S_n(f)(x) = \frac{1}{2}a_0(f) + \sum_{k=1}^{n} a_k(f)\cos kx + b_k(f)\sin kx. \tag{4.22}$$

Each $S_n(f)$ is a trigonometric polynomial; moreover, considering the functions

$$D_n(t) := 1 + 2\sum_{k=1}^{n} \cos kt \quad (t \in \mathbb{R}), \tag{4.23}$$

we also get

$$S_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,\mathrm{d}t \quad (x \in \mathbb{R}). \tag{4.24}$$

The function D_n is called the n**-th Dirichlet kernel**.

By multiplying (4.23) by $\sin t/2$, we obtain

$$\begin{aligned}\sin\frac{t}{2}D_n(t) &= \sin\frac{t}{2} + \sum_{k=1}^{n} \sin\left(\frac{1+2k}{2}t\right) - \sin\left(\frac{2k-1}{2}t\right)\\ &= \sin\frac{1+2n}{2}t,\end{aligned}$$

so that

$$D_n(t) = \begin{cases} \dfrac{\sin(1+2n)t/2}{\sin t/2} & \text{if } t \text{ is not a multiple of } \pi,\\ 2n+1 & \text{if } t \text{ is a multiple of } \pi.\end{cases} \tag{4.25}$$

D_n is not positive and $(D_n)_{n\geq 1}$ is not an approximate identity ([38, Prop. 1.2.3]). Moreover, there exists $f \in C_{2\pi}(\mathbb{R})$ such that $(S_n(f))_{n\geq 1}$ does not converge uniformly (nor pointwise) to f, i.e., the Fourier series of f does not converge uniformly (nor pointwise) to f.

For every $n \in \mathbb{N}$, put

$$F_n(f) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f). \tag{4.26}$$

$F_n(f)$ is a trigonometric polynomial. Moreover, from the identity

$$(\sin\frac{t}{2})\sum_{k=0}^{n-1}\sin\frac{2k+1}{2}t = \sin^2\frac{n}{2}t \quad (t\in\mathbb{R}), \tag{4.27}$$

it follows that for every $x \in \mathbb{R}$

$$\begin{aligned}F_n(f)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{\sin((2k+1)(x-t)/2)}{\sin((x-t)/2)}\,\mathrm{d}t \qquad (4.28)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{(n+1)}\frac{\sin^2((n+1)(x-t)/2)}{\sin^2((x-t)/2)}\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\varphi_n(x-t)\,\mathrm{d}t,\end{aligned}$$

where

$$\varphi_n(x) := \begin{cases}\frac{\sin^2((n+1)x/2)}{(n+1)\sin^2(x/2)} & \text{if } x \text{ is not a multiple of } 2\pi,\\ n+1 & \text{if } x \text{ is a multiple of } 2\pi.\end{cases} \tag{4.29}$$

Actually, the sequence $(\varphi_n)_{n\geq 1}$ is a positive kernel which is called the **Fejér kernel**, and the corresponding operators F_n, $n \geq 1$, are called the **Fejér convolution operators**.

Theorem 4.5. *For every $f \in L^p_{2\pi}(\mathbb{R})$, $1 \leq p < +\infty$,*

$$\lim_{n\to\infty} F_n(f) = f \quad \text{in} \quad L^p_{2\pi}(\mathbb{R})$$

and, if $f \in C_{2\pi}(\mathbb{R})$,

$$\lim_{n\to\infty} F_n(f) = f \quad \text{in} \quad C_{2\pi}(\mathbb{R}).$$

Proof. Evaluating the Fourier coefficients of $\mathbf{1}$, cos and sin, and by using (4.26), we obtain, for $n \geq 1$,

$$F_n(\mathbf{1}) = \mathbf{1},\ F_n(\cos) = \frac{n}{n+1}\cos,\ F_n(\sin) = \frac{n}{n+1}\sin$$

and

$$\beta_n = \frac{1}{2(n+1)}.$$

The result now follows from Theorem 4.4 or, more directly, from Theorem 4.3. □

Theorem 4.5 is due to Fejér ([56-57]). It furnishes the first constructive proof of *the Weierstrass approximation theorem for periodic functions.*

Theorem 4.6. *If $f \in L^p_{2\pi}(\mathbb{R})$, $1 \leq p < +\infty$ (resp. $f \in C_{2\pi}(\mathbb{R})$) then there exists a sequence of trigonometric polynomials that converges to f in $L^p_{2\pi}(\mathbb{R})$ (resp. uniformly on $\mathbb{R}$).*

As in the "algebraic" case, we shall now show that from Weierstrass' approximation theorem, it is possible to deduce a "restricted" version of Theorem 4.3, where in addition it is required that each operator L_n maps $C_{2\pi}(\mathbb{R})$ into $B(\mathbb{R})$. We shall also refer to this version as the **restricted version of Korovkin's second theorem**.

Theorem 4.7. *The restricted version of Korovkin's second Theorem 4.3 and Weierstrass' Theorem 4.6 are equivalent.*

Proof. An inspection of the proof of Theorem 4.5 shows that Theorem 4.3 implies Theorem 4.5 and, hence, Theorem 4.6.

Conversely, assume that Theorem 4.6 is true and consider a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $B(\mathbb{R})$ such that $L_n(g) \longrightarrow g$ uniformly on $\mathbb{R}$ for every $g \in \{\mathbf{1}, \cos, \sin\}$. For every $m \geq 1$, set $f_m(x) := \cos mx$ and $g_m(x) := \sin mx$ ($x \in \mathbb{R}$). Since the subspace of all trigonometric polynomials is dense in $C_{2\pi}(\mathbb{R})$ and

$$\sup_{n\geq 1} \|L_n\| = \sup_{n\geq 1} \|L_n(\mathbf{1})\| < +\infty,$$

it is enough to show that $L_n(f_m) \to f_m$ and $L_n(g_m) \to g_m$ uniformly on $\mathbb{R}$ for every $m \geq 1$.

Given $x \in \mathbb{R}$, consider the function $\Phi_x(y) = \sin\frac{x-y}{2}$ ($y \in \mathbb{R}$). Then

$$\Phi^2(y) := \sin^2\frac{x-y}{2} = \frac{1}{2}(1 - \cos x\cos y - \sin x \sin y) \quad (y \in \mathbb{R})$$

and hence

$$L_n(\Phi_x)(x) \to 0 \quad \text{uniformly with respect to} \quad x \in \mathbb{R}.$$

On the other hand, for $m \geq 1$ and $x, y \in \mathbb{R}$, we get

$$|f_m(x) - f_m(y)| = 2\Big|\sin m\Big(\frac{x+y}{2}\Big)\Big|\Big|\sin m\Big(\frac{x-y}{2}\Big)\Big| \leq c_m\Big|\sin\Big(\frac{x-y}{2}\Big)\Big|$$

where $c_m := 2\sup_{\alpha\in\mathbb{R}} \frac{\sin m\alpha}{\sin\alpha}$, and hence

$$|f_m(x)L_n(\mathbf{1}) - L_n(f_m)| \leq c_m L_n(|\Phi_x|) \leq c_m\sqrt{L_n(\mathbf{1})}\sqrt{L_n(\Phi_x^2)}.$$

Therefore, $L_n(f_m)(x) - f_m(x) \to 0$ uniformly with respect to $x \in \mathbb{R}$.

A similar reasoning can be applied also to the functions $g_m, m \geq 1$, because

$$|g_m(x) - g_m(y)| = 2\Big|\cos m\Big(\frac{x+y}{2}\Big)\sin m\Big(\frac{x-y}{2}\Big)\Big| \leq K_m\Big|\sin\Big(\frac{x-y}{2}\Big)\Big|$$

and this completes the proof. □

For another short proof of Korovkin's second theorem that uses the trigonometric version of Weierstrass' theorem, see [117].

Fejér's Theorem 4.5 is noteworthy because it reveals an important property of the Fourier series. Actually, it shows that Fourier series are always *Cesaro-summable* to f in $L^p_{2\pi}(\mathbb{R})$ or in $C_{2\pi}(\mathbb{R})$, provided that $f \in L^p_{2\pi}(\mathbb{R})$ or $f \in C_{2\pi}(\mathbb{R})$. Another deeper theorem, ascribed to Fejér and Lebesgue, states that, if $f \in L^1_{2\pi}(\mathbb{R})$, then its Fourier series is Cesaro-summable to f a.e. on $\mathbb{R}$ ([112, Theorem 8.35]).

Below, we further discuss another regular summation method, namely the **Abel-summation method**, which applies to Fourier series.

We begin with the following equality

$$\frac{1+z}{1-z} = 1 + 2\sum_{k=1}^{\infty} z^k \quad (z \in \mathbb{C}, |z| < 1) \tag{4.30}$$

which holds uniformly on any compact subset of $\{z \in \mathbb{C} \mid |z| < 1\}$. Given $x \in \mathbb{R}$ and $0 \leq r < 1$, by applying (4.30) to $z = re^{\mathrm{i}x}$ and by taking the real parts of both sides, we get

$$1 + 2\sum_{k=1}^{\infty} r^k \cos kx = \frac{1-r^2}{1-2r\cos x + r^2} \tag{4.31}$$

and the identity holds uniformly when r ranges in a compact interval of $[0, 1[$.

The family of functions

$$P_r(t) := \frac{1-r^2}{1-2r\cos t + r^2} \quad (t \in \mathbb{R}) \tag{4.32}$$

$(0 \leq r < 1)$ is called the **Abel-Poisson kernel** and the corresponding operators

$$P_r(f)(x) := \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{1-2r\cos(x-t)+r^2}\,\mathrm{d}t \quad (x \in \mathbb{R}) \tag{4.33}$$

$(0 \leq r < 1,\ f \in L^1_{2\pi}(\mathbb{R}))$ are called the **Abel-Poisson convolution operators**. Taking (4.31) into account, it is not difficult to show that

$$P_r(f)(x) = \frac{1}{2}a_0(f) + \sum_{k=1}^{\infty} r^k(a_k(f)\cos kx + b_k(f)\sin kx) \tag{4.34}$$

where the coefficients $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of f defined by (4.18)–(4.20).

Theorem 4.8. *If $f \in L^p_{2\pi}(\mathbb{R}), 1 \leq p < +\infty$, then*

$$\lim_{r\to 1^-} P_r(f) = f \quad \textit{in} \quad L^p_{2\pi}(\mathbb{R})$$

and, if $f \in C_{2\pi}(\mathbb{R})$,

$$\lim_{r\to 1^-} P_r(f) = f \quad \textit{uniformly on} \quad \mathbb{R}.$$

Proof. The kernels p_r, $0 \le r < 1$, are positive. Moreover, from (4.34), we get

$$P_r(\mathbf{1}) = \mathbf{1},\ P_r(\cos) = r\cos,\ P_r(\sin) = r\sin$$

and hence

$$\beta_r = \frac{1-r}{2}.$$

Therefore, the result follows from Theorem 4.4 (or from Theorem 4.5). □

According to (4.34), Theorem 4.8 claims that the Fourier series of a function $f \in L^p_{2\pi}(\mathbb{R})$ (resp. $f \in C_{2\pi}(\mathbb{R})$) is Abel summable to f in $L^p_{2\pi}(\mathbb{R})$ (resp. uniformly on $\mathbb{R}$).

For further applications of Korovkin's second theorem to approximation by convolution operators and summation processes, we refer, e.g., to [8, Section 5.4], [38], [41], [78].

We finally point out the relevance of Theorem 4.8 in the study of **the Dirichlet problem for the unit disk** $\mathbb{D} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$. Given $F \in C(\partial\mathbb{D}) = C(\mathbb{T}) \equiv C_{2\pi}(\mathbb{R})$, this problem consists in finding a function $U \in C(\mathbb{D})$ possessing second partial derivatives on the interior of $\mathbb{D}$ such that

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2}(x,y) + \dfrac{\partial^2 U}{\partial y^2}(x,y) = 0 & (x^2+y^2<1), \\ U(x,y) = F(x,y) & (x^2+y^2=1). \end{cases} \tag{4.35}$$

By using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ $(0 \le r < 1,\ \theta \in \mathbb{R})$ and the functions $f(\theta) := F(\cos\theta, \sin\theta)$ $(\theta \in \mathbb{R})$ and $u(r,\theta) := U(r\cos\theta, r\sin\theta)$ $(0 \le r < 1,\ \theta \in \mathbb{R})$, problem (4.35) turns into

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}(r,\theta) + \dfrac{1}{r}\dfrac{\partial u}{\partial r}(r,\theta) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0 & 0<r<1,\ \theta\in\mathbb{R}, \\ u(0,\theta) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(t)\,\mathrm{d}t & \theta\in\mathbb{R}, \\ \lim\limits_{r\to 1^-} u(r,\theta) = f(\theta) & \text{uniformly} \quad \text{w.r.t.} \quad \theta\in\mathbb{R}. \end{cases} \tag{4.36}$$

.

With the help of Theorem 4.8 it is not difficult to show that a solution to problem (4.36) is given by

$$u(r,\theta) = P_r(f)(\theta) \qquad (0 \le r < 1,\ \theta \in \mathbb{R}). \tag{4.37}$$

Accordingly, the function

$$U(x,y) := \begin{cases} u(r,\theta) & \text{if } x = r\cos\theta,\ y = r\sin\theta \text{ and } x^2+y^2<1, \\ F(x,y) & \text{if } x^2+y^2=1, \end{cases}$$

is a solution to problem (4.35). Furthermore, as it is well-known, the solution to (4.35) is unique. A similar result also holds if $F : \mathbb{T} \to \mathbb{R}$ is a Borel-measurable function such that $\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid F(\cos t, \sin t) \mid^p dt < +\infty$, $1 \le p < +\infty$. For more details we refer, e.g., to [38, Proposition 1.2.10 and Theorem 7.1.3] and to [96, Section 1.2].

We end this section by presenting an application of Theorem 4.1. Consider the d-dimensional simplex

$$K_d := \{x = (x_i)_{1\le i\le d} \in \mathbb{R}^d \mid x_i \ge 0, 1 \le i \le d, \text{and} \sum_{i=1}^{d} x_i \le 1\} \tag{4.38}$$

and for every $n \geq 1, f \in C(K_d)$ and $x = (x_i)_{1\leq i\leq d} \in K_d$, set

$$B_n(f)(x) := \sum_{\substack{h_1,\dots,h_d=0,\dots,n\\ h_1+\cdots+h_d\leq n}} f\Big(\frac{h_1}{n},\dots,\frac{h_d}{n}\Big)\frac{n!}{h_1!\cdots h_d!(n-h_1-\cdots-h_d)!} \times x_1^{h_1}\cdots x_d^{h_d}(1-x_1-\cdots-x_d)^{n-h_1-\cdots-h_d}. \tag{4.39}$$

$B_n(f)$ is a polynomial and it is usually called the n-th **Bernstein polynomial on the d-dimensional simplex** associated with f. These polynomials were first studied by Dinghas ([43]).

Theorem 4.9. *For every $f \in C(K_d)$,*

$$\lim_{n\to\infty} B_n(f) = f \quad \textit{uniformly on} \quad K_d.$$

Proof. From the multinomial theorem, it follows that $B_n(\mathbf{1}) = \mathbf{1}$. Consider now the first coordinate function pr_1 (see (4.2)). Then, for $x = (x_i)_{1\leq i\leq d} \in K_d$ and $n \geq 1$,

$$\begin{aligned}
B_n(pr_1)(x) &= \sum_{\substack{h_1,\dots,h_d=0,\dots,n\\ h_1+\cdots+h_d\leq n}} \frac{h_1}{n}\frac{n!}{h_1!\cdots h_d!(n-h_1-\cdots-h_d)!}\\
&\qquad \times x_1^{h_1}\cdots x_d^{h_d}(1-x_1-\cdots-x_d)^{n-h_1-\cdots-h_d}\\
&= x_1 \sum_{\substack{k_1,h_2,\dots,h_d=0,\dots,n\\ k_1+h_2+\cdots+h_d\leq n-1}} \frac{(n-1)!}{k_1!h_2!\cdots h_d!((n-1)-k_1-h_2-\cdots-h_d)!}\\
&\qquad \times x_1^{k_1}x_2^{h_2}\cdots x_d^{h_d}(1-x_1-\cdots-x_d)^{(n-1)-k_1-h_2-\cdots-h_d}\\
&= x_1.
\end{aligned}$$

On the other hand, for $n \geq 2$,

$$\begin{aligned}
B_n(pr_1^2)(x) &= \sum_{\substack{h_1,\dots,h_d=0,\dots,n\\ h_1+h_2+\cdots+h_d\leq n}} \frac{h_1^2}{n^2}\frac{n!}{h_1!\cdots h_d!(n-h_1-\cdots-h_d)!}\\
&\qquad \times x_1^{h_1}\cdots x_d^{h_d}(1-x_1-\cdots-x_d)^{n-h_1-\cdots-h_d}\\
&= \sum_{h_1+\cdots+h_d\leq n} \frac{h_1}{n}\frac{(n-1)!}{(h_1-1)!h_2!\cdots h_d!(n-h_1-\cdots-h_d)!}\\
&\qquad \times x_1^{h_1}\cdots x_d^{h_d}(1-x_1-\cdots-x_d)^{n-h_1-\cdots-h_d}\\
&= \sum_{h_1+\cdots+h_d\leq n} \frac{n-1}{n}\frac{h_1-1}{n-1}\frac{(n-1)!}{(h_1-1)!h_2!\cdots h_d!(n-h_1-\cdots-h_d)!}\\
&\qquad \times x_1^{h_1}\cdots x_d^{h_d}(1-x_1-\cdots-x_d)^{n-h_1-\cdots-h_d} \quad + \quad \frac{x_1}{n}\\
&= \frac{n-1}{n}x_1^2 \quad + \quad \frac{x_1}{n}.
\end{aligned}$$

Similar considerations apply to the other coordinate functions. Thus for every $j = 1,\dots,d$,

$$B_n(pr_j) = pr_j \quad \text{and} \quad B_n(pr_j^2) = \frac{n-1}{n}pr_j^2 + \frac{1}{n}pr_j. \tag{4.40}$$

Applying Theorem 4.2 to the sequence of positive linear operators $(B_n)_{n\geq 1}$, we get the result. $\square$

Remark 4.10. Theorem 4.9 gives a constructive proof of the multidimensional versions of the Weierstrass approximation theorem, namely

For every $f \in C(K_d)$, there exists a sequence of algebraic polynomials that converges to f uniformly on K_d.

Another useful generalization of the one-dimensional Bernstein polynomial is discussed below. Consider the hypercube $Q_d = [0,1]^d$ and for every $n \geq 1, f \in C(Q_d)$ and $x = (x_i)_{1\leq i\leq d}$, set

$$B_n(f)(x) := \sum_{h_1,\dots,h_d=0}^{n} f\Big(\frac{h_1}{n},\dots,\frac{h_d}{n}\Big)\binom{n}{h_1}\cdots\binom{n}{h_d} x_1^{h_1}(1-x_1)^{n-h_1}\cdots x_d^{h_d}(1-x_d)^{n-h_d}. \tag{4.41}$$

The polynomials $B_n(f)$, $n \geq 1$, are called the **Bernstein polynomials on the hypercube** associated with f. They were first studied by Hildebrandt and Schoenberg ([70]) and Butzer ([37]). The proof of Theorem 4.9 works also for these operators giving the same formula (4.40). Therefore, again by Theorem 4.2, we obtain

Theorem 4.11. *For every $f \in C(Q_d)$,*

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on} \quad Q_d.$$

We end the section by discussing some applications of Theorem 4.1 for noncompact subsets of $\mathbb{R}^d$.

The first application is concerned with the interval $[0,+\infty[$. Set

$$\begin{aligned} E := \{f \in & C([0,+\infty[) \mid \text{there exist } \alpha \geq 0 \text{ and } M \geq 0 \\ & \text{such that } |f(x)| \leq M\exp(\alpha x)\ (x \geq 0)\}, \end{aligned} \tag{4.42}$$

and for every $f \in E, n \geq 1$, and $x \geq 0$, define

$$M_n(f)(x) := \exp(-nx)\sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big)\frac{n^k x^k}{k!}\,. \tag{4.43}$$

The operator M_n is linear and positive and is called the n-th **Szász-Mirakjan operator** The sequence $(M_n)_{n\geq 1}$ was first introduced and studied by Mirakjan ([86]), Favard ([55]) and Szász ([113]) and is one of the most studied sequences of positive linear operators on function spaces on $[0,+\infty[$.

A simple calculation shows that

$$M_n(\mathbf{1}) = \mathbf{1}, \quad M_n(e_1) = e_1$$

and

$$M_n(e_2) = e_2 + \frac{1}{n}e_1\,.$$

Therefore, from Theorem 4.1, we get that

Theorem 4.12. *For every* $f \in C_b([0, +\infty[)$,

$$\lim_{n\to\infty} M_n(f) = f$$

uniformly on compact subsets of $[0, +\infty[$.

The next application concerns the case $X = \mathbb{R}^d$, $d \geq 1$. For every $n \geq 1$ and for every real-valued Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, set

$$G_n(f)(x) := \Big(\frac{n}{4\pi}\Big)^{d/2} \int_{\mathbb{R}^d} f(t) \exp(-\frac{n}{4}\|t-x\|^2)\,\mathrm{d}t \tag{4.44}$$

$(x \in \mathbb{R}^d)$.

We shall consider the operators $G_n, n \geq 1$, defined on the lattice subspace of all Borel-measurable functions $f \in F(\mathbb{R}^d)$ for which the integral (4.44) is absolutely convergent. They are referred to as the **Gauss-Weierstrass convolution operators on** $\mathbb{R}^d$. Among other things, they are involved in the study of the heat equation on $\mathbb{R}^d$ (see, e.g., [8], [38]).

By using the following formulae:

$$\Big(\frac{1}{2\pi\sigma^2}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big(-\frac{(t-\alpha)^2}{2\sigma^2}\Big)\,\mathrm{d}t = 1, \tag{4.45}$$

$$\Big(\frac{1}{2\pi\sigma^2}\Big)^{1/2} \int_{\mathbb{R}} t\exp\Big(-\frac{(t-\alpha)^2}{2\sigma^2}\Big)\,\mathrm{d}t = \alpha, \tag{4.46}$$

and

$$\Big(\frac{1}{2\pi\sigma^2}\Big)^{1/2} \int_{\mathbb{R}} t^2\exp\Big(-\frac{(t-\alpha)^2}{2\sigma^2}\Big)\,\mathrm{d}t \;=\; \alpha^2 + \sigma^2, \tag{4.47}$$

$(\alpha \in \mathbb{R}, \sigma > 0)$ (see [30, formulae (4.14) and (4.15)]) and appealing to Fubini's theorem, we immediately obtain for every $i = 1, \ldots, d$ and $x \in \mathbb{R}^d$,

$$G_n(\mathbf{1})(x) := \prod_{j=1}^{d} \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} \exp(-\frac{n}{4}(t_j - x_j)^2)\,\mathrm{d}t_j \;=\; 1, \tag{4.48}$$

$$\begin{aligned} G_n(pr_i)(x) \;&=\; \prod_{\substack{j=1\\ j\neq i}}^{d} \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big(-\frac{n}{4}(t_j - x_j)^2\Big)\,\mathrm{d}t_j \times \\ &\times\; \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} t_i \exp\Big(-\frac{n}{4}(t_i - x_i)\Big)^2\,\mathrm{d}t_i \;=\; x_i \;=\; pr_i(x) \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} G_n(pr_i^2)(x) \;&=\; \prod_{\substack{j=1\\ j\neq i}}^{d} \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big(-\frac{n}{4}(t_j - x_j)^2\Big)\,\mathrm{d}t_j \times \\ &\times\; \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} t_i^2 \exp\Big(-\frac{n}{4}(t_i - x_i)^2\Big)\,\mathrm{d}t_i \;=\; x_i^2 + \frac{2}{n} \;=\; pr_i^2(x) + \frac{2}{n}\,. \end{aligned} \tag{4.50}$$

From Theorem 4.1, we then obtain

Theorem 4.13. *For every $f \in C_b(\mathbb{R}^d)$,*

$$\lim_{n\to\infty} G_n(f) = f \quad \textit{uniformly on compact subsets of} \quad \mathbb{R}^d.$$

For additional properties of Gauss-Weierstrass operators, we refer, e.g., to [38] (see also a recent generalization given in [21-23]).

5 Korovkin-type theorems for positive linear operators

After the discovery of Korovkin's theorem, several mathematicians tried to extend it in several directions with the aim, for instance:

(i) to find other subsets of functions satisfying the same property as $\{\mathbf{1}, e_1, e_2\}$;

(ii) to establish results like Theorem 3.1 in other function spaces or in abstract Banach spaces;

(iii) to establish results like Theorem 3.1 for other classes of linear operators.

As a consequence of these investigations, a new theory was created which is nowadays called "Korovkin-type Approximation Theory". This theory has strong and fruitful connections not only with classical approximation theory but also with real analysis, functional analysis, harmonic analysis, probability theory and partial differential equations. We refer to [8] for a rather detailed description of the development of this theory.

We shall next discuss some of the main results of the theory obtained in the framework of the spaces $C_0(X)$ (X locally compact noncompact space), $C(X)$ (X compact space), occasionally in $L^p(X, \widetilde{\mu})$ spaces, $1 \le p < +\infty$, and in weighted continuous function spaces. These spaces play a central role in the theory and they are the most useful in applications.

In addition, it transpires that the elementary methods used in the previous sections are not appropriate to give a wider possibility to determine other test functions (like $\{\mathbf{1}, e_1, e_2\}$ in $C([0,1])$), let alone to characterize them. To this aim, it seems to be unavoidable to use a more precise analysis involving the point topology of the underlying space and the positive linear functionals on the relevant continuous function spaces.

It turns out that the right framework is the local compactness which we choose to consider in its generality because the class of locally compact spaces includes, other than the Euclidean spaces $\mathbb{R}^d$, $d \ge 1$, and their open or closed subsets, many other topological spaces which are important in their own right. The reader who is not interested in this level of generality, may replace everywhere our locally compact spaces with a space $\mathbb{R}^d$, $d \ge 1$, or with an open or a closed subset of it, or with the intersection of an open subset and a closed subset of $\mathbb{R}^d$. However, this restriction does not produce any simplification in the proofs or in the methods.

In the sequel, given a locally compact Hausdorff space X, we shall denote by

$$K(X)$$

the linear subspace of all real-valued continuous functions on X having compact support. Then $K(X) \subset C_b(X)$. We shall denote by

$$C_0(X)$$

the closure of $K(X)$ with respect to the sup-norm $\|\cdot\|_\infty$ (see (2.3)). Thus, $C_0(X)$ is a closed subspace of $C_b(X)$ and hence, endowed with the norm $\|\cdot\|_\infty$, is a Banach space.

If X is compact, then $C_0(X) = C(X)$. If X is not compact, then a function $f \in C(X)$ belongs to $C_0(X)$ if and only if

$$\text{for every } \varepsilon > 0 \text{ there exists a compact subset } K \text{ of } X \\ \text{such that } |f(x)| \le \varepsilon \text{ for every } x \in X \setminus K.$$

For additional topological and analytical properties of locally compact spaces and of some relevant continuous function spaces on them, we refer to the Appendix.

The spaces $C_0(X)$ and $C(X)$, (X compact), endowed with the natural pointwise ordering and the sup-norm, become Banach lattices. Similarly $L^p(X, \widetilde{\mu})$, endowed with the natural norm $\|\cdot\|_p$ and the ordering

$$f \le g \quad \text{if} \quad f(x) \le g(x) \quad \text{for} \quad \widetilde{\mu}\text{-a.e. } x \in X,$$

is a Banach lattice (for more details on $(L^p(X, \widetilde{\mu}), \|\cdot\|_p)$ spaces, we refer to the Appendix (formulae (11.11) and (11.12))).

For the reader's convenience, we recall that a **Banach lattice** E is a vector space endowed with a norm $\|\cdot\|$ and an ordering $\le$ on E such that

(i) $(E, \|\cdot\|)$ is a Banach space;

(ii) $(E, \le)$ is a vector lattice;

(iii) If $f, g \in E$ and $|f| \le |g|$ then $\|f\| \le \|g\|$.

(where $|f| := \sup(f, -f)$ for every $f \in E$).

Actually it is convenient to state the main definitions of the theory in the framework of Banach lattices. However, the reader not accustomed to this terminology may replace our abstract spaces with the concrete ones such as $C_0(X)$, $C(X)$ or $L^p(X, \widetilde{\mu})$. For more details on Banach lattices, we refer, e.g., to [2].

If E and F are Banach lattices, a linear operator $L : E \longrightarrow F$ is said to be **positive** if

$$L(f) \ge 0 \quad \text{for every} \quad f \in E,\ f \ge 0.$$

Every positive linear operator $L : E \longrightarrow F$ is continuous, (see, e.g., [2, Theorem 12.3]). Moreover, if $E = C(X)$, X compact, then $\|L\| = \|L(\mathbf{1})\|$.

A **lattice homomorphism** $S : E \longrightarrow F$ is a linear operator satisfying $|S(f)| = S(|f|)$ for every $f \in E$. Equivalently, this means that S preserves the finite lattice operation, i.e., for every $f_1, \dots, f_n \in E,\ n \ge 2$,

$$S\left(\inf_{1 \le i \le n} f_i\right) = \inf_{1 \le i \le n} S(f_i) \text{ and } S\left(\sup_{1 \le i \le n} f_i\right) = \sup_{1 \le i \le n} S(f_i).$$

For instance, if X is a locally compact Hausdorff space, $\widetilde{\mu}$ a regular finite Borel measure on X and $1 \le p < +\infty$, then the natural embedding $J_p : C_0(X) \longrightarrow L^p(X, \widetilde{\mu})$ defined by $J_p(f) := f$ $(f \in C_0(X))$ is a lattice homomorphism.

Analogously, if X and Y are compact spaces and $\varphi : Y \longrightarrow X$ is a continuous mapping, then the composition operator $T_\varphi(f) := f \circ \varphi, (f \in C(X))$, is a lattice homomorphism from $C(X)$ into $C(Y)$. Every lattice homomorphism is positive and hence continuous. A linear bijection $S : E \longrightarrow F$ is a lattice homomorphism if and only if S and its inverse S^{-1} are both positive [2, Theorem 7.3]. In this case, we also say that S is a **lattice isomorphism**. When there exists a lattice isomorphism between E and F, then we say that E and F are **lattice isomorphic**.

The following definition, which is one of the most important of the theory, is clearly motivated by the Korovkin theorem and was first formulated by V. A. Baskakov ([28]).

Definition 5.1. *A subset M of a Banach lattice E is said to be a* **Korovkin subset** *of E if for every sequence $(L_n)_{n\geq 1}$ of positive linear operators from E into E satisfying*

(i) $\sup_{n\geq 1}\|L_n\| < +\infty$,

and

(ii) $\lim_{n\to\infty} L_n(g) = g$ *for every* $g \in M$,

it turns out that

$$\lim_{n\to\infty} L_n(f) = f \quad \text{for every} \quad f \in E.$$

Note that, if $E = C(X)$, X compact space, and the constant function $\mathbf{1}$ belongs to the linear subspace $\mathcal{L}(M)$ generated by M, then condition (i) is superfluous because it is a consequence of (ii).

According to Definition 5.1, we may restate Korovkin's Theorem 3.1 by saying that $\{\mathbf{1}, e_1, e_2\}$ is a Korovkin set in $C([0,1])$.

We also point out that a subset M is a Korovkin subset of E if and only if the linear subspace $\mathcal{L}(M)$ generated by M is a Korovkin subset. In the sequel, a linear subspace that is a Korovkin subset will be referred to as a **Korovkin subspace** of E. If E and F are lattice isomorphic and if $S : E \longrightarrow F$ is a lattice isomorphism, then a subset M of E is a Korovkin subset in E if and only if $S(M)$ is a Korovkin subset in F.

Korovkin sets (when they exist) are useful for investigating the convergence of equibounded sequences of positive linear operators towards the identity operator or, from the point of view of approximation theory, the approximation of every element $f \in E$ by means of $(L_n(f))_{n\geq 1}$.

According to Lorentz ([80]), who first proposed a possible generalization, it seems to be equally interesting to study the following more general concept.

Definition 5.2. *Let E and F be Banach lattices and consider a positive linear operator $T : E \longrightarrow F$. A subset M of E is said to be a* **Korovkin subset** *of E for T if for every sequence $(L_n)_{n\geq 1}$ of positive linear operators from E into F satisfying*

(i) $\sup_{n\geq 1}\|L_n\| < +\infty$

and

(ii) $\lim_{n\to\infty} L_n(g) = T(g)$ *for every* $g \in M$,

it turns out that

$$\lim_{n\to\infty} L_n(f) = T(f) \quad \text{for every} \quad f \in E.$$

Thus, such subsets can be used to investigate the convergence of equibounded sequences of positive linear operators towards a given positive linear operator $T : E \longrightarrow F$ or to approximate weakly T by means of (generally, simpler) linear operators L_n, $n \geq 1$.

In the light of the above definition, two problems arise quite naturally:

Problem 5.3. Given a positive linear operator $T : E \longrightarrow F$, find conditions under which there exists a nontrivial (i.e., the linear subspace generated by it is not dense) Korovkin subset for T. In this case, try to determine some or all of them.

Problem 5.4. Given a subset M of E, try to determine some or all of the positive linear operators $T : E \to F$ (if they exist) for which M is a Korovkin subset.

In the next sections, we shall discuss some aspects related to Problem 5.3 (for further details, we refer to [8, Sections 3.3 and 3.4]). As regards Problem 5.4, very few results are available (see, e.g., [71], [72-73], [74], [114-116]).

The next result furnishes a complete characterization of Korovkin subsets for positive linear operators in the setting of $C_0(X)$ spaces. It was obtained by Yu. A. Shashkin ([110]) in the case when $X = Y$, X compact metric space and $T = I$ the identity operator, by H. Berens and G. G. Lorentz ([33]) when $X = Y$, X topological compact space, $T = I$, by H. Bauer and K. Donner ([31]) when $X = Y$, X locally compact space, $T = I$, by C. A. Micchelli ([85]) and M. D. Rusk ([105]) when $X = Y$, X compact, and by F. Altomare ([4]) in the general form below.

We recall that $\mathcal{M}_b^+(X)$ denotes the cone of all bounded Radon measures on X (see Appendix).

Theorem 5.5. *Let X and Y be locally compact Hausdorff spaces. Further, assume that X has a countable base and Y is metrizable. Given a positive linear operator $T : C_0(X) \longrightarrow C_0(Y)$ and a subset M of $C_0(X)$, the following statements are equivalent:*

(i) *M is a Korovkin subset of $C_0(X)$ for T.*

(ii) *If $\mu \in \mathcal{M}_b^+(X)$ and $y \in Y$ satisfy $\mu(g) = T(g)(y)$ for every $g \in M$, then $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$.*

Proof. (i)$\Rightarrow$(ii). Fix $\mu \in \mathcal{M}_b^+(X)$ and $y \in Y$ satisfying $\mu(g) = T(g)(y)$ for every $g \in M$. Consider a decreasing countable base $(U_n)_{n\geq 1}$ of open neighborhoods of y in Y and, for every $n \geq 1$, choose $\varphi_n \in K(Y)$ such that: $0 \leq \varphi_n \leq 1, \varphi_n(y) = 1$ and $\mathrm{supp}(\varphi_n) \subset U_n$ (see Theorem 11.1 of the Appendix).

Accordingly, define $L_n : C_0(X) \longrightarrow C_0(Y)$ by

$$L_n(f) := \mu(f)\varphi_n + T(f)(1 - \varphi_n) \quad (f \in C_0(X)).$$

Each L_n is linear, positive and $\|L_n\| \leq \|\mu\| + \|T\|$. Moreover, if $g \in M$, then

$$\lim_{n\to\infty} L_n(g) = T(g) \quad \text{in} \quad C_0(Y),$$

because, given $\varepsilon > 0$, there exists $v \in \mathbb{N}$ such that

$$|T(g)(z) - T(g)(y)| \leq \varepsilon \quad \text{for every} \quad z \in U_v.$$

Hence, since for every $n \geq v$ (thus $U_n \subset U_v$) and for every $z \in Y$

$$|L_n(g)(z) - T(g)(z)| = \varphi_n(z)|T(g)(z) - T(g)(y)|,$$

we get

$$|L_n(g)(z) - T(g)(z)| = \begin{cases} 0 & \text{if} \quad z \notin U_n, \\ \leq \varepsilon & \text{if} \quad z \in U_n, \end{cases}$$

and so $\|L_n(g) - T(g)\| \leq \varepsilon$.

Since M is a Korovkin subset for T, it turns out that, for every $f \in C_0(X)$, $\lim\limits_{n\to\infty} L_n(f) = T(f)$ and hence $\lim\limits_{n\to\infty} L_n(f)(y) = T(f)(y)$. But, for every $n \geq 1$, $L_n(f)(y) = \mu(f)$ and this completes the proof of (ii).

(ii)$\Rightarrow$(i). Our proof starts with the observation that from statement (ii), it follows that

$$\text{if} \quad \mu \in \mathcal{M}_b^+(X) \quad \text{and} \quad \mu(g) = 0 \quad \text{for every} \quad g \in M, \quad \text{then} \quad \mu = 0. \tag{5.1}$$

Moreover, since X has a countable base, every bounded sequence in $\mathcal{M}_b^+(X)$ has a vaguely convergent subsequence (see Theorem 11.8 of Appendix). Consider now a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C_0(X)$ into $C_0(Y)$ satisfying properties (i) and (ii) of Definition 5.2 and assume that for some $f_0 \in C_0(X)$ the sequence $(L_n(f_0))_{n\geq 1}$ does not converge uniformly to $T(f_0)$.

Therefore, there exist $\varepsilon_0 > 0$, a subsequence $(L_{k(n)})_{n\geq 1}$ of $(L_n)_{n\geq 1}$ and a sequence $(y_n)_{n\geq 1}$ in Y such that

$$|L_{k(n)}(f_0)(y_n) - T(f_0)(y_n)| \geq \varepsilon_0 \quad \text{for every} \quad n \geq 1. \tag{5.2}$$

We discuss separately the two cases when $(y_n)_{n\geq 1}$ is converging to the point at infinity of Y or not (see the Appendix).

In the first case (which can only occur when Y is noncompact), we have $\lim\limits_{n\to\infty} h(y_n) = 0$ for every $h \in C_0(Y)$.

For every $n \geq 1$, define $\mu_n \in \mathcal{M}_b^+(X)$ by

$$\mu_n(f) := L_{k(n)}(f)(y_n) \quad (f \in C_0(X)).$$

Since $\|\mu_n\| \leq \|L_{k(n)}\| \leq M := \sup\limits_{n\geq 1}\|L_n\|$, replacing, if necessary, the sequence $(\mu_n)_{n\geq 1}$ with a suitable subsequence, we may assume that there exists $\mu \in \mathcal{M}_b^+(X)$ such that $\mu_n \longrightarrow \mu$ vaguely. But, if $g \in M$, then

$$\begin{aligned} |\mu_n(g)| &\leq |L_{k(n)}(g)(y_n) - T(g)(y_n)| + |T(g)(y_n)| \leq \\ &\leq \|L_{k(n)}(g) - T(g)\| + |T(g)(y_n)| \end{aligned}$$

so that

$$\mu(g) = \lim_{n\to\infty} \mu_n(g) = 0.$$

From (5.1) it turns out that $\mu(f_0) = 0$ as well and hence

$$|L_{k(n)}(f_0)(y_n) - T(f_0)(y_n)| = |\mu_n(f_0) - T(f_0)(y_n)| \longrightarrow 0$$

which contradicts (5.2).

Consider now the case where the sequence $(y_n)_{n\geq 1}$ does not converge to the point at infinity of Y. Then, by replacing it with a suitable subsequence, we may assume that it converges to some $y \in Y$.

Again, consider for every $n \geq 1$

$$\mu_n(f) := L_{k(n)}(f)(y_n) \quad (f \in C_0(X)).$$

As in the previous reasoning, we may assume that there exists $\mu \in \mathcal{M}_b^+(X)$ such that $\mu_n \longrightarrow \mu$ vaguely. Then, for $g \in M$, since

$$|\mu_n(g) - T(g)(y_n)| \leq \|L_{k(n)}(g) - T(g)\| \longrightarrow 0,$$

we get $\mu(g) = T(g)(y)$. Therefore, assumption (ii) implies $\mu(f_0) = T(f_0)(y)$, or equivalently,

$$\lim_{n\to\infty}\Big(L_{k(n)}(f_0)(y_n) - T(f_0)(y_n)\Big) = 0$$

which is impossible because of (5.2). $\square$

Some applications of Theorem 4.5 will be shown in subsequent sections.

6 Korovkin-type theorems for the identity operator in $C_0(X)$

In this section, we discuss more closely those subsets of $C_0(X)$ that are Korovkin subsets in $C_0(X)$ (see Definition 4.1), i.e., that are Korovkin subsets for the identity operator on $C_0(X)$.

Throughout the whole section, we shall fix a locally compact Hausdorff space with a countable base, which is then metrizable as well. The next result immediately follows from Theorem 4.5.

Theorem 6.1. *([31]). Given a subset M of $C_0(X)$, the following statements are equivalent:*

(i) M is a Korovkin subset of $C_0(X)$.

(ii) If $\mu \in \mathcal{M}_b^+(X)$ and $x \in X$ satisfy $\mu(g) = g(x)$ for every $g \in M$, then $\mu(f) = f(x)$ for every $f \in C_0(X)$, i.e., $\mu = \delta_x$ (see (2.11)).

In order to discuss a first application of Theorem 6.1, we recall that a mapping $\varphi : Y \longrightarrow X$ between two locally compact Hausdorff spaces Y and X is said to be **proper** if for every compact subset $K \in X$, the inverse image $\varphi^{-1}(K) := \{y \in Y \mid \varphi(y) \in K\}$ is compact in Y. In this case, $f \circ \varphi \in C_0(Y)$ for every $f \in C_0(X)$.

Corollary 6.2. *Let Y be a metrizable locally compact Hausdorff space. If M is a Korovkin subset of $C_0(X)$, then M is a Korovkin subset for any positive linear operator $T : C_0(X) \longrightarrow C_0(Y)$ of the form*

$$T(f) := \lambda(f \circ \varphi) \quad (f \in C_0(X))$$

where $\lambda \in C_b(Y), \lambda \geq 0$, and $\varphi : Y \longrightarrow X$ is a proper mapping.

Proof. According to Theorem 5.5, we have to show that, if $\mu \in \mathcal{M}_b^+(X)$ and $y \in Y$ satisfy $\mu(g) = \lambda(y)g(\varphi(y))$ for every $g \in M$, then $\mu(f) = \lambda(y)f(\varphi(y))$ for every $f \in C_0(X)$.

If $\lambda(y) = 0$, then $\mu = 0$ on M and hence $\mu = 0$ by Theorem 6.1 and property (1) in the proof $(ii) \Longrightarrow (i)$ of Theorem 5.5. If $\lambda(y) > 0$, it suffices to apply Theorem 6.1 to $\dfrac{1}{\lambda(y)}\mu$ and $\varphi(y)$. $\square$

Note that, if Y is a closed subset of X, then the canonical mapping $\varphi : Y \longrightarrow X$ defined by $\varphi(y) := y$ $(y \in Y)$ is proper. Therefore, from Corollary 5.2, we get:

Corollary 6.3. *Let M be a Korovkin subset of $C_0(X)$. Consider an equibounded sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C_0(X)$ into $C_0(X)$. Then the following properties hold.*

1) Given a closed subset Y of X and $\lambda \in C_b(Y)$, $\lambda \geq 0$, if $\lim\limits_{n\to\infty} L_n(g) = \lambda g$ uniformly on Y for every $g \in M$, then

$$\lim_{n\to\infty} L_n(f) = \lambda f \quad \text{uniformly on} \quad Y \quad \text{for every} \quad f \in C_0(X).$$

2) If $\lim_{n\to\infty} L_n(g) = g$ uniformly on compact subsets of X for every $g \in M$, then $\lim_{n\to\infty} L_n(f) = f$ uniformly on compact subsets of X for every $f \in C_0(X)$.

Next, we proceed to investigate some useful criteria to explicitly determine Korovkin subsets of $C_0(X)$. We begin with the following result which is at the root of all subsequent results. We recall that, if M is a subset of $C_0(M)$, then the symbol $\mathcal{L}(M)$ denotes the linear subspace of $C_0(X)$ generated by M.

Proposition 6.4. *Let M be a subset of $C_0(X)$ and assume that for every $x, y \in X, x \neq y$, there exists $h \in \mathcal{L}(M), h \geq 0$, such that $h(x) = 0$ and $h(y) > 0$. Then M is a Korovkin subset of $C_0(X)$.*

Proof. We shall verify condition (ii) of Theorem 6.1. Therefore, consider $\mu \in \mathcal{M}_b^+(X)$ and $x \in X$ satisfying $\mu(g) = g(x)$ for every $g \in M$ and, hence, for every $g \in \mathcal{L}(M)$.

If $y \in X, y \neq x$, then there exists $h \in \mathcal{L}(M)$, $h \geq 0$, such that $h(y) > 0$ and $0 = h(x) = \mu(h)$. Therefore, by Theorem 11.7 of the Appendix, there exists $\alpha \geq 0$ such that $\mu = \alpha\delta_x$. Choosing $h \in \mathcal{L}(M)$ such that $h(x) > 0$, we get $\alpha h(x) = \mu(h) = h(x)$, so $\alpha = 1$ and $\mu = \delta_x$. □

Below, we state an important consequence of Proposition 6.4. In the sequel, if M is a subset of $C(X)$ and for $f_0 \in C(X)$, we shall set

$$f_0 M := \{f_0 \cdot f \mid f \in M\} \tag{6.1}$$

and

$$f_0 M^2 := \{f_0 \cdot f^2 \mid f \in M\}. \tag{6.2}$$

Theorem 6.5. *Consider a strictly positive function $f_0 \in C_0(X)$ and a subset M of $C(X)$ that separates the points of X. Furthermore, assume that $f_0 M \cup f_0 M^2 \subset C_0(X)$. Then $\{f_0\} \cup f_0 M \cup f_0 M^2$ is a Korovkin subset of $C_0(X)$.*

If, in addition, M is finite, say $M = \{f_1, \ldots, f_n\}, n \geq 1$, then

$$\left\{ f_0, f_0 f_1, \ldots, f_0 f_n, f_0 \sum_{i=1}^{n} f_i^2 \right\}$$

is a Korovkin subset of $C_0(X)$.

Proof. If $x, y \in X, x \neq y$, then there exists $f \in M$ such that $f(x) \neq f(y)$. Therefore, the function $h := f_0(f - f(x))^2$ belongs to $\mathcal{L}(\{f_0\} \cup f_0 M \cup f_0 M^2)$, it is positive and $h(x) = 0 < h(y)$. Thus the result follows from Proposition 6.4.

If $M = \{f_1, \ldots, f_n\}$ is finite, then in the above reasoning one can consider the function $h := f_0 \sum_{i=1}^{n} (f_i - f_i(x))^2$. □

The next result is an obvious consequence of Theorem 6.5 but is worth being stated explicitly because of its connection with the Stone-Weierstrass theorem (see Section 9).

Theorem 6.6. *Consider a strictly positive function $f_0 \in C_0(X)$ and a subset M of $C_0(X)$ that separates the points of X. Then $\{f_0\} \cup f_0 M \cup f_0 M^2$ is a Korovkin subset of $C_0(X)$.*

Moreover, if M is finite, say $M = \{f_1, \ldots, f_n\}, n \geq 1$, then

$$\left\{ f_0, f_0 f_1, \ldots, f_0 f_n, f_0 \sum_{i=1}^{n} f_i^2 \right\}$$

is a Korovkin subset of $C_0(X)$.

Finally, if f_0 *is also injective, then* $\{f_0, f_0^2, f_0^3\}$ *is a Korovkin subset of* $C_0(X)$.

From Theorem 6.6, the following result immediately follows.

Corollary 6.7. *(1)* $\{e_1, e_2, e_3\}$ *is a Korovkin subset of* $C_0(]0,1])$.

(2) $\{e_{-1}, e_{-2}, e_{-3}\}$ *is a Korovkin subset of* $C_0([1,+\infty[)$, *where* $e_{-k}(x) := x^{-k}$ *for every* $x \in [1,+\infty[$ *and* $k = 1,2,3$.

(3) $\{f_1, f_2, f_3\}$ *is a Korovkin subset of* $C_0([0,+\infty[)$ *where* $f_k(x) := \exp(-kx)$ *for every* $x \in [0,+\infty[$ *and* $k = 1,2,3$.

(4) $\{\Phi, pr_1\Phi, \ldots, pr_d\Phi, \|\cdot\|^2\Phi\}$ *is a Korovkin subset of* $C_0(\mathbb{R}^d)$, $d \geq 1$, *where* $\Phi(x) := exp(-\|x\|^2)$ *for every* $x \in \mathbb{R}^d$.

A useful generalization of the previous result is presented below.

Proposition 6.8. *Given* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $0 < \lambda_1 < \lambda_2 < \lambda_3$, *then*

(1) $\{e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3}\}$ *is a Korovkin subset of* $C_0(]0,1])$ *where* $e_{\lambda_k}(x) := x^{\lambda_k}$ *for every* $x \in]0,1]$ *and* $k = 1,2,3$.

(2) $\{e_{-\lambda_1}, e_{-\lambda_2}, e_{-\lambda_3}\}$ *is a Korovkin subset of* $C_0([1,+\infty[)$ *where* $e_{-\lambda_k}(x) := x^{-\lambda_k}$ *for every* $x \in [1,+\infty[$ *and* $k = 1,2,3$.

(3) $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ *is a Korovkin subset of* $C_0([0,+\infty[)$ *where* $f_{\lambda_k}(x) := \exp(-\lambda_k x)$ *for every* $x \in [0,+\infty[$ *and* $k = 1,2,3$.

Proof. We give the proof only for (3); the proofs of the other statements are left to the reader (see, e.g., [8, Proposition 4.2.4]). We shall apply Proposition 6.4 and to this end fix $x_0 \in [0,+\infty[$. Then, by using differential calculus, it is not difficult to show that the function

$$h(x) := \exp(-\lambda_1 x) + \alpha\exp(-\lambda_2 x) + \beta\exp(-\lambda_3 x) \quad (x \geq 0)$$

where $\alpha := \dfrac{\lambda_1 - \lambda_3}{\lambda_3 - \lambda_2}\exp((\lambda_2 - \lambda_1)x_0)$ and $\beta := \dfrac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2}\exp((\lambda_3 - \lambda_1)x_0)$, satisfies $h(x_0) = 0 < h(y)$ for every $y \in [0,+\infty[$, $y \neq x_0$. □

Here, we discuss some applications of Proposition 6.8. The first application is taken from [31, Proposition 4.1].

We begin by recalling that, if $\varphi \in L^1([0,+\infty[)$, then the **Laplace transform** of φ on $[0,+\infty[$ is defined by

$$\mathcal{L}(\varphi)(\lambda) := \int_0^{+\infty} \mathrm{e}^{-\lambda t}\varphi(t)\,\mathrm{d}t \quad (\lambda \geq 0). \tag{6.3}$$

By Lebesgue's dominated convergence theorem, $\mathcal{L}(\varphi) \in C_0([0,+\infty[)$ and $\|\mathcal{L}(\varphi)\| \leq \|\varphi\|_1$. By means of φ, we may naturally define a linear operator $L_\varphi : C_0([0,+\infty[) \to C_0([0,+\infty[)$ by setting for every $f \in C_0([0,+\infty[)$ and $x \geq 0$

$$L_\varphi(f)(x) := \int_0^{+\infty} f(x+y)\varphi(y)\,\mathrm{d}y = \int_x^{+\infty} f(u)\varphi(x-u)\,\mathrm{d}u. \tag{6.4}$$

Note that $L_\varphi(f) \in C_0([0,+\infty[)$ by virtue of Lebesgue's dominated convergence theorem. Moreover L_φ is bounded and $\|L_\varphi\| \le \|\varphi\|_1$. Finally, if $\varphi \ge 0$, then L_φ is positive as well.

For every $\lambda > 0$, denoting by f_λ the function

$$f_\lambda(x) := \exp(-\lambda x) \quad (x \ge 0), \tag{6.5}$$

we get

$$L_\varphi(f_\lambda) = \mathcal{L}(\varphi)(\lambda) f_\lambda. \tag{6.6}$$

Proposition 6.9. *Consider a sequence $(\varphi_n)_{n\ge1}$ of positive functions in $L^1([0,+\infty[)$ such that $\sup\limits_{n\ge1}\|\varphi_n\|_1 < +\infty$ and, for every $n \ge 1$, denote by*

$$L_n : C_0([0,+\infty[) \to C_0([0,+\infty[)$$

the positive linear operator associated with φ_n defined by (6.4). Then the following statements are equivalent:

(i) $\lim\limits_{n\to\infty} L_n(f) = f$ uniformly on $[0,+\infty[$ for every $f \in C_0([0,+\infty[)$.

(ii) $\lim\limits_{n\to\infty} \mathcal{L}(\varphi_n)(\lambda) = 1$ for every $\lambda > 0$.

(iii) There exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},\ 0 < \lambda_1 < \lambda_2 < \lambda_3$ such that

$$\lim_{n\to\infty} \mathcal{L}(\varphi_n)(\lambda_k) = 1 \quad \textit{for} \quad k = 1,2,3.$$

Proof. The implication (i)⇒(ii) follows from (6.6). The implication (ii)⇒(iii) being obvious, the only point remaining concerns (iii)⇒(i).

Since $(\varphi_n)_{n\ge1}$ is bounded in $L^1([0,+\infty[)$, we have $\sup\limits_{n\ge1}\|L_n\| < +\infty$. Moreover, (iii) means that $\lim\limits_{n\to\infty} L_n(f_{\lambda_k}) = f_{\lambda_k}$ uniformly on $[0,+\infty[)$ by (6.6). Therefore, (i) follows by applying part (3) of Proposition 6.8. □

As another application of the previous results, we shall study the behaviour of the Szász-Mirakjan operators (see (4.43)) on $C_0([0,+\infty[)$ and on continuous function spaces on $[0,+\infty[$ with polynomial weights.

We recall that these operators are defined by

$$M_n(f)(x) := \exp(-nx) \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{n^k x^k}{k!} \tag{6.7}$$

for $n \ge 1, x \ge 0$ and $f \in C([0,+\infty[)$ such that $|f(x)| \le M \exp(\alpha x)$ $(x \ge 0)$ for some $M \ge 0$ and $\alpha > 0$.

Lemma 6.10. *If $f \in C_0([0,+\infty[)$, then $M_n(f) \in C_0([0,+\infty[)$ and $\|M_n(f)\| \le \|f\|$ for every $n \ge 1$.*

Proof. The function $M_n(f)$ is continuous because the series (6.7) is uniformly convergent on compact subsets of $[0,+\infty[$. Moreover, for every $x \ge 0$

$$|M_n(f)(x)| \le \|f\| \exp(-nx) \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} = \|f\|.$$

In order to show that $M_n(f) \in C_0([0,+\infty[)$, given $\varepsilon > 0$, choose $v \in \mathbb{N}$ such that $|f\left(\frac{k}{n}\right)| \leq \varepsilon$ for every $k \geq v$. For sufficiently large $x \geq 0$ we get

$$\exp(-nx)\sum_{k=0}^{v}\left|f\left(\frac{k}{n}\right)\right|\frac{(nx)^k}{k!} \leq \varepsilon$$

so that

$$\begin{aligned}\left|M_n(f)(x)\right| &\leq \exp(-nx)\sum_{k=0}^{v}\left|f\left(\frac{k}{n}\right)\right|\frac{(nx)^k}{k!} + \exp(-nx)\sum_{k=v+1}^{\infty}\left|f\left(\frac{k}{n}\right)\right|\frac{(nx)^k}{k!} \\ &\leq 2\varepsilon.\end{aligned}$$

□

Theorem 6.11. *For every $f \in C_0([0,+\infty[)$,*

$$\lim_{n\to\infty} M_n(f) = f \quad \textit{uniformly on} \quad [0,+\infty[.$$

Proof. Since $(M_n)_{n\geq 1}$ is an equibounded sequence of positive linear operators from $C_0([0,+\infty[)$ into $C_0([0,+\infty[)$, by Proposition 6.8, (3), it is sufficient to show that, for every $\lambda \geq 0$,

$$\lim_{n\to\infty} M_n(f_\lambda) = f_\lambda \quad \text{uniformly on} \quad [0,+\infty[$$

where f_λ is defined by (6.5).

A simple calculation indeed shows that

$$M_n(f_\lambda)(x) = \exp\Big[-\lambda x\Big(\frac{1-\exp(-\lambda/n)}{\lambda/n}\Big)\Big]$$

for every $x \geq 0$. Since $\lim\limits_{n\to\infty}\frac{1-\exp(-\lambda/n)}{\lambda/n} = 1$, the sequence $(M_n(f_\lambda))_{n\geq 1}$ converges pointwise to f_λ and is decreasing. Moreover, each $M_n(f_\lambda)$ and f_λ vanishes at $+\infty$ and so, by Dini's theorem applied in the framework of the compactification $[0,+\infty]$, we obtain the uniform convergence as well. □

Our next aim is to discuss the behaviour of the operators M_n on continuous function spaces with polynomial weights. This naturally leads to investigate some Korovkin-type results in weighted continuous function spaces. For further results in this respect, we also refer to the next Section 8 and to [10-11], [14], [60-62], [104].

Consider again an arbitrary locally compact Hausdorff space X having a countable base. Let w be a continuous weight on X, i.e., $w \in C(X)$ and $w(x) > 0$ for every $x \in X$, and set

$$C_0^w(X) := \{f \in C(X) \mid wf \in C_0(X)\}. \tag{6.8}$$

The space $C_0^w(X)$, endowed with the natural (pointwise) order and the weighted norm

$$\|f\|_w := \|wf\|_\infty \quad (f \in C_0^w(X)) \tag{6.9}$$

is a Banach lattice

If $w \in C_b(X)$, then $C_0(X) \subset C_0^w(X)$ and, if $w \in C_0(X)$, then $C_b(X) \subset C_0^w(X)$. The spaces $C_0^w(X)$ and $C_0(X)$ are lattice isomorphic, a lattice isomorphism between $C_0(X)$ and $C_0^w(X)$ being the linear operator $S : C_0(X) \to C_0^w(X)$ defined by

$$S(f) := w^{-1}f \qquad (f \in C_0(X)). \tag{6.10}$$

Therefore, *if M is a subset of $C_0^w(X)$, then M is a Korovkin subset of $C_0^w(X)$ if and only if*

$$wM := \{wf \mid f \in M\} \tag{6.11}$$

is a Korovkin subset of $C_0(X)$.

Accordingly, from Proposition 6.4 and Theorem 6.5, we immediately obtain the following result.

Corollary 6.12. *Given a subset M of $C_0^w(X)$, the following statements hold:*

(1) If for every $x, y \in X$, $x \neq y$, there exists $h \in \mathcal{L}(M), h \geq 0$, such that $h(x) = 0 < h(y)$, then M is a Korovkin subset of $C_0^w(X)$.

(2) Assume that $w \in C_0(X)$. If $M^2 \subset C_0^w(X)$ and M separates the points of X, then $\{\mathbf{1}\} \cup M \cup M^2$ is a Korovkin subset of $C_0^w(X)$. Moreover, if $M = \{f_1, \dots, f_n\}, n \geq 1$, is finite, then $\left\{\mathbf{1}, f_1, \dots, f_n, \sum_{i=1}^{n} f_i^2\right\}$ is a Korovkin subset of $C_0^w(X)$.

Applying the above part (2) to a subset X of $\mathbb{R}^d$, $d \geq 1$, and $M = \{pr_1, \dots, pr_d\}$, we obtain:

Corollary 6.13. *Consider a locally compact subset X of $\mathbb{R}^d$, $d \geq 1$, and $w \in C_0(X)$ such that $\|\cdot\|^2 w \in C_0(X)$. Then $\{\mathbf{1}, pr_1, \dots, pr_d, \|\cdot\|^2\}$ is a Korovkin subset of $C_0^w(X)$.*

From the previous Corollary 6.13, it follows in particular that, if X is a noncompact real interval, then $\{\mathbf{1}, e_1, e_2\}$ is a Korovkin subset of $C_0^w(X)$ for any $w \in C_0(X)$ such that $e_2 w \in C_0(X)$. When $X \subset [0, +\infty[$, this result can be considerably generalized.

Corollary 6.14. *Let X be a noncompact subinterval of $[0, +\infty[$ and let $w \in C_0(X)$ be a weight on X. Consider $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $0 < \lambda_1 < \lambda_2$ and $e_{\lambda_2} \in C_0^w(X)$. Then $\{\mathbf{1}, e_{\lambda_1}, e_{\lambda_2}\}$ is a Korovkin subset of $C_0^w(X)$.*

Proof. First note that $e_{\lambda_1} \in C_0^w(X)$ because $x^{\lambda_1} \leq 1 + x^{\lambda_2}$ for every $x \geq 0$. In order to get the result, we shall apply part (1) of Corollary 6.12 by showing that for a given $x_0 \in X$ there exists $h \in \mathcal{L}(\{\mathbf{1}, e_{\lambda_1}, e_{\lambda_2})$ such that $h(x_0) = 0 < h(y)$ for any $y \in X, y \neq x_0$.

Set $a := \inf X \geq 0$. If $x_0 = a$, then it is sufficient to consider $h(x) = x^{\lambda_1} - a^{\lambda_1}$ $(x \in X)$. If X is upper bounded and $x_0 = \sup X$, then we may consider $h(x) = x_0^{\lambda_1} - x^{\lambda_1}$ $(x \in X)$. Finally if x_0 belongs to the interior of X, then by means of differential calculus it is not difficult to show that the function

$$h(x) := (\lambda_2 - \lambda_1)x_0^{\lambda_2} - \lambda_2 x_0^{\lambda_2 - \lambda_1} x^{\lambda_1} + \lambda_1 x^{\lambda_2} \quad (x \in X)$$

satisfies the required properties. □

From (6.11) and Proposition (6.8), (3), we also get:

Corollary 6.15. *Consider the functions $g_\beta(x) := \exp(\beta x)$ and $g_\gamma(x) := \exp(\gamma x)$ $(x \in [0, +\infty[)$, where $0 < \beta < \gamma$. Then for every $\alpha > \gamma$, the subset $\{\mathbf{1}, g_\beta, g_\gamma\}$ is a Korovkin subset of the space*

$$E_0^\alpha := \{f \in C([0, +\infty[) \mid \lim_{x\to+\infty} \exp(-\alpha x) f(x) = 0\}$$

endowed with the weighted norm $\|f\|_\alpha := \sup\limits_{x\geq 0} \exp(-\alpha x)|f(x)|$ $(f \in E_0^\alpha)$).

In the case when $w \in C_b(X)$, another easy method of finding Korovkin subsets of $C_0^w(X)$ is indicated below.

Proposition 6.16. *Consider a continuous bounded weight $w \in C_b(X)$. Then every Korovkin subset of $C_0(X)$ is a Korovkin subset of $C_0^w(X)$ as well.*

Proof. Let M be a Korovkin subset of $C_0(X)$. In order to show that M is a Korovkin subset of $C_0^w(X)$, we shall show that wM is a Korovkin subset of $C_0(X)$. To this end, fix $\mu \in \mathcal{M}^+(X)$ and $x \in X$ such that $\mu(wg) = w(x)g(x)$ for every $g \in M$, and consider the positive linear form $\nu : C_0(X) \longrightarrow \mathbb{R}$ defined by $\nu(f) := \mu(wf)/w$ $(f \in C_0(X))$. Then $\nu \in \mathcal{M}_b^+(X)$ and $\nu(g) = g(x)$ for every $g \in M$. From Theorem 6.1, we then conclude that $\nu(f) = f(x)$ for every $f \in C_0(X)$, i.e., $\mu(wf) = w(x)f(x)$ for every $f \in C_0(X)$.

Note that if $\varphi \in K(X)$, after setting $f := \varphi/w \in K(X)$, we get $\varphi = wf$ and hence $\mu(\varphi) = \varphi(x)$. By density, we conclude that $\mu(f) = f(x)$ for every $f \in C_0(X)$ and hence the desired result follows from Theorem 6.1. $\square$

Two simple applications of Corollary 6.13 and Proposition 6.16 are shown below. The first one is concerned again with the Szász-Mirakjan operators (6.7) on the weighted continuous function spaces

$$C_0^{w_m}([0, +\infty[) := \left\{f \in C([0, +\infty[) \mid \lim_{x\to\infty} \frac{f(x)}{1 + x^m} = 0\right\}, \tag{6.12}$$

$m \geq 1$, where $w_m(x) := \frac{1}{1+x^m}$ $(x \geq 0)$.

Theorem 6.17. *For every $m \geq 1$ and $n \geq 1$ and for every $f \in C_0^{w_m}([0, +\infty[)$, $M_n(f) \in C_0^{w_m}([0, +\infty[)$ and $\lim\limits_{n\to\infty} M_n(f) = f$ on $[0, +\infty[$ with respect to the weighted norm $\|\cdot\|_{w_m}$ and, hence, uniformly on compact subsets of $[0, +\infty[$.*

Proof. In [32, Lemma 5], it was shown that every M_n is a bounded linear operator from $C_0^{w_m}$ into $C_0^{w_m}$ and that $\sup\limits_{x\geq 0}\|M_n\| < +\infty$. For every $\lambda > 0$ consider again the function $f_\lambda(x) := \exp(-\lambda x)$ $(x \geq 0)$. By Propositions 6.8 and 6.16, the subset $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ is a Korovkin subset of $\mathcal{C}_0^{w_m}([0, +\infty[)$ provided that $0 < \lambda_1 < \lambda_2 < \lambda_3$. On the other hand, for each function f_λ, we have already shown that $\lim\limits_{n\to\infty} M_n(f_\lambda) = f_\lambda$ uniformly on $[0, +\infty[$ (see the proof of Theorem 6.11) and hence the same limit relationship holds with respect to $\|\cdot\|_{w_m}$ because $\|\cdot\|_{w_m} \leq \|\cdot\|_\infty$ on $C_0([0, +\infty[)$ and hence the result follows. $\square$

Remarks 6.18.

1. The behaviour of Szász-Mirakjan operators on locally convex weighted function spaces has been further investigated in [11].

2. A result similar to Theorem 6.17 can be also proved for the Baskakov operators [27]

$$B_n(f)(x) := \frac{1}{(1+x)^n} \sum_{h=0}^{n} \binom{n+h-1}{h} f\Big(\frac{h}{n}\Big)\Big(\frac{x}{1+x}\Big)^h \tag{6.13}$$

($f \in C_0^{w_m}([0,+\infty[)$, $x \geq 0$, $m \geq 1$, $n \geq 1$).

For the details, we refer to [20, Theorem 2.3] (see also [8, pp. 341-344] and [32]).

As another application consider $X = \mathbb{R}$ and the weight $w_m(x) := \frac{1}{1+x^m}$ $(x \in \mathbb{R})$ with $m \geq 4$, m even. Consider the sequence of the Gauss-Weierstrass operators defined by (4.44) for $p = 1$. Hence

$$G_n(f)(x) := \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} f(t) \exp\Big(-\frac{n}{4}(t-x)^2\Big)\, \mathrm{d}t \tag{6.14}$$

for $n \geq 1$ and $x \in \mathbb{R}$ where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is any Borel measurable function for which the integral (6.14) is absolutely convergent. In particular, the operators G_n are well-defined on the function space

$$C_0^{w_m}(\mathbb{R}) := \Big\{ f \in C(\mathbb{R}) \mid \lim_{|x|\to\infty} \frac{f(x)}{1+x^m} = 0 \Big\}. \tag{6.15}$$

Theorem 6.19. *Under the above hypotheses, $G_n(C_0^{w_n}(\mathbb{R})) \subset C_0^{w_m}(\mathbb{R})$ and*

$$\lim_{n\to\infty} G_n(f) = f \quad \textit{with respect to } \|\cdot\|_{w_m}$$

(and hence uniformly on compact subsets of $\mathbb{R}$) for any $f \in C_0^{w_m}(\mathbb{R})$.

Proof. Consider the function $e_m(x) = x^m$ $(x \in \mathbb{R})$. Then, for any $n \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned} G_n(e_m)(x) &= \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{\mathbb{R}} t^m \exp\Big(-\frac{n}{4}(t-x)^2\Big)\, \mathrm{d}t \\ &= \sum_{k=0}^{m} \binom{m}{k} M_k \Big(\frac{2}{n}\Big)^{k/2} x^{m-k} \end{aligned}$$

where $M_0 := 1$ and

$$M_k := \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1\cdot 3 \cdots (k-1) & \text{if } k \text{ is even,} \end{cases}$$

(see [29, formulae (4.20) and (4.21)]). Therefore,

$$G_n(e_m)(x) \;\leq\; \sum_{k=0}^{m} \binom{m}{k} M_k \Big(\frac{2}{n}\Big)^{k/2} |x|^{m-k} \leq x^m + \sum_{k=1}^{m} \binom{m}{k} M_k 2^{k/2} |x|^{m-k}$$

and

$$w_m(x) G_n(1+e_m)(x) \leq \frac{1}{1+x^m}\Big(x^m + \sum_{k=1}^{m} \binom{m}{k} M_k 2^{k/2}|x|^{m-k}\Big) \leq M,$$

M being a suitable positive constant independent of m.

Consider now $f \in C_0^{w_m}(\mathbb{R})$. Thus to each $\varepsilon > 0$ there corresponds $\delta_1 > 0$ such that for any $x \in \mathbb{R}, |x| \geq \delta_1$,

$$w_m(x)|f(x)| \leq \frac{\varepsilon}{2M}.$$

Furthermore, choose $\delta \geq \delta_1$ such that

$$w_m(x) \leq \frac{\varepsilon}{2M_\delta} \quad \text{for} \quad x \in \mathbb{R}, |x| \geq \delta,$$

where $M_\delta := \sup\limits_{|x|\leq\delta} |f(t)|$. Then, for any $x \in \mathbb{R}$, $|x| \geq \delta$,

$$\begin{aligned}
|G_n(f)(x)| &\leq \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{|t|\leq\delta} |f(t)| \exp\Big(-\frac{n}{4}(t-x)^2\Big)\,\mathrm{d}t \\
&+ \Big(\frac{n}{4\pi}\Big)^{1/2} \int_{|t|\geq\delta} |f(t)| \exp\Big(-\frac{n}{4}(t-x)^2\Big)\,\mathrm{d}t \\
&\leq M_\delta + \frac{\varepsilon}{2M}\Big(\frac{n}{4\pi}\Big)^{1/2} \int_{|t|\geq\delta} (1+e_m(t)) \exp\Big(-\frac{n}{4}(t-x)^2\Big)\,\mathrm{d}t \\
&\leq M_\delta + \frac{\varepsilon}{2M} G_n(1+e_m)(x)
\end{aligned}$$

so that

$$w_m(x)|G_n(f)(x)| \leq \varepsilon$$

because of the previous estimates.

This proves that $G_n(f) \in C_0^{w_m}(\mathbb{R})$. Moreover, note that, since $|f| \leq \|f\|_{w_m}(1+e_m)$, then

$$w_m|G_n(f)| \leq \|f\|_{w_m} w_m G_n(1+e_m)$$

and hence $\|G_n(f)\|_{w_m} \leq M\|f\|_{w_m}$, that is, $\|G_n\| \leq M$ for any $n \geq 1$.

On the other hand, from Corollary 6.13, we infer that $\{\mathbf{1}, e_1, e_2\}$ is a Korovkin subset of $C_0^{w_m}(\mathbb{R})$, where $e_k(x) := x^k$ ($k = 1, 2$, $x \in \mathbb{R}$), and for every $h \in \{\mathbf{1}, e_1, e_2\}$ clearly $G_n(h) \to h$ as $n \to \infty$ with respect to $\|\cdot\|_{w_n}$ because of (4.48)-(4.50). Therefore, the result follows. □

Remark 6.20. A generalization of Theorem 6.19 can be found in [21, Theorem 3.1] and [23].

7 Korovkin-type theorems for the identity operator on $C(X)$, X compact

The results of the previous section also apply when X is a compact metric space (and hence $C_0(X) = C(X)$). However, in this particular case, some of them have a particular relevance and hence are worth an explicit description.

From Theorem 6.5, by replacing f_0 with the constant function $\mathbf{1}$, the following result immediately follows.

Theorem 7.1. *If M is a subset of $C(X)$ that separates the points of X, then $\{\mathbf{1}\} \cup M \cup M^2$ is a Korovkin subset of $C(X)$.*

Moreover, if $M = \{f_1, \ldots, f_n\}$, $n \geq 1$, is finite, then $\left\{\mathbf{1}, f_1, \ldots, f_n, \sum\limits_{i=1}^{n} f_i^2\right\}$ is a Korovkin subset of $C(X)$.

In particular, if $f \in C(X)$ is injective, then $\{\mathbf{1}, f, f^2\}$ is a Korovkin subset of $C(X)$.

Theorem 7.1 extends both the Korovkin Theorems 3.1 and 4.3 as well as Volkov's Theorem 4.2. It was obtained by Freud ([59]) for finite subsets M and by Schempp ([106]) in the general case. By adapting the proof of Proposition 6.8 and by using Proposition 6.4, it is not difficult to show that:

Proposition 7.2. *(1) If $0 < \lambda_1 < \lambda_2$, then $\{\mathbf{1}, e_{\lambda_1}, e_{\lambda_2}\}$ is a Korovkin subset of $C([0,1])$, where $e_{\lambda_k}(x) := x^{\lambda_k}$ for every $x \in [0,1]$ and $k = 1, 2$.*

(2) If $u \in C([0,1])$ is strictly convex, then $\{\mathbf{1}, e_1, u\}$ is a Korovkin subset of $C([0,1])$.

Next, we shall discuss some applications of Theorem 7.1 by considering a convex compact subset K of a locally convex space E. We denote by

$$A(K)$$

the subspace of all real-valued continuous affine functions on K. We recall that a function $u : K \to \mathbb{R}$ is said to be affine if

$$u(\lambda x + (1-\lambda)y) = \lambda u(x) + (1-\lambda)u(y) \tag{7.1}$$

for every $x, y \in K$ and $\lambda \in [0,1]$, or, equivalently, if

$$u(\sum_{i=1}^{n} \lambda_i x_i) = \sum_{i=1}^{n} \lambda_i u(x_i) \tag{7.2}$$

for every $n \geq 2$, $x_1, \ldots, x_n \in K$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Note that $A(K)$ contains the constant functions as well as the restrictions to K of every continuous linear functional on E so that, by the Hahn-Banach theorem, $A(K)$ separates the points of K. Therefore, from Theorem 7.1, we immediately get:

Corollary 7.3. *$A(K) \cup A(K)^2$ is a Korovkin subset of $C(K)$.*

Consider now a continuous selection $(\widetilde{\mu}_x)_{x \in K}$ of probability Borel measures on K, i.e., for every $f \in C(K)$ the function $x \mapsto \int_K f \, d\widetilde{\mu}_x$ is continuous on K. Such a function will be denoted by $T(f)$, that is

$$T(f)(x) := \int_K f \, d\widetilde{\mu}_x \quad (x \in K). \tag{7.3}$$

Note that T can be viewed as a positive linear operator from $C(K)$ into $C(K)$ and $T(\mathbf{1}) = \mathbf{1}$.

Conversely, by the Riesz representation theorem (see Theorem 11.3 of the Appendix), each positive linear operator $T : C(K) \longrightarrow C(K)$ generates a continuous selection of probability Borel measures on K satisfying (7.3). Such a selection will be referred to as the **canonical continuous selection associated with** T.

From now on, we fix a continuous selection $(\widetilde{\mu}_x)_{x \in K}$ of probability Borel measures on K satisfying

$$\int_K u \, d\widetilde{\mu}_x = u(x) \quad \text{for every} \quad u \in A(K), \tag{7.4}$$

namely,

$$T(u) = u \quad \text{for every} \quad u \in A(K). \tag{7.5}$$

Property (7.4) means that each $x \in K$ is the barycenter or the resultant of $\widetilde{\mu}_x$ ([39, Vol. II, Def. 26.2]).

For every $n \geq 1$ and $x \in K$, denote by

$$\widetilde{\mu}_x^{(n)}$$

the product measure on K^n of $\widetilde{\mu}_x$ with itself n times (see, e.g., [30, §22]) and set, for every $f \in C(K)$,

$$B_n(f)(x) := \int_{K^n} f\Big(\frac{x_1 + \cdots + x_n}{n}\Big)\, \mathrm{d}\widetilde{\mu}_x^{(n)}(x_1, \dots, x_n). \tag{7.6}$$

By Fubini's theorem, we can also write

$$B_n(f)(x) = \int_K \cdots \int_K f\Big(\frac{x_1 + \cdots + x_n}{n}\Big)\, \mathrm{d}\widetilde{\mu}_x(x_1) \cdots d\widetilde{\mu}_x(x_n). \tag{7.7}$$

By the continuity property of the product measure (see [39, Vol. I, Proposition 13.12] and [30, Theorem 30.8]) and the continuity of the selection, it follows that $B_n(f) \in C(K)$.

The positive linear operator $B_n : C(K) \longrightarrow C(K)$ is referred to as the **n-th Bernstein-Schnabl operator** associated with the given selection $(\widetilde{\mu}_x)_{x\in K}$ (or the given positive linear operator $T : C(K) \longrightarrow C(K)$).

Bernstein-Schnabl operators were introduced for the first time in 1968 by Schnabl ([107-108]) in the context of the set of all probability Radon measures on a compact Hausdorff space and, as we shall see next, they generalize the classical Bernstein operators (3.5). Subsequently, Grossman ([66]) introduced the general definition (7.6) (or (7.7)). In the particular case of Bauer simplices (see, e.g., [8, Section 1.5, p. 59]), these operators have been extensively studied by Nishishiraho ([87-91]). Another particular class of them has been also studied by Altomare ([5]). Their construction essentially involves positive projections and, in this case, they satisfy many additional properties useful in the study of evolution problems (see the end of Section 10). For some shape preserving properties of these operators, we also refer to Raşa ([97-99]).

For a comprehensive survey on these operators, we refer to [8, Chapter 6] and to the references contained in the relevant notes. More recent results can be also found in [7], [18], [19], [24-25], [100-102], [103].

Below, we discuss some examples.

Examples 7.4.

1. Consider $K = [0, 1]$ and set $\widetilde{\mu}_x := x\delta_1 + (1 - x)\delta_0$ for all $x \in [0, 1]$. Then $(\widetilde{\mu}_x)_{0\le x\le 1}$ is a continuous selection and the corresponding Bernstein-Schnabl operators turn into the Bernstein operators (3.5) (use an induction argument).

2. Let $\alpha, \beta, \gamma \in C([0, 1])$ and assume that $0 \le \alpha \le 1,\ 0 \le \beta \le 1,\ 0 \le \gamma \le 1,\ \alpha + \beta + \gamma = \mathbf{1}$ and $\beta(x)/2 + \gamma(x) = x$ for every $x \in [0, 1]$. Put $\widetilde{\mu}_x := \alpha(x)\delta_0 + \beta(x)\delta_{1/2} + \gamma(x)\delta_1$. Then the Bernstein-Schnabl operators associated with $(\widetilde{\mu}_x)_{0\le x\le 1}$ on $[0, 1]$ are given by

$$B_n(f)(x) = \sum_{h=0}^{n}\sum_{k=0}^{n} \binom{n}{h}\binom{n-h}{k} \alpha(x)^{n-h-k}\beta(x)^h\gamma(x)^k f\Big(\frac{h + 2k}{2n}\Big) \tag{7.8}$$

   $(f \in C([0, 1]),\ 0 \le x \le 1)$.

3. Let $a = a_0 < a_1 < \cdots < a_p = b,\ p \ge 1$, be a subdivision of the compact real interval $[a, b]$. For every $x \in [a, b]$, set

$$\widetilde{\mu}_x := \frac{x - a_k}{a_{k+1} - a_k}\delta_{a_{k+1}} + \frac{a_{k+1} - x}{a_{k+1} - a_k}\delta_{a_k}, \tag{7.9}$$

whenever $x \in [a_k, a_{k+1}]$, $0 \le k \le p-1$. Then the Bernstein-Schnabl operators are given by

$$B_n(f)(x) = \frac{1}{(a_{k+1}-a_k)^n} \sum_{r=0}^{n} \binom{n}{r} (x-a_k)^r (a_{k+1}-x)^{n-r} f\Big(\frac{r}{n} a_{k+1} + \frac{n-r}{n} a_k\Big) \tag{7.10}$$

whenever $x \in [a_k, a_{k+1}],\ 0 \le k \le p-1$ $(f \in C([a,b]),\ n \ge 1)$.

4. Consider the d-dimensional simplex K_d of $\mathbb{R}^d$ defined by (4.38) and for every $x = (x_1, \dots, x_d) \in K_d$, set
$$\widetilde{\mu}_x := \Big(1 - \sum_{i=1}^{d} x_i\Big)\delta_0 + \sum_{i=1}^{d} x_i \delta_{a_i}$$
where $a_i = (\delta_{ij})_{1 \le j \le d}$ for every $i = 1, \dots, d$, δ_{ij} being the Kronecker symbol. Then the Bernstein-Schnabl operators associated with $(\widetilde{\mu}_x)_{x \in K_d}$ turn into the Bernstein operators on the simplex K_d defined by (4.38).

5. Consider the hypercube $Q_d := [0,1]^d$ of $\mathbb{R}^d$ and for every $x = (x_1, \dots, x_d) \in Q_d$, set
$$\widetilde{\mu}_x := \sum_{h_1, \dots, h_d = 0}^{1} x_1^{h_1}(1-x_1)^{1-h_1} \cdots x_d^{h_d}(1-x_d)^{1-h_d} \delta_{b_{h_1,\dots,h_d}},$$
where $b_{h_1,\dots,h_d} = (\delta_{h_1 1}, \dots, \delta_{h_d 1})$ $\quad (h_1, \dots, h_d \in \{0,1\})$. Then the Bernstein-Schnabl operators are the Bernstein operators on Q_d defined by (4.41).

Many other significant examples can be described in the framework of other finite dimensional subsets such as balls, or more generally, ellipsoids of $\mathbb{R}^d$, or in the setting of infinite-dimensional Bauer simplices (see [8, Chapter 6], [24], [103]).

In order to discuss the approximation properties of Bernstein-Schnabl operators, note that for every $n \ge 1$
$$B_n(\mathbf{1}) = \mathbf{1}. \tag{7.11}$$
Moreover, if $u \in A(K)$, then for every $x_1, \dots, x_n \in K$
$$u\Big(\frac{x_1 + \cdots + x_n}{n}\Big) = \frac{u(x_1) + \cdots + u(x_n)}{n}$$
and
$$u^2\Big(\frac{x_1 + \cdots + x_n}{n}\Big) = \frac{1}{n^2}\Big[\sum_{i=1}^{n} u^2(x_i) + 2 \sum_{1 \le i < j \le n} u(x_i)u(x_j)\Big].$$
Therefore, because of (7.4), for every $x \in K$, we get
$$B_n(u)(x) = u(x) \tag{7.12}$$
and
$$B_n(u^2)(x) = \frac{1}{n} T(u^2)(x) + \frac{n-1}{n} u^2(x). \tag{7.13}$$
In other words, for every $h \in A(K) \cup A(K)^2$, $B_n(h) \to h$ uniformly on K and hence by virtue of Corollary 7.3, we obtain that:

Theorem 7.5. *For every $f \in C(K)$,*

$$\lim_{n\to\infty} B_n(f) = f \quad \textit{uniformly on} \quad K.$$

For additional properties of Bernstein-Schnabl operators, including estimates of the rate of convergence, asymptotic formulae, shape-preserving properties and, especially, their connections with the approximation of positive semigroups as well as of the solutions of evolution equations, we refer to the references we cited before Examples 7.4.

8 Korovkin-type theorems in weighted continuous function spaces and in $L^p(X, \widetilde{\mu})$ spaces

By using Korovkin-type theorems in spaces of continuous functions it is also possible to get some results for $L^p(X, \widetilde{\mu})$-spaces, $1 \le p < +\infty$. In this regard, other than the space $C_0(X)$, also weighted continuous function spaces can be efficiently used. In this section, we shall develop this method by also showing some additional Korovkin-type results in weighted continuous function spaces that complement the ones already discussed at the end of Section 6, and then by presenting the corresponding ones in $L^p(X, \widetilde{\mu})$-spaces.

The method relies on a fundamental characterization of Korovkin subspaces which is due to Bauer and Donner ([31]) (see also [8, Theorem 4.1.2]). Its counterpart for Korovkin subspaces for positive linear operators was obtained in [8, Theorem 3.1.4]. An extension to locally convex function spaces was obtained in [10]. For the sake of brevity, we omit its proof.

As in the previous section, X denotes a fixed locally compact Hausdorff space with a countable base.

Theorem 8.1. *([31]) Given a linear subspace H of $C_0(X)$, the following statements are equivalent:*

(i) H is a Korovkin subspace of $C_0(X)$;

(ii) For every $f \in C_0(X)$ and for every $\varepsilon > 0$, there exist finitely many functions $h_0, \dots, h_n \in H$, $k_0, \dots, k_n \in H$ and $u, v \in C_0(X), u, v \ge 0$ such that $\|u\| \le \varepsilon, \|v\| \le \varepsilon$ and

$$\left\| \inf_{0\le j\le n} k_j - \sup_{0\le i\le n} h_i \right\| \le \varepsilon \quad \textit{and} \quad \sup_{0\le i\le n} h_i - u \le f \le \inf_{0\le j\le n} k_j + v.$$

Remark 8.2. If X is compact and $\mathbf{1} \in H$, then Theorem 8.1 can be stated in a simpler form (see [8, Theorem 4.1.4]).

Let us mention two important consequences of Theorem 8.1.

Theorem 8.3. *([31]). Let H be a Korovkin subspace of $C_0(X)$. If E is a Banach lattice and if $S : C_0(X) \to E$ is a lattice homomorphism, then H is a Korovkin subspace of $C_0(X)$ for S.*

Proof. Consider a sequence $(L_n)_{n\ge1}$ of positive linear operators from $C_0(X)$ into E such that $M := \sup_{n\ge1}\|L_n\| < +\infty$ and $\lim_{n\to\infty} L_n(h) = S(h)$ for every $h \in H$. Given $f \in C_0(X)$ and $\varepsilon > 0$, choose $\delta \in]0, \varepsilon[$ such that $\|S(g)\| \le \varepsilon$ for every $g \in C_0(X)$, $\|g\| \le \delta$. By Theorem 8.1, there exist $h_0, \dots, h_p, k_0, \dots, k_p \in H$ and $u, v \in C_0(X)$, $u, v \ge 0$, such that $\|u\| \le \delta$, $\|v\| \le \delta$ and

$$\|v\| \le \varepsilon$$

as well as

$$\left\| \inf_{0\le j\le n} S(k_j) - \sup_{0\le i\le n} s(h_i) \right\| \le \delta \text{ and } \sup_{0\le i\le p} h_i - u \le f \le \inf_{0\le j\le p} k_j + v.$$

Therefore,

$$\left\| \inf_{0\le j\le p} S(k_j) - \sup_{0\le i\le p} S(h_i) \right\| \le \varepsilon.$$

Moreover, for every $n \ge 1$,

$$\sup_{0\le i\le p} L_n(h_i) - L_n(u) \le L_n(f) \le \inf_{0\le j\le p} L_n(k_j) + L_n(v)$$

and

$$\sup_{0\le i\le p} S(h_i) - S(u) \le S(f) \le \inf_{0\le j\le p} S(k_j) + S(v)\,.$$

Accordingly,

$$L_n(f) - S(f) \le \sum_{j=0}^{p} |L_n(k_j) - S(k_j)| + |\inf_{0\le j\le p} S(k_j) - \sup_{0\le i\le p} S(h_i)| + L_n(v) + S(u)$$

and

$$S(f) - L_n(f) \le \sum_{i=0}^{p} |L_n(h_i) - S(h_i)| + |\inf_{0\le j\le p} S(k_j) - \sup_{0\le i\le p} S(h_i)| + +L_n(u) + S(v),$$

so that

$$\begin{aligned} |S(f) - L_n(f)| &\le \sum_{i=0}^{p} |L_n(h_i) - S(h_i)| + \sum_{j=0}^{p} |L_n(k_j) - S(k_j)| \\ &+ |\inf_{0\le j\le p} S(k_j) - \sup_{0\le i\le p} S(h_i)| + L_n(u) + L_n(v) + S(u) + S(v) \\ &\le \sum_{i=0}^{p} |L_n(h_i) - S(h_i)| + \sum_{j=0}^{p} |L_n(k_j) - S(k_j)| + 2M\varepsilon + 2\varepsilon. \end{aligned}$$

It is now easy to conclude that $L_n(f) \to S(f)$ as $n \to \infty$ because $L_n(h_i) \to S(h_i)$ and $L_n(k_j) \to S(k_j)$ as $n \to \infty$ for every $i, j = 0, \dots, p$. □

Remark 8.4. From Theorem 8.3, it follows that, if M is a Korovkin subset of $C_0(X)$, then M is a Korovkin subset for the natural embedding from $C_0(X)$ into an arbitrary closed lattice subspace E of $B(X)$ containing $C_0(X)$ (for instance, $E = C_b(X)$ or $E = B(X)$).

Another consequence of Theorem 8.3 is indicated below.

Proposition 8.5. *([120]). Let E be a Banach lattice and consider a lattice homomorphism $S : C_0(X) \to E$ such that $S(C_0(X))$ is dense in E. If M is a Korovkin subspace of $C_0(X)$, then $S(M)$ is a Korovkin subspace of E.*

Proof. Consider an equibounded sequence $(L_n)_{n\geq 1}$ of positive linear operators on E and assume that $L_n(k) \to k$ as $n \to \infty$ for every $k \in S(M)$. This means that $L_n(S(h)) \to S(h)$ as $n \to \infty$ for each $h \in M$ and hence for each $h \in H := \mathcal{L}(M)$. So, by Theorem 8.3, $L_n(S(f)) \longrightarrow S(f)$ for every $f \in C_0(X)$. The result now follows from the assumption that $S(C_0(X))$ is dense in E and $(L_n)_{n\geq 1}$ is equibounded. □

Remark 8.6. A typical situation where Proposition 8.5 can be applied concerns the case when E is a Banach lattice containing $C_0(X)$ as a dense sublattice and S is the natural embedding from $C_0(X)$ into E. Thus, in this case,

every Korovkin subspace of $C_0(X)$ is a Korovkin subspace of E as well.

After these preliminaries, we can now proceed to discuss some Korovkin-type results for $L^p(X, \widetilde{\mu})$ spaces.

Consider a Borel measure $\widetilde{\mu}$ on X and, given $p \in [1, +\infty[$, consider the space

$$L^p(X, \widetilde{\mu})$$

endowed with its natural norm $\|\cdot\|_p$ (for more details, we refer to the Appendix (formulae (11.11) and (11.12))).

Since $\widetilde{\mu}$ is regular, $K(X)$ is dense in $L^p(X, \widetilde{\mu})$ with respect to the convergence in p-th mean (see Theorem 11.4 of Appendix). Therefore, we get the following useful results.

Corollary 8.7. *Let M be a Korovkin subset of $C_0(X)$. Furthermore, let Y be a locally compact Hausdorff space with a countable base and consider a Borel measure $\widetilde{\mu}$ on Y and $p \in [1, +\infty[$. Let $S : C_0(X) \longrightarrow L^p(Y, \widetilde{\mu})$ be a lattice homomorphism such that $K(Y) \subset S(C_0(X))$. Then $S(M)$ is a Korovkin subset of $L^p(Y, \widetilde{\mu})$.*

In particular, if $\widetilde{\mu}$ is a finite Borel measure on X, then M is also a Korovkin subset of $L^p(X, \widetilde{\mu})$.

Corollary 8.8. *Let X be a compact subset of $\mathbb{R}^d$, $d \geq 1$, and consider a finite Borel measure $\widetilde{\mu}$ on X. Then $\{\mathbf{1}, pr_1, \ldots, pr_d, \sum\limits_{i=1}^{d} pr_i^2\}$ is a Korovkin subset of $L^p(X, \widetilde{\mu})$ for every $p \in [1, +\infty[$.*

If, in addition, X is contained in some sphere of $\mathbb{R}^d$, then $\{\mathbf{1}, pr_1, \ldots, pr_d\}$ is a Korovkin subset of $L^p(X, \widetilde{\mu})$.

Corollary 8.9. *Consider $0 < \lambda_1 < \lambda_2 < \lambda_3$ and $p \in [1, +\infty[$. Then*

(1) $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ is a Korovkin subset of $L^p([0, +\infty[)$, where $f_{\lambda_k}(x) := \exp(-\lambda_k x)$ for every $x \in [0, +\infty[$ and $k = 1, 2, 3$.

(2) If $u : \mathbb{R} \longrightarrow]0, 1[$ is a strictly increasing continuous function satisfying $\lim\limits_{x\to+\infty} u(x) = 0$ and $\lim\limits_{x\to-\infty} u(x) = 1$, then

$$\{\Phi, \Phi u^{\lambda_1}, \Phi u^{\lambda_2}\}$$

is a Korovkin subset of $L^p(\mathbb{R})$, where $\Phi(x) := \exp(-x^2)$ $(x \in \mathbb{R})$.

Proof. (1) Consider the lattice homomorphism $S : C([0, 1]) \to L^p([0, +\infty[)$ defined by

$$S(f)(x) := \exp(-\lambda_1 x) f(\exp(-x)) \quad (f \in C([0, 1]),\ x \geq 0).$$

Then S maps the subset $M := \{\mathbf{1}, e_{\lambda_2-\lambda_1}, e_{\lambda_3-\lambda_1}\}$ into

$$\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}.$$

Hence the result follows from Corollary 8.7 and Proposition 7.2, (1).

(2) A similar reasoning can be used by considering now the lattice homomorphism $S : C([0,1]) \to L^p(\mathbb{R})$ defined by

$$S(f)(x) := \exp(-x^2) f(u(x)) \quad (f \in C([0,1]),\ x \in \mathbb{R}).$$

□

We now proceed to state a result analogous to Corollary 8.7 for more general weighted function spaces (see Section 6). Consider a continuous weight w on X and the relevant weighted space $C_0^w(X)$ defined by (6.8). Note that, if $\widetilde{\mu}$ is a Borel measure on X and if

$$w^{-1} \in L^p(X, \widetilde{\mu}) \qquad \text{for some } p \in [1, +\infty[. \tag{8.1}$$

then $C_0^w(X) \subset L^p(X, \widetilde{\mu})$ and it is dense with respect to the convergence in the p-th mean.

Corollary 8.10. *Consider a Korovkin subset M of $C_0^w(X)$, i.e., wM is a Korovkin subset of $C_0(X)$ (see (6.11)), and a Borel measure $\widetilde{\mu}$ on X. If (8.1) holds, then M is a Korovkin subset of $L^p(X, \widetilde{\mu})$.*

Proof. Consider the lattice homomorphism $S : C_0(X) \longrightarrow L^p(X, \widetilde{\mu})$ defined by $S(f) := w^{-1} f$ ($f \in C_0(X)$). Then $M = S(wM)$ and $S(C_0(X)) = C_0^w(X)$. Therefore, the result follows from Corollary 8.7. □

The previous Corollary together with Corollaries 6.12 - 6.15 furnish a simple but useful method to construct Korovkin subsets in $L^p(X, \widetilde{\mu})$-spaces.

By using similar methods, we can extend some of the previous results to more general weighted function spaces which often occur in the applications. Some of the subsequent results are taken from [22, Section 2].

In what follows, we shall assume that X is noncompact and we shall denote by

$$X_\infty := X \cup \{\infty\}$$

the Alexandrov one-point compactification of X (see, e.g., [30, §27]). A function $f \in C(X)$ is said to be convergent at infinity if there exists a (unique) $l \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a compact subset K of X such that $|f(x) - l| \le \varepsilon$ for each $x \in X \setminus K$. In such a case, we also write $\lim_{x\to\infty} f(x) = l$. Similarly, we shall write $\lim_{x\to\infty} f(x) = \infty$ to mean that for every $M \ge 0$ there exists a compact subset K of X such that $f(x) \ge M$ for every $x \in X \setminus K$.

Given a weight $w \in C(X)$, consider the Banach lattice

$$C_*^w(X) := \{f \in C(X) \mid wf \text{ is convergent at infinity }\} \tag{8.2}$$

endowed with the natural (pointwise) order and the weighted norm

$$\|f\|_w := \|wf\|_\infty \qquad (f \in C_*^w(X)). \tag{8.3}$$

For every $f \in C_*^w(X)$, denote by $T(f)$ the function on X_∞ defined by

$$T(f)(z) := \begin{cases} w(z) f(z) & \text{if } z \in X, \\ \lim\limits_{x\to\infty} w(x) f(x) & \text{if } z = \infty. \end{cases} \tag{8.4}$$

Then $T(f) \in C(X_\infty)$ and $\|T(f)\|_\infty = \|f\|_\infty$. Moreover, the linear operator $T : C_*^w(X) \to C(X_\infty)$ is a lattice isomorphism whose inverse we shall denote by

$$S : C(X_\infty) \to C_*^w(X).$$

Thus a subset M of $C_*^w(X)$ is a Korovkin subset of $C_*^w(X)$ if and only if $T(M)$ is a Korovkin subset of $C(X_\infty)$.

Consider now a Borel measure $\widetilde{\mu}$ on X and assume that (8.1) holds. Then $C_*^w(X) \subset L^p(X, \widetilde{\mu})$ and $C_*^w(X)$ is dense in $L^p(X, \widetilde{\mu})$ because $K(X) \subset C_*^w(X)$.

Proposition 8.11. *Under assumption (8.1), each Korovkin subset of $C_*^w(X)$ is a Korovkin subset of $L^p(X, \widetilde{\mu})$ as well.*

Proof. The above lattice isomorphism S can be considered as a lattice homomorphism from $C(X_\infty)$ into $L^p(X, \widetilde{\mu})$ and its range is $C_*^w(X)$ which is dense in $L^p(X, \widetilde{\mu})$. Since $H := T(M)$ is a Korovkin subset of $C(X_\infty)$, then $M = S(H)$ is a Korovkin subset of $L^p(X, \widetilde{\mu})$ by Corollary 8.7. □

Below, we state some consequences of Proposition 8.11.

Proposition 8.12. *Let M be a subset of $C_*^w(X)$ and assume that*

(i) for every $x_0, y_0 \in X$, $x_0 \neq y_0$, there exists $h \in \mathcal{L}(M)$, $h \geq 0$, such that $h(x_0) = 0$ and $h(y_0) > 0$;

(ii) for every $x_0 \in X$ there exist positive functions $h, k \in \mathcal{L}(M)$ such that

$$h(x_0) = 0, \quad \lim_{x \to \infty} w(x)h(x) > 0,$$

and

$$k(x_0) > 0, \quad \lim_{x \to \infty} w(x)k(x) = 0.$$

Then M is a Korovkin subset of $C_*^w(X)$ and hence of $L^p(X, \widetilde{\mu})$ for every Borel measure $\widetilde{\mu}$ on X and for every $p \in [1, +\infty[$ satisfying (8.1).

Proof. Conditions (i) and (ii) mean that $T(M)$ satisfies the hypotheses of Propositions 6.4 in $C(X_\infty)$. Therefore, the result follows from Propositions 6.4 and 8.11. □

Corollary 8.13. *Consider a subset M of $C(X)$ that separates the points of X and a strictly positive function $f_0 \in C_0^w(X)$. Further, assume that*

(1) $f_0 M \cup f_0 M^2 \subset C_*^w(X)$ (see (6.1) and (6.2)),

(2) there exists $g \in M$ such that

$$\lim_{x \to \infty} w(x) f_0(x) g(x) = 0 \text{ and } \lim_{x \to \infty} w(x) f_0(x) g^2(x) > 0.$$

Then $\{f_0\} \cup f_0 M \cup f_0 M^2$ is a Korovkin subset of $C_*^w(X)$ and hence of $L^p(X, \widetilde{\mu})$ for every Borel measure $\widetilde{\mu}$ on X and for every $p \in [1, +\infty[$ satisfying (8.1).

Proof. We shall verify conditions (i) and (ii) of Proposition 8.12. Given $x_0, y_0 \in X, x_0 \neq y_0$, there exists $f \in M$ such that $f(x_0) \neq f(y_0)$. Therefore, the function $h := f_0(f - f(x_0))^2 \in \mathcal{L}(\{f_0\} \cup f_0M \cup f_0M^2)$ satisfies $h(x_0) = 0$ and $h(y_0) > 0$ and hence property (i) follows.

As regards condition (ii) of Proposition 8.12, considering a function $g \in M$ satisfying assumption (2), then for every $x_0 \in X$ the functions $h := f_0(g - g(x_0))^2$ and $k := f_0$ satisfy condition (ii) of Proposition 8.12. $\square$

Corollary 8.14. *Consider $f_1, \ldots, f_n \in C(X), n \geq 1$, that separate the points of X, and a strictly positive function $f_0 \in C_0^w(X)$. Further, assume that*

(i) $f_0 f_i \in C_0^w(X)$ for every $i = 1, \ldots, n$, as well as $f_0 \sum\limits_{i=1}^{n} f_i^2 \in C_*^w(X)$;

(ii) $\lim\limits_{x\to\infty} w(x)f_0(x) \sum\limits_{i=1}^{n} f_i^2(x) > 0$.

Then $\left\{f_0, f_0f_1, \ldots, f_0f_n, f_0 \sum\limits_{i=1}^{n} f_i^2\right\}$ is a Korovkin subset $C_*^w(X)$ and hence in $L^p(X, \widetilde{\mu})$ for every Borel measure $\widetilde{\mu}$ on X and for every $p \in [1, +\infty[$ satisfying (8.1).

Proof. The proof is similar to that of Corollary 8.13, except that in this case, the function h must be chosen as $h := f_0 \sum\limits_{i=1}^{n} (f_i - f_i(x_0))^2$. $\square$

The particular case of Corollary 8.14 where $f_0 = \mathbf{1}$ is worth stating separately.

Corollary 8.15. *Consider $f_1, \ldots, f_n \in C(X), n \geq 1$, that separate the points of X a weight such that and $w \in C_0(X)$ a weight such that*

(i) $f_i \in C_0^w(X)$ for every $i = 1, \ldots, n$, as well as $\sum\limits_{i=1}^{n} f_i^2 \in C_*^w(X)$;

(ii) $\lim\limits_{x\to\infty} w(x) \sum\limits_{i=1}^{n} f_i^2(x) > 0$.

Then $\left\{\mathbf{1}, f_1, \ldots, f_n \sum\limits_{i=1}^{n} f_i^2\right\}$ is a Korovkin subset in $C_*^w(X)$ and hence in $L^p(X, \widetilde{\mu})$ for every finite Borel measure $\widetilde{\mu}$ on X and for every $p \in [1, +\infty[$ satisfying (8.1).

A simple situation when the assumptions of Corollaries 8.14 and 8.15 are satisfied is indicated below.

Corollary 8.16. *Consider $f_1, \ldots, f_n \in C(X), n \geq 1$, that separate the points of X and assume that $\lim\limits_{x\to\infty} \sum\limits_{i=1}^{n} f_i^2(x) = +\infty$. The following statements hold:*

(1) If $f_0 \in C(X)$ is a strictly positive function, then $\left\{f_0, f_0f_1, \ldots, f_0f_n, f_0 \sum\limits_{i=1}^{n} f_i^2\right\}$ is a Korovkin subset in $C_*^w(X)$, where

$$w := \frac{1}{f_0\left(1 + \sum\limits_{i=1}^{n} f_i^2\right)}.$$

Therefore, if $\widetilde{\mu}$ is a Borel measure on X and $1 \leq p < +\infty$, and if $f_0 \in L^p(X, \widetilde{\mu})$ as well as $f_0 f_i^2 \in L^p(X, \widetilde{\mu})$ for every $i = 1, \dots, n$, then $\left\{ f_0, f_0 f_1, \dots, f_0 f_n, f_0 \sum_{i=1}^{n} f_i^2 \right\}$ is a Korovkin subset in $L^p(X, \widetilde{\mu})$.

(2) *In particular, $\left\{ \mathbf{1}, f_1, \dots, f_n, \sum_{i=1}^{n} f_i^2 \right\}$ is a Korovkin subset in $C_*^w(X)$, where*

$$w := \frac{1}{1 + \sum_{i=1}^{n} f_i^2}.$$

Therefore, if $\widetilde{\mu}$ is a finite Borel measure on X and $1 \leq p < +\infty$, and if $f_i^2 \in L^p(X, \widetilde{\mu})$ for every $i = 1, \dots, n$, then $\left\{ \mathbf{1}, f_1, \dots, f_n, \sum_{i=1}^{n} f_i^2 \right\}$ is a Korovkin subset in $L^p(X, \widetilde{\mu})$.

The following particular case of Corollary 8.16 will be used next.

Corollary 8.17. *Let X be an unbounded closed subset of $\mathbb{R}^d, d \geq 1$. The following statements hold:*

(1) *If $f_0 \in C(X)$ is a strictly positive function, then $\left\{ f_0, f_0 pr_1, \dots, f_0 pr_d, f_0 \| \cdot \|^2 \right\}$ is a Korovkin subset in $C_*^w(X)$, where*

$$w := \frac{1}{f_0 \Big(1 + \| \cdot \|^2 \Big)}.$$

Therefore, if $\widetilde{\mu}$ is a Borel measure on X and $1 \leq p < +\infty$, and if $f_0 \in L^p(X, \widetilde{\mu})$ as well as $\| \cdot \|^2 f_0 \in L^p(X, \widetilde{\mu})$, then $\left\{ f_0, f_0 pr_1, \dots, f_0 pr_d, f_0 \| \cdot \|^2 \right\}$ is a Korovkin subset in $L^p(X, \widetilde{\mu})$.

(2) *In particular, $\left\{ \mathbf{1}, pr_1, \dots, pr_d, \| \cdot \|^2 \right\}$ is a Korovkin subset in $C_*^w(X)$, where*

$$w := \frac{1}{1 + \| \cdot \|^2}.$$

Therefore, if $\widetilde{\mu}$ is a finite Borel measure on X and $1 \leq p < +\infty$, and if $\| \cdot \|^2 \in L^p(X, \widetilde{\mu})$, then $\left\{ \mathbf{1}, pr_1, \dots, pr_d, \| \cdot \|^2 \right\}$ is a Korovkin subset in $L^p(X, \widetilde{\mu})$.

Proof. It is sufficient to apply Corollary 8.16 with $f_i = pr_i$, $1 \leq i \leq d$. □

Below, we list some additional examples.

Examples 8.18.

1) Let I be a noncompact real interval and set $r_1 := \inf I \in \mathbb{R} \cup \{-\infty\}$ and $r_2 := \sup I \in \mathbb{R} \cup \{+\infty\}$. Consider a strictly positive injective function $f_0 \in C(I)$.

   Assume that for some strictly positive function $w \in C(I)$ the following properties hold:

   (i) $\lim\limits_{x \to r_i} w(x) f_0(x) = \lim\limits_{x \to r_i} w(x) f_0^2(x) = 0$ for every $i = 1, 2$ such that $r_i \notin I$;

   (ii) there exists $l > 0$ such that $\lim\limits_{x \to r_i} w(x) f_0^3(x) = l$ for every $i = 1, 2$ such that $r_i \notin I$.

Then $\{f_0, f_0^2, f_0^3\}$ is a Korovkin subset in $C_*^w(I)$ and hence in $L^p(I,\widetilde{\mu})$ for every Borel measure $\widetilde{\mu}$ on I and $p \in [1,+\infty[$ such that $1/w \in L^p(I,\widetilde{\mu})$.

The result is a direct consequence of Corollary 8.14 with $n = 1$ and $f_1 = f_0$.

2) Let I be an unbounded closed real interval. The following statements hold:

(1) If $f_0 \in C(I)$ is a strictly positive function, then $\{f_0, f_0e_1, f_0e_2\}$ is a Korovkin subset in $C_*^w(I)$, where

$$w := \frac{1}{f_0\Big(1+e_2\Big)}.$$

Therefore, if $\widetilde{\mu}$ is a Borel measure on I and $1 \le p < +\infty$, and if $f_0 \in L^p(I,\widetilde{\mu})$ as well as $f_0e_2 \in L^p(X,\widetilde{\mu})$, then $\{f_0, f_0e_1, f_0e_2\}$ is a Korovkin subset in $L^p(X,\widetilde{\mu})$.

(2) In particular, $\{\mathbf{1}, e_1, e_2\}$ is a Korovkin subset in $C_*^w(I)$, where

$$w := \frac{1}{1+e_2}.$$

Therefore, if $\widetilde{\mu}$ is a finite Borel measure on I and $1 \le p < +\infty$, and if $e_2 \in L^p(X,\widetilde{\mu})$, then $\{\mathbf{1}, e_1, e_2\}$ is a Korovkin subset in $L^p(X,\widetilde{\mu})$.

For a rather complete list of references on Korovkin-type theorems in L^p-spaces, we refer to [8, Appendix D.2.3].

Next, we discuss a simple application of the above results (more precisely, of Corollary 8.17) concerning the operators $G_n, n \ge 1$, defined by (4.44) as

$$G_n(f)(x) := \Big(\frac{n}{4}\Big)^{d/2} \int_{\mathbb{R}^d} f(t) \exp\Big(-\frac{n}{4}\|t-x\|^2\Big)\,\mathrm{d}t \quad (x \in \mathbb{R}^d) \tag{8.5}$$

for every Borel measurable function $f : \mathbb{R}^d \to \mathbb{R}$ for which the integral (8.5) is absolutely convergent $(d \ge 1)$.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable strictly positive function which we assume to be integrable with respect to the d-dimensional Lebesgue measure λ_d. We denote by $\widetilde{\mu}_\varphi := \varphi\lambda_d$ the finite Borel measure on $\mathbb{R}^d$ having density φ with respect to λ_d, i.e.,

$$\widetilde{\mu}_\varphi(B) = \int_B \varphi(x)\,\mathrm{d}x \tag{8.6}$$

for every Borel subset B of $\mathbb{R}^d$. If $p \in [1,+\infty[$ and $f \in L^p(\mathbb{R}^d, \widetilde{\mu}_\varphi)$, we shall set

$$\|f\|_{\varphi,p} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \varphi(x)\,\mathrm{d}x\Big)^{1/p}. \tag{8.7}$$

Theorem 8.19. *Assume that*

$$C_\varphi := \sup_{n\ge 1,\, t\in\mathbb{R}^d} \Big\{\frac{1}{\varphi(t)}\Big(\frac{n}{4\pi}\Big)^{d/2} \int_{\mathbb{R}^d} \varphi(x) \exp\Big(-\frac{n}{4}\|t-x\|^2\Big)\,\mathrm{d}x\Big\} < +\infty. \tag{8.8}$$

Then for every $p \in [1,+\infty[$ and $n \ge 1$ and for every $f \in L^p(\mathbb{R}^d, \widetilde{\mu}_\varphi)$, the integral (8.5) converges absolutely for a.e. $x \in \mathbb{R}^d$.

Moreover $G_n(f) \in L^p(\mathbb{R}^d, \widetilde{\mu}_\varphi)$ *and*

$$\|G_n(f)\|_{\varphi,p} \le \max\{1, C_\varphi\}\|f\|_{\varphi,p}. \tag{8.9}$$

Finally, if $pr_i^2 \in L^p(\mathbb{R}^d, \widetilde{\mu}_\varphi)$ *for every* $i = 1, \dots, d$, *then*

$$\lim_{n\to\infty} G_n(f) = f \quad \text{with respect to} \quad \|\cdot\|_{\varphi,p}. \tag{8.10}$$

Proof. Setting, for every $n \ge 1$,

$$K_n(x,t) := \frac{1}{\varphi(t)}\Big(\frac{n}{4\pi}\Big)^{d/2} \exp\Big(-\frac{n}{4}\|t-x\|^2\Big), \quad (t, x \in \mathbb{R}^d),$$

we get

$$G_n(f)(x) = \int_{\mathbb{R}} K_n(t,x) f(t)\, \mathrm{d}\widetilde{\mu}_\varphi(t) \quad (x \in \mathbb{R}^d).$$

Moreover, if $x \in \mathbb{R}^n$ is fixed, then

$$\int_{\mathbb{R}^d} K_n(t,x)\, \mathrm{d}\widetilde{\mu}_\varphi(t) = \Big(\frac{n}{4\pi}\Big)^{d/2} \int_{\mathbb{R}^d} \exp\Big(-\frac{n}{4}\|t-x\|^2\Big)\, \mathrm{d}t = 1$$

and, for $t \in \mathbb{R}^d$ fixed,

$$\int_{\mathbb{R}^d} K_n(t,x)\, \mathrm{d}\widetilde{\mu}_\varphi(x) \le C_\varphi.$$

Therefore, the first part of the statement follows from Fubini's and Tonelli's theorems and from Hölder's inequality (see also [58, Theorem 6.18]). As regards the final part, note that each pr_i belongs to $L^p(\mathbb{R}^d, \widetilde{\mu}_\varphi)$ too because $|pr_i| \le 1 + pr_i^2, \quad (i = 1, \dots, d)$.

From (8.9), it follows that the sequence $(G_n)_{n\ge1}$ is equibounded from $L^p(\mathbb{R}^d, \widetilde{\mu}_\varphi)$ into itself. Moreover, formulae (4.48)–(4.50) imply that

$$G_n(h) \to h \text{ in } L^p(\mathbb{R}^d, \tilde{\mu}_\varphi)$$

for every $h \in \Big\{\mathbf{1}, pr_1, \dots, p_d, \sum_{i=1}^d pr_i^2\Big\}$ and hence the result follows from Corollary 8.17, (2). □

Remark 8.20. It is not difficult to show that condition (8.8) is satisfied, for instance, by the functions $\varphi_m(x) := (1 + \|x\|^m)^{-1}$ or $\varphi_m(x) = \exp(-m\|x\|)$ $(x \in \mathbb{R}^d)$ for every $m \ge 1$. For more details, we refer to the papers [21, Section 3], [22] and [23, Section 4] where an extension of Theorem 8.19 has been established for a sequence of more general integral operators and where the operators G_n, $n \ge 1$, have been studied also in weighted continuous function spaces.

9 Korovkin-type theorems and Stone-Weierstrass theorems

In this section, we shall deepen the connections between Korovkin-type theorems and Stone-Weierstrass theorems, which have already been pointed out with Theorems 3.8 and 4.7.

We start by giving a new proof of a generalization of Weierstrass' theorem due to Stone ([111]) by means of Theorem 6.6. We first need the following result.

Lemma 9.1. *Every closed subalgebra A of $C_b(X)$ is a lattice subspace (i.e., $|f| \in A$ for every $f \in A$).*

Proof. It is well-known that

$$t^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (t-1)^n = \lim_{n\to\infty} p_n(t)$$

uniformly with respect to $t \in [0,2]$, where

$$p_n(t) := \sum_{k=0}^{n} \binom{1/2}{k} (t-1)^k \quad (n \geq 1,\, t \in [0,2]).$$

Since $\lim\limits_{n\to\infty} p_n(0) = 0$, we also get

$$t^{1/2} = \lim_{n\to\infty} q_n(t) \quad \text{uniformly with respect to} \quad t \in [0,2],$$

where $q_n := p_n - p_n(0)$, $n \geq 1$. Therefore, for any $f \in A$, $f \neq 0$,

$$|f| = \|f\| \left(\frac{f^2}{\|f\|^2}\right)^{1/2} = \|f\| \lim_{n\to\infty} q_n \left(\frac{f^2}{\|f\|^2}\right) \in \overline{A} = A.$$

$\square$

Before stating the Stone approximation theorem, we recall that a subset M of $C_0(X)$ is said to **separate strongly the points of** X if it separates the points of X and if for every $x \in X$ there exists $f \in M$ such that $f(x) \neq 0$.

Theorem 9.2. *Let X be a locally compact Hausdorff space with a countable base and let A be a closed subalgebra of $C_0(X)$ that separates strongly the points of X. Then $A = C_0(X)$.*

Proof. We first prove that there exists a strictly positive function f_0 in A. For this, note that A is separable since $C_0(X)$ is. Denoting by $\{h_n \mid n \geq 1\}$ a dense subset of A, then for every $x \in X$ there exists $n \geq 1$ such that $h_n(x) \neq 0$. Hence the function $f_0 := \sum\limits_{n=1}^{\infty} \frac{|h_n|}{2^n \|h_n\|}$ lies in A, because of Lemma 9.1, and is strictly positive on X.

Given such a function $f_0 \in A$, we then have that $\{f_0\} \cup f_0 A \cup f_0 A^2 \subset A$ and hence, by Theorem 6.6, A is a Korovkin subspace in $C_0(X)$.

Considering $f \in C_0(X)$ and $\varepsilon > 0$, by Theorem 8.1 there exist $h_0, \ldots, h_n \in A$, $k_0, \ldots, k_n \in A$ as well as $u, v \in C_0(X), u \geq 0, v \geq 0$, such that $\|u\| \leq \varepsilon, \|v\| \leq \varepsilon$, and, finally,

$$\left\| \inf_{0\leq j\leq n} k_j - \sup_{0\leq i\leq n} h_i \right\| \leq \varepsilon \quad \text{and} \quad \sup_{0\leq i\leq n} h_i - u \leq f \leq \inf_{0\leq j\leq n} k_j + v.$$

Then $f - \inf\limits_{0\leq j\leq n} k_j \leq v$ and

$$\inf_{0\leq j\leq n} k_j - f \leq |\inf_{0\leq j\leq n} k_j - \sup_{0\leq i\leq n} h_i| + u.$$

Therefore,

$$|f - \inf_{0\leq j\leq n} k_j| \leq |\sup_{0\leq i\leq n} h_i - \inf_{0\leq j\leq n} k_j| + u + v$$

and then

$$\|f - \inf_{0\le j\le n} k_j\| \le 3\varepsilon$$

which shows that $f \in \overline{A} = A$ (since, by Lemma 9.1, $\inf_{0\le j\le n} k_j \in A$) and this completes the proof. $\square$

As in the case of the interval $[0,1]$ (see Theorem 3.8), we actually show that Theorems 6.6 and 9.2 are equivalent. The next result will be very useful for our purposes.

Theorem 9.3. *Let X be a locally compact Hausdorff space and consider a subset M of $C_0(X)$ such that the linear subspace generated by it contains a strictly positive function $f_0 \in C_0(X)$. Given $x_0 \in X$, denote by $A(M, x_0)$ the subspace of all functions $f \in C_0(X)$ such that $\mu(f) = f(x_0)$ for every $\mu \in \mathcal{M}_b^+(X)$ satisfying $\mu(g) = g(x_0)$ for every $g \in \{f_0\} \cup f_0M \cup f_0M^2$. Then $A(M, x_0)$ is a closed subalgebra of $C_0(X)$ which contains M.*

Proof. $A(M, x_0)$ is a closed subspace of $C_0(X)$. Set

$$\begin{aligned} M(x_0) &:= \{x \in X \mid g(x) = g(x_0) \quad \text{for every} \quad g \in M\} \\ &= \{x \in X \mid f_0(x)(g(x) - g(x_0))^2 = 0 \quad \text{for every} \quad g \in M\} \end{aligned}$$

and fix $f \in A(M, x_0)$ and $\mu \in \mathcal{M}_b^+(X)$ such that $\mu(g) = g(x_0)$ for every $g \in \{f_0\} \cup f_0M \cup f_0M^2$. In particular, $\mu(f_0(g - g(x_0))^2) = 0$ for each $g \in M$. On the other hand, if $x \in M(x_0)$, then $\delta_x(g) = g(x_0)$ for every $g \in \{f_0\} \cup f_0M \cup f_0M^2$, so that $f(x) = f(x_0)$ and hence $f_0(x_0)f^2 = f^2(x_0)f_0$ on $M(x_0)$.

Therefore, by Corollary 11.6 in the Appendix, we get that $\mu(f_0(x_0)f^2) = f^2(x_0)\mu(f_0) = f^2(x_0)f_0(x_0)$ and hence $\mu(f^2) = f^2(x_0)$. Therefore, $f^2 \in A(M, x_0)$ and hence $A(M, x_0)$ is a subalgebra of $C_0(X)$.

Finally note that, if $h \in M$, then $f_0(x_0)h = h(x_0)f_0$ on $M(x_0)$. Therefore, if we again fix $\mu \in \mathcal{M}_b^+(X)$ such that $\mu(g) = g(x_0)$ for each $g \in \{f_0\} \cup f_0M \cup f_0M^2$, by applying Corollary 11.6 we at once obtain that $f_0(x_0)\mu(h) = h(x_0)\mu(f_0) = h(x_0)f_0(x_0)$, so $\mu(h) = h(x_0)$ and hence $h \in A(M, x_0)$. $\square$

With the help of the preceding theorem, it is easy to show this next result.

Theorem 9.4. *The Korovkin-type Theorem 6.6 and the Stone-Weierstrass Theorem 9.2 are equivalent.*

Proof. In light of the proof of Theorem 9.2, we have only to show that Theorem 9.2 implies Theorem 6.6. So, consider a subset M of $C_0(X)$ that separates the points of X and such that the linear subspace generated by it contains a strictly positive function $f_0 \in C_0(X)$.

Given $x_0 \in X$, consider the subspace $A(M, x_0)$ defined in the preceding theorem. By virtue of Theorems 9.2 and 9.3, we then infer that $A(M, x_0) = C_0(X)$. In other words, we have shown that for every $\mu \in \mathcal{M}_b^+(X)$ and for every $x_0 \in X$ satisfying $\mu(g) = g(x_0)$ for every $g \in \{f_0\} \cup f_0M \cup f_0M^2$, we also have $\mu(f) = f(x_0)$ for every $f \in C_0(X)$ and this means that $\{f_0\} \cup f_0M \cup f_0M^2$ is a Korovkin subset in $C_0(X)$ because of Theorem 6.1. $\square$

For a further deepening of the relationship between Korovkin-type theorems and Stone-Weierstrass theorems, we refer to [8, Section 4.4] and [12-13].

10 Korovkin-type theorems for positive projections

In this last section, we shall discuss some Korovkin-type theorems for a class of positive linear projections by showing a nontrivial application of the general theorem proved in Section 5.

Consider a compact metric space X and a **positive linear projection** $T : C(X) \to C(X)$, i.e., T is a positive linear operator such that $T(T(f)) = T(f)$ for every $f \in C(X)$. We shall denote by H_T the range of T, i.e.,

$$H_T := T(C(X)) = \{h \in C(X) \mid T(h) = h\}\,. \tag{10.1}$$

We shall also assume that $\mathbf{1} \in H_T$ (hence $T(\mathbf{1}) = \mathbf{1}$) and that H_T separates the points of X. In the sequel, we shall present some examples of such projections.

For every $x \in X$ denote by $\mu_x \in \mathcal{M}^+(X)$ the Radon measure defined by

$$\mu_x(f) := T(f)(x) \quad (f \in C(X)) \tag{10.2}$$

and by $\widetilde{\mu}_x$ the unique probability Borel measure on X that corresponds to μ_x via the Riesz representation theorem, i.e.,

$$T(f)(x) = \mu_x(f) = \int_X f \,\mathrm{d}\widetilde{\mu}_x \quad (f \in C(X)). \tag{10.3}$$

Thus, $(\widetilde{\mu}_x)_{x\in X}$ is the canonical continuous selection associated with T.

By the Cauchy-Schwarz inequality (2.16), for every $h \in H_T$, we get

$$|h| = |T(h)| \leq \sqrt{T(\mathbf{1})T(h^2)} = \sqrt{T(h^2)}$$

so that

$$h^2 \leq T(h^2)\,. \tag{10.4}$$

Set

$$Y_T := \{x \in X \mid T(f)(x) = f(x) \text{ for every } f \in C(X)\}. \tag{10.5}$$

Y_T is closed and it is actually equal to the so-called Choquet boundary of H_T and hence it is not empty (see [8, Proposition 3.3.1 and Corollary 2.6.5]).

Proposition 10.1. *If M is a subset of H_T that separates the points of X, then*

$$Y_T = \{x \in X \mid T(h^2)(x) = h^2(x) \text{ for every } h \in M\}. \tag{10.6}$$

Moreover, if $(h_n)_{n\geq 1}$ is a finite or countable family of H_T that separates the points of X and if the series $u := \sum\limits_{n=1}^{\infty} h_n^2$ is uniformly convergent on X, then $u \leq T(u)$ and

$$Y_T := \{x \in X \mid Tu(x) = u(x)\}. \tag{10.7}$$

Proof. Consider $x \in X$ such that $T(h^2)(x) = h^2(x)$, i.e., $\mu_x(h^2) = h^2(x)$ for each $h \in M$. Since $M \subset H_T$, we also get $\mu_x(h) = h(x)$ $(h \in M)$ and $\{\mathbf{1}\} \cup M \cup M^2$ is a Korovkin subset of $C(X)$ by Theorem 7.1. According to Theorem 6.1, we then obtain that $f(x) = \mu_x(f) = T(f)(x)$ for every $f \in C(X)$.

As regards the second part of the statement, note that $T(u) = \sum_{n=1}^{\infty} T(h_n^2)$ and hence $u \le T(u)$ by (10.4). Moreover, if $x \in X$ and $T(u)(x) = u(x)$, then

$$\sum_{n=1}^{\infty} \Big(T(h_n^2)(x) - h_n^2(x) \Big) = 0.$$

Hence, on account of (10.4), $T(h_n^2)(x) = h_n^2(x)$ for every $n \ge 1$ and so $x \in Y_T$ by (10.6). □

Remark 10.2. Note that there always exists a countable family $(h_n)_{n\ge1}$ of H_T that separates the points of X and such that the series $\sum_{n=1}^{\infty} h_n^2$ is uniformly convergent on X. Actually, since $C(X)$ is separable, H_T is separable as well so that, considering a countable dense family $(\varphi_n)_{n\ge1}$ of H_T, it is enough to put $h_n := \frac{\varphi_n}{2^n(1+\|\varphi_n\|)}$ $(n \ge 1)$.

Theorem 10.3. *If M is a subset of H_T that separates the points of X, then*

$$H_T \cup M^2 \quad \textit{is a Korovkin subset for} \quad T.$$

Moreover, if $u \in C(X)$ satisfies $u \le T(u)$ and if (10.7) holds, then

$$H_T \cup \{u\} \quad \textit{is a Korovkin subset for} \quad T.$$

In particular, the above statement holds for $u = \sum_{n=1}^{\infty} h_n^2$, where $(h_n)_{n\ge1}$ is an arbitrary sequence in H_T that separates the points of X and such that the series $\sum_{n=1}^{\infty} h_n^2$ is uniformly convergent on X.

Proof. In order to apply Theorem 5.5, consider $\mu \in \mathcal{M}^+(X)$ and $x \in X$ such that $\mu(h) = T(h)(x) = (\mu_x(h))$ for every $h \in H_T \cup M^2$. Accordingly, if $h \in M$,

$$\mu(T(h^2) - h^2) = T(h^2)(x) - \mu_x(h^2) = 0$$

and $T(h^2) - h^2 \ge 0$. Therefore, by using (10.6) and by applying Theorem 11.5 and Corollary 11.6 of the Appendix, we see that, for every $f \in C(X)$, $\mu(T(f)) = \mu(f)$ because $Tf = f$ on Y_T. Hence $\mu(f) = \mu(T(f)) = \mu_x(Tf) = Tf(x)$ and this finishes the proof.

A similar reasoning can be used to show the second part of the statement by using (10.7) instead of (10.6). □

Theorem 10.3 is due to Altomare ([3], [5]; see also [8, Section 3.3]). For an extension to so called adapted spaces, we refer to [9]. Below, we show some examples.

Example 10.4. Consider the d-dimensional simplex K_d of $\mathbb{R}^d$, $d \ge 1$, defined by (4.38) and the positive projection $T_d : C(K_p) \to C(K_p)$ defined by

$$T_d(f)(x) := \Big(1 - \sum_{i=1}^{d} x_i\Big) f(0) + \sum_{i=1}^{d} x_i f(a_i) \tag{10.8}$$

$(f \in C(K_d),\ x = (x_i)_{1\le i\le d} \in K_d)$, where $a_i := (\delta_{ij})_{1\le j\le d}$ for every $i = 1, \dots, d$.

In this case, H_{T_d} is the subspace $A(K_d)$ (see Section 7, formula (7.1)) which in turn is generated by $M := \{\mathbf{1}, pr_1, \ldots, pr_d\}$, and hence, by Theorem 10.3,

$$\left\{\mathbf{1}, pr_1, \ldots, pr_d, \sum_{i=1}^{d} pr_i^2\right\} \text{ is a Korovkin subset for } T_d. \tag{10.9}$$

More generally, if K is an infinite-dimensional Bauer simplex (see, e.g., [8, Section 1.5, p. 59]), then there exists a unique linear positive projection T on $C(K)$ whose range is $A(K)$. In this case, for every strictly convex $u \in C(K)$, we get that $A(K) \cup \{u\}$ is a Korovkin subset for T (see [8, Corollary 3.3.4]).

Example 10.5. Consider the hypercube $Q_d := [0,1]^d$ of $\mathbb{R}^d$, $d \geq 1$, and the positive projection $S_d : C(Q_d) \to C(Q_d)$ defined by

$$S_d(f)(x) := \sum_{h_1,\ldots,h_d=0}^{1} x_1^{h_1}(1-x_1)^{1-h_1} \cdots x_d^{h_d}(1-x_d)^{1-h_d} f(b_{h_1,\ldots,h_d}) \tag{10.10}$$

where $b_{h_1,\ldots,h_d} := (\delta_{h_i 1})_{1\leq i\leq d} \quad (h_1, \ldots, h_d \in \{0,1\})$.

In this case, H_{S_d} is the subspace of $C(Q_d)$ generated by $\{\mathbf{1}\} \cup \{\prod_{i\in J} pr_i \mid J \subset \{1,\ldots,d\}\}$. Therefore, by Theorem 10.3, we obtain that

$$\{\mathbf{1}\} \cup \left\{\prod_{i\in J} pr_i \mid J \subset \{1,\ldots,d\}\right\} \cup \left\{\sum_{i=1}^{d} pr_i^2\right\} \quad \text{is a Korovkin subset for } S_d. \tag{10.11}$$

For an extension of the above result, see [8, Corollary 3.3.9].

Example 10.6. Consider a bounded open subset Ω of $\mathbb{R}^d$, $d \geq 2$, which we assume to be regular in the sense of potential theory (see, e.g., [68, Section 8.3] or [8, pp. 125-128]) (for instance, each convex open subset of $\mathbb{R}^d$ is regular). Denote by $H(\Omega)$ the subspace of all $u \in C(\overline{\Omega})$ that are harmonic on Ω.

By the regularity of Ω, it follows that for every $f \in C(\overline{\Omega})$ there exists a unique $u_f \in H(\Omega)$ such that $u_f|_{\partial\Omega} = f|_{\partial\Omega}$, i.e., u_f is the unique solution of the Dirichlet problem

$$\begin{cases} \triangle u := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{on } \Omega, \\ u|_{\partial\Omega} = f|_{\partial\Omega} \end{cases} \quad (u \in C(\overline{\Omega}) \cap C^2(\Omega)). \tag{10.12}$$

Then the **Poisson operator** $T : C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$ defined by

$$T(f) := u_f \quad (f \in C(\overline{\Omega})) \tag{10.13}$$

is a positive projection whose range is $H(\Omega)$.

Therefore, from Theorem 10.3, we get that

$$H(\Omega) \cup \left\{\sum_{i=1}^{d} pr_i^2\right\} \text{ is a Korovkin subset for } T. \tag{10.14}$$

For further examples of positive linear projections, we refer to [24] and [103].

We now discuss an application of Theorem 10.3. Consider a metrizable convex compact subset K of a locally convex space. For every $f \in C(K)$, $z \in K$ and $\alpha \in [0,1]$, we denote by $f_{z,\alpha} \in C(K)$ the function defined by

$$f_{z,\alpha}(x) := f(\alpha x + (1-\alpha)z) \quad (x \in K). \tag{10.15}$$

Consider a positive linear projection $T : C(K) \longrightarrow C(K)$, T different from the identity operator, and assume that

$$A(K) \subset H_T := T(C(K)), \tag{10.16}$$

i.e.,

$$T(u) = u \text{ for every } u \in A(K), \tag{10.17}$$

and

$$h_{z,\alpha} \in H_T \text{ for every } h \in H_T,\ z \in K,\ 0 \le \alpha \le 1. \tag{10.18}$$

Here, once again, we use the symbol $A(K)$ to designate the subspace of all continuous affine functions on K (see (7.1)). Moreover, note that in all the examples 10.4-10.6, assumptions (10.16)–(10.18) are satisfied.

Moreover, from Theorem 10.3, it also follows that

$$H_T \cup A(K)^2 \text{ is a Korovkin subset for } T. \tag{10.19}$$

Consider the sequence of Bernstein-Schnabl operators $(B_n)_{n\ge 1}$ associated with T according to (7.6) and (7.7), i.e., associated with the continuous selection $(\widetilde{\mu}_x)_{x\in X}$ defined by (10.3).

Note that, if T is the projection T_p defined by (10.8) (resp., the projection S_p defined by (10.10)), then the corresponding Bernstein-Schnabl operators are the Bernstein operators (4.39) (resp., the Bernstein operators (4.41)). By Theorem 7.5, we already know that

$$\lim_{n\to\infty} B_n(f) = f \text{ uniformly on } K \quad (f \in C(K)). \tag{10.20}$$

In this particular case, we can also study the iterates of the operators B_n.

For every $n, m \ge 1$, we set

$$B_n^m := \begin{cases} B_n & \text{if } m = 1, \\ B_n \circ B_n^{m-1} & \text{if } m \ge 2. \end{cases} \tag{10.21}$$

From (10.18), it follows that, if $h \in H_T$, then $B_n(h) = h$ and hence

$$B_n^m(h) = h = T(h). \tag{10.22}$$

On the other hand, if $u \in A(K)$, then by applying an induction argument it is not difficult to obtain from (7.13) that

$$B_n(u^2) = \Big(1 - \Big(\frac{n-1}{n}\Big)^m\Big)T(u^2) + \Big(\frac{n-1}{n}\Big)^m u^2. \tag{10.23}$$

Now, we can easily prove the next result.

Theorem 10.7. *Let $f \in C(K)$. Then*

(1) $\lim_{n\to\infty} B_n^m(f) = f$ *uniformly on K for every $m \geq 1$.*

(2) $\lim_{m\to\infty} B_n^m(f) = T(f)$ *uniformly on K for every $n \geq 1$.*

(3) If $(k(n))_{n\geq 1}$ is a sequence of positive integers, then

$$\lim_{n\to\infty} B_n^{k(n)}(f) = \begin{cases} f & \text{uniformly on } K \text{ if } \frac{k(n)}{n} \to 0, \\ T(f) & \text{uniformly on } K \text{ if } \frac{k(n)}{n} \to +\infty. \end{cases}$$

Proof. It is sufficient to apply Corollary 7.3 and statement (10.19) taking (10.22) and (10.23) into account as well as the elementary formula

$$\left(\frac{n-1}{n}\right)^m = \exp\left(m\log\left(1-\frac{1}{n}\right)\right) \quad (n, m \geq 2).$$

□

It is worth remarking that under some additional assumptions on T, the sequence $(B_n^{k(n)}(f))_{n\geq 1}$ $(f \in \mathcal{C}(K))$ converges uniformly also when $\frac{k(n)}{n} \longrightarrow t \in]0, +\infty[$. More precisely, for every $t \geq 0$ there exists a positive linear operator $T(t) : C(K) \longrightarrow C(K)$ such that for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\frac{k(n)}{n} \longrightarrow t$, and for every $f \in C(K)$,

$$T(t)f = \lim_{n\to\infty} B_n^{k(n)}(f). \tag{10.24}$$

Moreover, the family $(T(t))_{t\geq 0}$ is a strongly continuous semigroup of operators whose generator $(A, D(A))$ is the closure of the operator $(Z, D(Z))$ where

$$D(Z) := \left\{u \in C(K) \mid \lim_{n\to\infty} n(B_n(u) - u) \quad \text{exists in} \quad C(K)\right\} \tag{10.25}$$

and, for every $u \in D(Z) \subset D(A)$,

$$A(u) = Z(u) = \lim_{n\to\infty} n(B_n(u) - u). \tag{10.26}$$

If K is a subset of $\mathbb{R}^d$, $d \geq 1$, with nonempty interior and if T maps the subspace of all polynomials of degree m into itself for every $m \geq 1$, then $C^2(K) \subset D(Z) \subset D(A)$ and, for every $u \in C^2(K)$,

$$Au(x) = Zu(x) = \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_i} \tag{10.27}$$

$(x = (x_i)_{1\leq i\leq d})$, where for every $i, j = 1, \ldots, d$,

$$\alpha_{ij}(x) := T(pr_i pr_j)(x) - x_i x_j. \tag{10.28}$$

The differential operator (10.27) is an elliptic second order differential operator which degenerates on the subset Y_T defined by (10.5) (which contains all the extreme points of K).

Moreover, for every $u_0 \in D(A)$, the (abstract) Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = A(u(\cdot,t))(x) & x \in K, t \geq 0, \\ u(x,0) = u_0(x) & x \in K, \\ u(\cdot,t) \in D(A) & t \geq 0, \end{cases} \tag{10.29}$$

has a unique solution $u : K \times [0, +\infty[\to \mathbb{R}$ given by

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} B_n^{k(n)}(u_0)(x) \quad (x \in K, t \geq 0) \tag{10.30}$$

and the limit is uniform with respect to $x \in K$, where $\frac{k(n)}{n} \to t$.

These results were discovered by Altomare ([5], see also [8, Sections 6.2 and 6.3]) and opened the door to a series of researches whose main aims are, first, the approximation of the solutions of initial boundary differential problem associated with degenerate evolution equations (like (10.29)) by means of iterates of positive linear operators (like (10.30)), and, second, both a numerical and a qualitative analysis of the solutions by means of formula (10.30).

These researches are documented in several papers. Here, we content ourselves to cite, other than Chapter 6 of [8], also the papers [6], [17] and [18] and the references therein.

11 Appendix: A short review of locally compact spaces and of some continuous function spaces on them

For the convenience of the reader, in this Appendix, we collect some basic definitions and results concerning locally compact Hausdorff spaces, some continuous function spaces and Radon measures on them. For more details, we refer the reader to Chapter IV of [30] or to Chapter 3 of [54].

We start by recalling that a topological space X is said to be **compact** if every open cover of X has a finite subcover. A subset of a topological space is said to be compact if it is compact in the relative topology. A topological space is said to be **locally compact** if each of its points possesses a compact neighborhood.

Actually, if X is locally compact and **Hausdorff** (i.e., for every pair of distinct points $x_1, x_2 \in X$ there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, such that $U_1 \cap U_2 = \emptyset$), then each point of X has a fundamental system of compact neighborhoods.

Every compact space is locally compact. The spaces $\mathbb{R}^d$, $d \geq 1$, are fundamental examples of (noncompact) locally compact spaces. Furthermore, if X is locally compact, then every open subset of X and every closed subset of X, endowed with the relative topology, is locally compact. More generally, a subset of a locally compact Hausdorff space, endowed with the relative topology, is locally compact if and only if it is the intersection of an open subset of X with a closed subset of X (see [54, Corollary 3.3.10]). Therefore, every real interval is locally compact.

A topological space X is said to be **metrizable** if its topology is induced by a metric on X. In this case, we say that X is **complete** if such a metric is complete. Note that every compact metrizable space is complete and **separable**, i.e., it contains a dense countable subset.

A special role in the measure theory on topological spaces (and in the Korovkin-type approximation theory) is played by locally compact Hausdorff spaces with a **countable base** (or basis), i.e., with a countable family of open subsets such that every open subset is the union of some subfamily of it. Such spaces are metrizable, complete and separable. Actually, a metrizable space has a countable base if and only if it is separable. The spaces $\mathbb{R}^d$, $d \geq 1$, and each open or closed subset of them are locally compact Hausdorff spaces with a countable base.

From now on, X will stand for a fixed locally compact Hausdorff space. We denote by $K(X)$ the linear space of all real-valued continuous functions $f : X \longrightarrow \mathbb{R}$ whose (closed) support

$$\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$$

is compact. $K(X)$ is a lattice subspace of $C_b(X)$ and it coincides with $C(X)$ if X is compact.

The next result shows that there are sufficiently many functions in $K(X)$. (For a proof, see [30, Corollary 27.3].)

Theorem 11.1. *(***Urysohn's lemma***) For every compact subset K of X and for every open subset U containing K, there exists $\varphi \in K(X)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on K and $\operatorname{supp}(\varphi) \subset U$ (and hence $\varphi = 0$ on $X \backslash U$).*

Another fundamental function space is the space $C_0(X)$ which is defined as the closure of $K(X)$ in $C_b(X)$ with respect to the sup norm, in symbols

$$C_0(X) := \overline{K(X)}.$$

Thus, $C_0(X)$ is a closed linear subspace of $C_b(X)$ and hence, endowed with the sup-norm, is a Banach space.

By means of Urysohn's lemma, it is not difficult to prove the following characterization of functions lying in $C_0(X)$.

Theorem 11.2. *Assume that X is not compact. For a function $f \in C(X)$, the following statements are equivalent:*

(i) $f \in C_0(X)$;

(ii) $\{x \in X \mid |f(x)| \geq \varepsilon\}$ *is compact for every* $\varepsilon > 0$;

(iii) for every $\varepsilon > 0$ there exists a compact subset K of X such that $|f(x)| \leq \varepsilon$ for every $x \in X \backslash K$.

Because of the preceding theorem, the functions lying in $C_0(X)$ are said to vanish at infinity. If X is compact, then $C_0(X) = C(X)$. Moreover, $C_0(X)$ is a lattice subspace of $C_b(X)$ and, endowed with the sup-norm, is separable provided X has a countable base.

Another characterization of functions in $C_0(X)$ involves sequences of points of X that converge to the point at infinity of X. More precisely, assuming that X is noncompact, a sequence $(x_n)_{n\geq 1}$ in X is said to **converge to the point at infinity** of X if for every compact subset K of X there exists $\nu \in \mathbb{N}$ such that $x_n \in X \backslash K$ for every $n \geq \nu$. For any such sequence and for every $f \in C_0(X)$, we then have

$$\lim_{n\to\infty} f(x_n) = 0. \tag{11.1}$$

Conversely, a function $f \in C(X)$ satisfying (11.1) for every sequence $(x_n)_{n\geq 1}$ converging to the point at infinity of X necessarily lies in $C_0(X)$ provided that X is countable at infinity, i.e., it is the union of a sequence of compact subsets of X. Note also that X is countable at infinity if and only if there exists $f_0 \in C_0(X)$ such that $f_0(x) > 0$ for every $x \in X$. Moreover, if X has a countable basis, then X is countable at infinity.

A useful tool which plays an important role in Korovkin-type approximation theory is given by *Radon measures*. Actually, we shall only need to handle **positive bounded Radon measures** which are, by definition, positive linear functionals on $C_0(X)$. The set of all of them will be denoted by $\mathcal{M}_b^+(X)$.

Every $\mu \in \mathcal{M}_b^+(X)$, that is, every positive linear functional $\mu : C_0(X) \longrightarrow \mathbb{R}$, is continuous (with respect to the sup-norm), and its norm

$$\|\mu\| := \sup\{|\mu(f)| \mid f \in C_0(X), |f| \leq 1\} \tag{11.2}$$

is also called the **total mass** of μ.

A simple example of bounded positive Radon measure is furnished by the **Dirac measure** at a point $a \in X$, which is defined by

$$\delta_a(f) := f(a) \quad (f \in C_0(X)). \tag{11.3}$$

A positive linear combination of Dirac measures is called a (positive) **discrete measure**.

In other words, a Radon measure $\mu \in \mathcal{M}_b^+(X)$ is discrete if there exist finitely many points $a_1, \dots, a_n \in X, n \geq 1$, and finitely many positive real numbers $\lambda_1, \dots, \lambda_n$ such that

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{a_i}, \tag{11.4}$$

i.e.,

$$\mu(f) = \sum_{i=1}^{n} \lambda_i f(a_i) \quad \text{for every} \quad f \in C_0(X). \tag{11.5}$$

In this case, $\|\mu\| = \sum\limits_{i=1}^{n} \lambda_i$ and μ is also said to be **supported on** $\{a_1, \dots, a_n\}$.

There is a strong relationship between positive bounded Radon measures and (positive) finite Borel measures on X. In order to briefly describe it, we recall that the **Borel σ-algebra** in X is, by definition, the σ-algebra generated by the system of all open subsets of X. It will be denoted by $\mathcal{B}(X)$ and its elements are called Borel subsets of X. Open subsets, closed subsets and compact subsets of X are Borel subsets.

A **Borel measure** $\widetilde{\mu}$ on X is, by definition, a measure $\widetilde{\mu} : \mathcal{B}(X) \to [0, +\infty]$ such that

$$\widetilde{\mu}(K) < +\infty \quad \text{for every compact subset} \quad K \text{ of } X. \tag{11.6}$$

Every finite measure $\widetilde{\mu}$ on $\mathcal{B}(X)$, i.e., $\widetilde{\mu}(X) < +\infty$, is a Borel measure.

A measure $\widetilde{\mu} : \mathcal{B}(X) \longrightarrow [0, +\infty]$ is said to be **inner regular** if

$$\widetilde{\mu}(B) = \sup\{\widetilde{\mu}(K) \mid K \subset B, K \text{ compact}\} \text{ for every } B \in \mathcal{B}(X) \tag{11.7}$$

and **outer regular** if

$$\widetilde{\mu}(B) = \inf\{\widetilde{\mu}(U) \mid B \subset U, U \text{ open}\} \text{ for every } B \in \mathcal{B}(X). \tag{11.8}$$

A measure $\widetilde{\mu}$ is said to be **regular** if it is both inner regular and outer regular. The Lebesgue-Borel measure on $\mathbb{R}^d$, $d \geq 1$, is regular. Actually, if X has a countable base, then every Borel measures on X is regular ([30, Theorem 29.12]).

If $\widetilde{\mu}$ is a finite measure on $\mathcal{B}(X)$, then every $f \in C_b(X)$ is $\widetilde{\mu}$-integrable. Therefore, we can consider the positive bounded Radon measure $I_{\widetilde{\mu}}$ on X defined by

$$I_{\widetilde{\mu}}(f) := \int_X f \, \mathrm{d}\widetilde{\mu} \quad (f \in C_0(X)). \tag{11.9}$$

Then $\|I_{\widetilde{\mu}}\| \leq \widetilde{\mu}(X)$, and, if $\widetilde{\mu}$ is inner regular, $\|I_{\widetilde{\mu}}\| = \widetilde{\mu}(X)$.

As a matter of fact, formula (11.9) describes all the positive bounded Radon measures on X as the following fundamental result shows (see [30, Section 29]).

Theorem 11.3. (**Riesz representation theorem**) *If $\mu \in \mathcal{M}_b^+(X)$, then there exists a unique finite and regular Borel measure $\widetilde{\mu}$ on X such that*

$$\mu(f) = \int_X f \, \mathrm{d}\widetilde{\mu} \quad \textit{for every} \quad f \in C_0(X).$$

Moreover, $\|\mu\| = \widetilde{\mu}(X)$.

Another noteworthy and useful result concerning regular Borel measures is shown below. Consider a measure $\widetilde{\mu}$ on $\mathcal{B}(X)$ and $p \in [1, +\infty[$. As usual, we shall denote by $\mathcal{L}^p(X, \widetilde{\mu})$ the linear subspace of all $\mathcal{B}(X)$-measurable functions $f : X \longrightarrow \mathbb{R}$ such that $|f|^p$ is $\widetilde{\mu}$-integrable.

If $f \in \mathcal{L}^p(X, \widetilde{\mu})$, it is customary to set

$$N_p(f) := \left(\int_X |f|^p \, \mathrm{d}\widetilde{\mu} \right)^{1/p}. \tag{11.10}$$

The functional $N_p : \mathcal{L}^p(X, \widetilde{\mu}) \to \mathbb{R}$ is a seminorm and the convergence with respect to it is the usual **convergence in p^{th}-mean**. Setting

$$\begin{aligned} \mathcal{N} &:= \{f \in \mathcal{L}^p(X, \widetilde{\mu}) \mid N_p(f) = 0\} \\ &= \{f \in F(X) \mid f \text{ is } \mathcal{B}(X) \text{ measurable and } f = 0 \ \ \widetilde{\mu} \text{ a.e.}\}, \end{aligned}$$

the quotient linear space

$$L^p(X, \widetilde{\mu}) := \mathcal{L}^p(X, \widetilde{\mu}) / \mathcal{N} \tag{11.11}$$

endowed with the norm

$$\|\widetilde{f}\|_p := N_p(f) \quad (\widetilde{f} \in L^p(X, \widetilde{\mu})), \tag{11.12}$$

is a Banach space. (Here, $\widetilde{f} := \{g \in \mathcal{L}^p(X, \widetilde{\mu}) \mid f = g \ \ \widetilde{\mu} \text{ a.e.}\}$.) Note that, if $\widetilde{\mu}$ is a Borel measure, then $K(X) \subset \mathcal{L}^p(X, \widetilde{\mu})$ for every $p \in [1, +\infty[$.

If $\widetilde{\mu}$ is also regular, we can say much more (for a proof of the next result, see [30, Theorem 29.14]).

Theorem 11.4. *If $\widetilde{\mu}$ is a regular Borel measure on X, then, for every $p \in [1, +\infty[$, the space $K(X)$ is dense in $\mathcal{L}^p(X, \widetilde{\mu})$ with the respect to convergence in the p^{th}-mean (and hence in $L^p(X, \widetilde{\mu})$ with respect to $\|\cdot\|_p$).*

Next, we discuss a characterization of discrete Radon measures. Usually this characterization is proved by using the notion of support of Radon measures (see, e.g., [36, Chapter III, Section 2] and [39, Vol. I, Section 11]). Below, we present a simple and direct proof.

We start with the following result which is important in its own right.

Theorem 11.5. *Let $\mu \in \mathcal{M}_b^+(X)$ and consider a closed subset Y of X such that*

$$\mu(\varphi) = 0 \text{ for every } \varphi \in K(X), \operatorname{supp}(\varphi) \subset X \backslash Y. \tag{11.13}$$

Then $\mu(f) = \mu(g)$ for every $f, g \in C_0(X)$ such that $f = g$ on Y.

Proof. It suffices to show that, if $f \in C_0(X)$ and $f = 0$ on Y, then $\mu(f) = 0$. Consider such a function $f \in C_0(X)$ and, given $\varepsilon > 0$, set $U := \{x \in X \mid |f(x)| < \varepsilon\}$. Hence, by Theorem 11.2, $X \backslash U$ is compact and $X \backslash U \subset X \backslash Y$.

By Urysohn's lemma (Theorem 11.1), there exists $\varphi \in K(X)$, $0 \leq \varphi \leq 1$, such that $\varphi = 1$ on $X \backslash U$ and $\operatorname{supp}(\varphi) \subset X \backslash Y$. In particular, $\operatorname{supp}(f\varphi) \subset \operatorname{supp}(\varphi) \subset X \backslash Y$ and hence $\mu(f\varphi) = 0$.

Therefore,

$$|\mu(f)| = |\mu(f) - \mu(f\varphi)| \leq \|\mu\| \|f(1-\varphi)\| \leq \|\mu\|\varepsilon,$$

because $\|f(1-\varphi)\| \leq \varepsilon$ as we now confirm. For every $x \in X$, we have, indeed, $|f(x)(1-\varphi(x))| = 0$ if $x \notin U$ and, if $x \in U, |f(x)(1-\varphi(x))| \leq |f(x)| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that $\mu(f) = 0$. □

We point out that there always exists a closed subset Y of X satisfying (11.13). The smallest of them is called the support of the measure μ (see the references before Theorem 11.5). An important example of a subset Y satisfying (11.13) is given below.

Corollary 11.6. *Let $\mu \in \mathcal{M}_b^+(X)$ and consider an arbitrary family $(f_i)_{i \in I}$ of positive functions in $C_0(X)$ such that $\mu(f_i) = 0$ for every $i \in I$. Then the subset*

$$Y := \{x \in X \mid f_i(x) = 0 \text{ for every } i \in I\}$$

satisfies (11.13).

Therefore, if $f, g \in C_0(X)$ and if

$$f(x) = g(x) \quad \text{for every} \quad x \in Y,$$

then $\mu(f) = \mu(g)$.

Proof. Consider $\varphi \in K(X)$ such that

$$\operatorname{supp}(\varphi) \subset X \backslash Y = \{x \in X \mid \text{ there exists } i \in I \text{ with } f_i(x) > 0\}.$$

By using a compactness argument, we then find a finite subset J of I such that

$$\operatorname{supp}(\varphi) \subset \bigcup_{i \in J} \{x \in X \mid f_i(x) > 0\}.$$

If we set $\alpha := \min \left\{ \sum\limits_{i \in J} f_i(x) \mid x \in \operatorname{supp}(\varphi) \right\} > 0$, we then obtain

$$|\varphi| \leq \frac{\|\varphi\|}{\alpha} \sum_{i \in J} f_i$$

and hence $\mu(f) = 0$. □

By means of the previous result, it is easy to reach the announced characterization of discrete Radon measures.

Theorem 11.7. *Given $\mu \in \mathcal{M}_b^+(X)$ and different points $a_1, \dots, a_n \in X, n \geq 1$, the following statements are equivalent:*

(i) There exist $\lambda_1, \dots, \lambda_n \in [0, +\infty[$ such that $\mu = \sum\limits_{i=1}^{n} \lambda_i \delta_{a_i}$ (see (11.5));

(ii) if $\varphi \in K(X)$ and $\mathrm{supp}(\varphi) \cap \{a_1, \dots, a_n\} = \emptyset$, then $\mu(\varphi) = 0$;

(iii) for every $x \in X \backslash \{a_1, \dots, a_n\}$ there exists $f \in C_0(X)$, $f \geq 0$, such that $f(x) > 0$, $f(a_i) = 0$ for each $i = 1, \dots, n$, and $\mu(f) = 0$.

Proof. (i)⇒(ii). It is obvious.

(ii)⇒(iii). If $x \in X \backslash \{a_1, \dots, a_n\}$, by Urysohn's lemma, we can choose $\varphi \in K(X)$, $0 \leq \varphi \leq 1$, such that $\varphi(x) = 1$ and $\mathrm{supp}(\varphi) \subset X \backslash \{a_1, \dots, a_n\}$ so that $\mu(f) = 0$.

(iii)⇒(ii). Consider $\varphi \in K(X)$ such that $\mathrm{supp}(\varphi) \cap \{a_1, \dots, a_n\} = \emptyset$. By hypothesis, for every $x \in \mathrm{supp}(\varphi)$, there exists $f_x \in C_0(X)$, $f_x \geq 0$, such that $f_x(x) > 0$, $f_x(a_i) = 0$ for every $i = 1, \dots, n$, and $\mu(f_x) = 0$.

Since $\mathrm{supp}(\varphi)$ is compact, there exist $x_1, \dots, x_p \in \mathrm{supp}(\varphi)$ such that

$$\mathrm{supp}(\varphi) \subset \bigcup_{k=1}^{p} \{x \in X \mid f_{x_k}(x) > 0\}.$$

Therefore, the function $f := \sum\limits_{k=1}^{p} f_{x_k} \in C_0(X)$ is positive, it does not vanish at any point of $\mathrm{supp}(\varphi)$, and $\mu(f) = 0$.

If we set $m := \min\{f(x) \mid x \in \mathrm{supp}(\varphi)\} > 0$, it is immediate to verify that $m|\varphi| \leq \|\varphi\| f$ and hence $\mu(\varphi) = 0$.

(ii)⇒(i). For every $j = 1, \dots, n$, consider $\varphi_j \in K(X)$ such that $0 \leq \varphi_j \leq 1$, $\varphi_j(a_j) = 1$ and $\varphi_j(a_i) = 0$ for each $i = 1, \dots, n$, $i \neq j$.

If $f \in C_0(X)$, then

$$f = \sum_{i=1}^{n} f(a_i) \varphi_i \quad \text{on} \quad \{a_1, \dots, a_n\}.$$

On the other hand, the subset $\{a_1, \dots, a_n\}$ satisfies (11.13) and hence, by Theorem 11.5,

$$\mu(f) = \sum_{i=1}^{n} f(a_i) \mu(\varphi_i) = \sum_{i=1}^{n} \lambda_i f(a_i)$$

where $\lambda_i := \mu(\varphi_i) \geq 0 \quad (i = 1, \dots, n)$ and this completes the proof. □

We end the Appendix by discussing some aspects of vague convergence for Radon measures. For more details, we refer, e.g., to [30, §30] or to [39, Vol. I, §12].

A sequence $(\mu_n)_{n \geq 1}$ in $\mathcal{M}_b^+(X)$ is said to **converge vaguely** to $\mu \in \mathcal{M}_b^+(X)$ if

$$\lim_{n \to \infty} \mu_n(f) = \mu(f) \quad \text{for every} \quad f \in C_0(X). \tag{11.14}$$

Thus, (11.14) simply means that $\mu_n \to \mu$ weakly in the dual space of $C_0(X)$.

If, in addition, X has a countable base, then $C_0(X)$ is separable and hence, by Banach's theorem, the unit ball of the dual of $C_0(X)$ is weakly sequentially compact. Therefore

Theorem 11.8. *If X has a countable base, then every sequence in $\mathcal{M}_b^+(X)$ that is bounded with respect to the norm (11.2), has a subsequence that converges vaguely to some $\mu \in \mathcal{M}_b^+(X)$.*

References

[1] O. Agratini, On statistical approximation in spaces of continuous functions, *Positivity*, **13**(2009), no. 4, 735-743.

[2] C. D. Aliprantis and O. Burkinshaw, Positive Operators, Academic Press, New York, 1985.

[3] F. Altomare, Teoremi di approssimazione di tipo Korovkin in spazi di funzioni, *Rendiconti di Matematica*, (3) 1980, Vol. **13**, Serie VI, 409–429.

[4] F. Altomare, Approximation of finitely defined operators in function spaces, *Note Mat.*, **7** (1987), 211–229.

[5] F. Altomare, Limit semigroups of Bernstein-Schnabl operators associated with positive projections, *Ann. Sc. Norm. Sup. Pisa, Cl. Sci.* (4) **16** (1989)(2), 259–279.

[6] F. Altomare, Approximation theory methods for the study of diffusion equations, Approximation Theory, *Proc. IDOMAT 75 (M.W. Müller, M. Felten, D.H. Mache Eds.) Math. Res.*, Vol. **86**, pp. 9–26, Akademie Verlag, Berlin, 1995.

[7] F. Altomare, Asymptotic formulae for Bernstein-Schnabl operators and smoothness, *Boll. U.M.I.*, (9) II (2009), 135–150.

[8] F. Altomare and M. Campiti, Korovkin-type Approximation Theory and its Applications, de Gruyter Studies in Mathematics, **17**, Walter de Gruyter & Co., Berlin, 1994.

[9] F. Altomare and M. Cappelletti Montano, Affine projections on adapted subalgebras of continuous functions, *Positivity*, **9** (2005), no. 4, 625–643.

[10] F. Altomare and M. Cappelletti Montano, Korovkin-type theorems in regular locally convex vector lattices of continuous functions - Part I, *Studia Math.*, **171** (3) (2005), 239–260.

[11] F. Altomare and M. Cappelletti Montano, Korovkin-type theorems in regular locally convex vector lattices of continuous functions - Part II, *Studia Math.*, **172** (1) (2006), 69–90.

[12] F. Altomare and M. Cappelletti Montano, On some density theorems in regular vector lattices of continuous functions, *Collect. Math.*, **58**, 2 (2007), 131–149.

[13] F. Altomare and M. Cappelletti Montano, On a class of locally convex vector lattices of continuous function spaces and related approximation/density problems, Proc. Intern. Symp. on Banach and Function Spaces II (ISBFS 2006), Kitakyushu (Japan), 14–17 September, 2006, *Yokohama Publishers, Yokohama*, 2008, pp. 25–50.

[14] F. Altomare and S. Diomede, Contractive Korovkin subsets in weighted spaces of continuous functions, *Rend. Circ. Mat. Palermo, Serie II*, **50** (2001), pp. 547–568.

[15] F. Altomare and S. Diomede, Positive operators and approximation in function spaces on completely regular spaces, *Int. J. of Math. and Math. Sci.*, **61** (2003), 3841–3871.

[16] F. Altomare and V. Leonessa, On a sequence of positive linear operators associated with a continuous selection of Borel measures, *Mediterr. J. Math.*, **3** (2006), no. 3 - 4, 363–382.

[17] F. Altomare and V. Leonessa, An invitation to the study of evolution equations by means of positive linear operators, *Lecture Notes of Seminario Interdisciplinare di Matematica*, **8** (2009), 1–41.

[18] F. Altomare, V. Leonessa and S. Milella, Bernstein-Schnabl operators on noncompact real intervals, *Jaen J. Approx.*, **1** (2) (2009), 223–256.

[19] F. Altomare, V. Leonessa and I. Raşa, On Bernstein-Schnabl operators on the unit interval, *Z. Anal. Anwend.*, **27** (2008), 353–379.

[20] F. Altomare and E. Mangino, On a generalization of Baskakov operators, *Rev. Roumaine Math. Pures Appl.*, **44** (1999), 5-6, 683–705.

[21] F. Altomare and S. Milella, Integral-type operators on continuous function spaces on the real line, *J. of Approx. Theory*, **152** (2008), 107–124.

[22] F. Altomare and S. Milella, On a sequence of integral operators on weighted L^p spaces, *Analysis Math.*, **34** (2008), 237–259.

[23] F. Altomare, S. Milella and G. Musceo, Multiplicative perturbations of the Laplacian and related approximation problems, preprint, 2010.

[24] F. Altomare and I. Raşa, Towards a characterization of a class of differential operators associated with positive projection, *Atti Sem. Mat. Fis. Univ. Modena, Supplemento al n. XLVI*, (1998), 3–38.

[25] F. Altomare and I. Raşa, Feller semigroups, Bernstein type operators and generalized convexity associated with positive projections, in: New Developments in Approximation Theory (Dortmund 1998), 9–32, *Internat. Ser. Numer. Math.*, **132**, Birkhäuser Verlag, Basel, 1999.

[26] G. A. Anastassiou and S. G. Gal, Approximation Theory, Birkhäuser, Boston, 2000.

[27] V. A. Baskakov, An example of a sequence of linear positive operators in the space of continuous functions, *Dokl. Akad. Nauk. SSSR*, **113** (1957), 249–251.

[28] V. A. Baskakov, On various convergence criteria for linear positive operators (Russian), *Usp. Mat. Nauk*, **16** (1961), no. 1(97), 131–134.

[29] H. Bauer, Probability Theory, de Gruyter Studies in Mathematics, **23**, Walter de Gruyter & Co., Berlin, 1996.

[30] H. Bauer, Measure and Integration Theory, de Gruyter Studies in Mathematics, **26**, W. de Gruyter & Co., Berlin, 2001.

[31] H. Bauer and K. Donner, Korovkin approximation in $C_0(X)$, *Math. Ann.*, **236** (1978), no. 3, 225–237.

[32] M. Becker, Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.*, **27** (1978), no. 1,, 127–142.

[33] H. Berens and G. G. Lorentz, Theorems of Korovkin type for positive linear operators on Banach lattices, in: Approximation Theory (Proc. Internat. Sympos., Univ. Texas, Austin, Tex. 1973), 1–30; Academic Press, New York, 1973.

[34] S. N. Bernstein, Démostration du théorème de Weierstrass fondée sur le calcul de probabilités, *Comm. Soc. Math. Kharkow* (2), **13**, 1912-1913, 1–2.

[35] H. Bohman, On approximation of continuous and analytic functions, *Ark. Math.*, **2** (1952-54), 43–46.

[36] N. Bourbaki, Eléments de Mathématique, Livre VI, Integration, Ch. 1-9, Actualités Scientifiques et Industrielles, 1343 (1969), Hermann, Paris.

[37] P. L. Butzer, On two-dimensional Bernstein polynomials, *Canad. J. Math.*, **5** (1953), 107–113.

[38] P. L. Butzer and R. J. Nessel, Fourier Analysis and Approximation, Vol. 1, Academic Press, New York, 1971.

[39] G. Choquet, Lectures on Analysis, Vol. I and II, A. Benjamin Inc., New York-Amsterdam, 1969.

[40] M. M. Derrienic, Sur l'approximation de fonctions intégrable sur [0,1] par des polynômes de Bernstein modifies, *J. Approx. Theory*, **31** (1981), 325–343.

[41] R. A. DeVore, The Approximation of Continuous Functions by Positive Linear Operators, Lecture Notes in Math. **293**, Springer-Verlag, Heidelberg, Berlin, New York, 1972.

[42] R. A. DeVore and G. G. Lorentz, Constructive Approximation, Grundlehren der mathematischen Wissenschaften, **303**, Springer- Verlag, Berlin, Heidelberg, 1993.

[43] A. Dinghas, Über einige Identitäten von Bernsteinschem Typus, *Norske Vid. Selsk. Fohr. Trondheim*, **24** (1951), 96–97.

[44] Z. Ditzian, Polynomial approximation and $\omega_\varphi^r(f,t)$ twenty years later, *Surveys in Approximation Theory*, **3** (2007), 106–151.

[45] Z. Ditzian and V. Totik, Moduli of Smoothness, *Springer Ser. Comput. Math.* **9**, New York, Springer 1987.

[46] K. Donner, Extension of Positive Operators and Korovkin Theorems, Lecture Notes in Math. **904**, Springer-Verlag, Berlin, 1982.

[47] O. Duman, Statistical approximation for periodic functions, *Demonstratio Math.*, **36** (2003), 873–878.

[48] O. Duman, M. K. Khan and C. Orhan, A-statistical convergence of approximating operators, *Math. Inequal. Appl.*, **6** (2003), 689–699.

[49] O. Duman and C. Orhan, Statistical approximation in the space of locally integrable functions, *Publ. Math. Debrecen*, **63** (2003), 133–144.

[50] O. Duman and C. Orhan, Statistical approximation by positive linear operators, *Studia Math.*, **161** (2004), 187–197.

[51] O. Duman and C. Orhan, An abstract version of the Korovkin approximation theorem, *Publ. Math. Debrecen*, **69**/1-2 (2006), 33–46.

[52] O. Duman, M. A. Özarslan and E. Erkus-Duman, Rates of ideal convergence for approximation operators, to appear in *Mediterr. J. Math.*, 2010.

[53] J. L. Durrmeyer, Une formule d'inversion de la transformée de Laplace: Applications á la theorie des moments, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[54] R. Engelking, General Topology, Sigma Series in Pure Mathematics **6**, Heldermann Verlag, Berlin, 1989.

[55] J. Favard, Sur les multiplicateurs d'interpolation, *J. Math. Pures Appl.*, **23** (9)(1944), 219–247.

[56] L. Fejér, Sur les fonctions bornées et intégrables, C. Rendus Hebdomadaires, *Séances de l'Académie des Sciences Paris*, **131** (1900), 984–987.

[57] L. Fejér, Untersuchungen über Fouriersche Reihen, *Math. Ann.*, **58** (1904), 984–987.

[58] G. B. Folland, Real Analysis, Modern Techniques and their Applications, John Wiley & Sons, New York, 1999.

[59] G. Freud, Über positive linear Approximationsfolgen von stetigen reellen Funktionen auf kompakten Mengen, in: On Approximation Theory (Proc. Conf. Oberwolfach, 1963), 233–238; Birkhäuser, Basel, 1964.

[60] A. D. Gadjiev, A problem on the convergence of a sequence of positive linear operators on unbounded sets and theorems that are analogous to P. P. Korovkin's theorem (Russian), *Dokl. Akad. Nauk SSSR*, **218** (1974), no. 5, 1001–1004; translated in *Soviet Math. Dokl.*, **15** (1974), no. 5, 1433–1436.

[61] A. D. Gadjiev, Theorems of Korovkin type, *Math. Notes*, **20** (1976), 995–998.

[62] A. D. Gadjiev, Positive linear operators in weighted spaces of functions of several variables (Russian), *Izv. Akad. Nauk Azerbaijan SSR Ser. Fiz.-Tehn. Mat. Nauk*, **1** (1980), no. 4, 32–37.

[63] A. D. Gadjiev and C. Orhan, some approximation theorems via statistical convergence, *Rocky Mountain J. Math.*, **32** (2002), 129–138.

[64] S. Gal, Shape-Preserving Approximation by Real and Complex Polynomials, Birkhäuser, Boston, 2008.

[65] I. Gavrea, H. Gonska, R. Păltănea and G. Tachev, General estimates for the Ditzian-Totik modulus, *East J. Approx.*, **9** (2003)(2), 175–194.

[66] M. W. Grossman, Note on a generalized Bohman-Korovkin theorem, *J. Math. Anal. Appl.*, **45** (1974), 43–46.

[67] A. Guessab and G. Schmeisser, Two Korovkin-type theorems in multivariate approximation, *Banach J. Math. Anal.*, **2** (2008), no. 2, 121–128.

[68] L. L. Helms, Introduction to Potential Theory, Wiley-Interscience, New York, 1969.

[69] W. Heping, Korovkin-type theorem and application, *J. Approx. Theory*, **132** (2005), 258–264.

[70] T. H. Hildebrandt and I. J. Schoenberg, On linear functional operations and the moment problem for a finite interval in one or several dimensions, *Ann. of Math.*, (2), **34** (1933), 317–328.

[71] T. Ishii and K. Izuchi, BKW-operators for Chebyshev systems, *Tokyo J. Math.*, **22** (1999), no. 2, 375–389.

[72] K. Izuchi, H. Takagi and S. Watanabe, Sequential Korovkin type theorems and weighted composition operators, *Acta Sci. Math. (Szeged)* **62** (1996), 161–174.

[73] K. Izuchi, H. Takagi and S. Watanabe, Sequential BKW-operators and function algebras, *J. Approx. Theory*, **85** (1996), no. 2, 185–200.

[74] K. Izuchi and S.-E. Takahasi, BKW-operators on the interval [0,1], *Rend. Circ. Mat. Palermo, Serie II*, Tomo XLVI (1997), 477–489.

[75] L. V. Kantorovich, Sur certains développements suivant les polynômes de la forme de S. Bernstein, I, II, *C. R. Acad. URSS*, (1930), 563–568, 595–600.

[76] K. Keimel and W. Roth, Ordered Cones and Approximation, *Lecture Notes in Math.* **1517**, Springer-Verlag, Berlin, 1992.

[77] P. P. Korovkin, Convergence of linear positive operators in the spaces of continuous functions (Russian), *Doklady Akad. Nauk. SSSR (N.S.)*, **90** (1953), 961–964.

[78] P. P. Korovkin, Linear Operators and Approximation Theory, translated from the Russian ed. (1959), Russian Monographs and Texts on Advances Mathematics and Physics, Vol. III, Gordon and Breach Publishers, Inc. New York, Hindustan Publ. Corp. (India), Delhi, 1960.

[79] H. E. Lomelí and C. L. García, Variation on a theorem of Korovkin, *Amer. Math. Monthly*, Vol. **113**, no. 10 (Oct. 2006), 774–750.

[80] G. G. Lorentz, Korovkin sets (Sets of convergence), Regional Conference at the University of California, Riverside, June 15-19, 1972, Center for Numerical Analysis, no. **58**, The University of Texas at Austin, 1972.

[81] G. G. Lorentz, Approximation of Functions, 2nd Ed. Chelsea Publ. Comp., New York, N. Y. 1986.

[82] G. G. Lorentz, Bernstein Polynomials, 2nd. Ed. Chelsea Publ. Comp., New York, N. Y. 1986.

[83] G. G. Lorentz, M. v. Golitschek and Y. Makovoz, Constructive Approximation, Advanced Problems, Grundlehren der mathematischen Wissenschaften, **304**, Springer- Verlag, Berlin, Heidelberg, 1996.

[84] N. I. Mahmudov, Korovkin-type theorems and applications, *Cent. Eur. J. Math.*, **7**(2) (2009), 348–356.

[85] C. A. Micchelli, Convergence of positive linear operators on C(X), *J. Approx. Theory*, **13** (1975), 305–315.

[86] G. M. Mirakjan, Approximation of continuous functions with the aid of polynomials, (Russian), *Dokl. Akad. Nauk SSSR*, **31** (1941), 201–205.

[87] T. Nishishiraho, A generalization of the Bernstein polynomials and limit of its iterations, *Sci. Rep. Kanazawa Univ.*, **19** (1974), no. 1, 1–7.

[88] T. Nishishiraho, Saturation of positive linear operators, *Tôhoku Math. J.*, (2) **28** (1976), no. 2, 239–243.

[89] T. Nishishiraho, The degree of convergence of positive linear operators, *Tôhoku Math. J.*, (2) **29** (1977), no. 1, 81–89.

[90] T. Nishishiraho, Saturation of bounded linear operators, *Tôhoku Math. J.*, **30** (1978), 69–81.

[91] T. Nishishiraho, Convergence of positive linear approximation processes, *Tôhoku Math. J.*, (2)**35** (1983), no. 3, 441–458.

[92] R. Păltănea, Approximation Theory Using Positive Operators, Birkhäuser, Boston, 2004.

[93] A. Pinkus, Weierstrass and approximation theory, *J. Approx. Theory*, **107** (2000), 1–66.

[94] A. Pinkus, Density in approximation theory, *Surveys in Approximation Theory*, **1** (2005), 1–45.

[95] T. Popoviciu, Asupra demonstratiei teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare [On the proof of Weierstrass' theorem with the help of interpolation polynomials], Lucrările Sesiunii generale ştiinţifice (2-12 Iunie 1950), 1664–1667, Editura Academiei R. P. Române, Bucuresti, 1951.

[96] T. Ransford, Potential Theory in the Complex Plane, London Mathematical Society Student Texts, **28**, Cambridge University Press, Cambridge, 1995.

[97] I. Raşa, Generalized Bernstein operators and convex functions, *Studia Univ. Babes-Bolyai Math.*, **33** (1988)(2), 36–39.

[98] I. Raşa, On the monotonicity of sequences of Bernstein-Schnabl operators, *Anal. Numr. Thor. Approx.*, **17** (1988)(2), 185–187.

[99] I. Raşa, Altomare projections and Lototsky-Schnabl operators, *Suppl. Rend. Circ. Mat. Palermo*, **33** (1993), 439–451.

[100] I. Raşa, On some properties of Altomare projections, *Conf. Sem. Mat. Univ. Bari*, **253** (1993), 17 p.

[101] I. Raşa, Feller semigroups, elliptic operators and Altomare projections, *Rend. Circ. Mat. Palermo*, Serie II, Suppl. **68** (2002), 133–155.

[102] I. Raşa and T. Vladislav, Altomare projections and Lipschitz constants, *Rend. Circ. Mat. Palermo*, (2) Suppl. **52** (1998), 749–756.

[103] M. Romito, Lototsky-Schnabl operators associated with a strictly elliptic differential operator and their corresponding Feller semigroups, *Mh. Math.* **126** (1998), 329–352.

[104] W. Roth, A Korovkin type theorem for weighted spaces of continuous functions, *Bull. Austral. Math. Soc.*, **55** (1997). 239–248.

[105] M. D. Rusk, Korovkin type theorems for finitely defined operators, Dissertation, University of California, Riverside, 1975.

[106] W. Schempp, A note on Korovkin test families, *Arch. Math. (Basel)*, **23** (1972), 521–524.

[107] R. Schnabl, Eine Verallgemeinerung der Bernsteinpolynome, *Math. Ann.*, **179** (1968), 74–82.

[108] R. Schnabl, Zur Approximation durch Bernstein Polynome auf gewissen Räumen von Wahrscheinlichkeitsmassen, *Math. Ann.*, **180** (1969), 326–330.

[109] B. Sendov and V. A. Popov, The Average Moduli of Smoothness, Pure and Applied Mathematics, John Wiley & Sons, 1988.

[110] Yu. A. Shashkin, The Mil'man-Choquet boundary and the theory of approximation, *Functional Anal. i Priložen.*, **1** (1967), no.2, 95–96.

[111] M. H. Stone, The generalized Weierstrass approximation theorem, *Math. Mag.*, **21** (1948), 167–184.

[112] K. R. Stromberg, An Introduction to Classical Real Analysis, Wadsworth, Inc. Belmont, California, 1981.

[113] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. Standards*, **45** (1950), 239–245.

[114] S.-E. Takahasi, Bohman-Korovkin-Wulbert operators on normed spaces, *J. Approx. Theory*, **72** (1993), 174–184.

[115] S.-E. Takahasi, (T,E)-Korovkin closures in normed spaces and BKW-operators, *J. Approx. Theory*, **82** (1995), 340–351.

[116] S.-E. Takahasi, Bohman-Korovkin-Wulbert operators from a function space into a commutative C*-algebra for special test functions, *Tôhoku Math. J.*, (2) **48** (1996), no. 1, 139–148.

[117] M. Uchiyama, Proofs of Korovkin's theorems via inequalities, *Amer. Math. Monthly*, **110**, no. 4 (Apr. 2003), 334–336.

[118] V. I. Volkov, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables (Russian), *Dokl. Akad. Nauk. SSSR (N.S.)*, **115** (1957), 17–19.

[119] K. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen, Sitzungberichte der Akademie zu Berlin, 1885, 633–639, 789–805.

[120] M. Wolff, On the theory of approximation by positive linear operators in vector lattices, in: Functional Analysis: Surveys and Recent Results (Proc. Conf. Paderborn, 1976), 73–87; North-Holland Math. Studies, Vol. 27; Notas de Mat., no. **63**, North-Holland, Amsterdam, 1977.

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