Müntz Type Theorems I

J. M. Almira

18 October, 2007

Abstract

In this paper, we concentrate our attention on the Müntz problem in the univariate setting and for the uniform norm.

MSC: 41-01, 41-02

Contents

1 Introduction

In his seminal paper [3] of 1912, the Russian mathematician S. N. Bernstein (one of the greatest approximation theorists of the last century) asked under which conditions on an increasing sequence

Surveys in Approximation Theory
Volume 3, 2007. pp. 152–194.

ISSN 1555-578X

$\Lambda = (0 = \lambda_0 < \lambda_1 < \cdots)$ one can guarantee that the vector space

$$\Pi(\Lambda) := \operatorname{span}\{x^{\lambda_k} : k = 0, 1, \ldots\} \tag{1}$$

spanned by the monomials x^{λ_k} is a dense subset of $C[0,1]$. He specifically proved that the condition

$$\sum_{\lambda_k > 0} \frac{1 + \log \lambda_k}{\lambda_k} = \infty$$

is necessary and the condition

$$\lim_{k\to\infty} \frac{\lambda_k}{k \log k} = 0$$

is sufficient, and conjectured that a necessary and sufficient condition to have $\overline{\Pi(\Lambda)} = C[0,1]$ is

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

This conjecture was proved by Müntz [25] in 1914. In his proof, he used the method of Gram determinants to compute the distance of $x^{\lambda}$from $\Pi(\lambda_0, \ldots, \lambda_n))_2$ in the $L^2(0,1)$–metric. The determinants that appear in this problem are of the form

$$\det(1/(1 + a_i + a_j))_{0 \le i,j \le n},$$

and their explicit expression was obtained in the 19th century by Cauchy.

For the sake of clarity, let us give a precise formulation of the classical Müntz Theorem.

Theorem 1 (Müntz, 1914) *Let $\Lambda = (\lambda_i)_{k=0}^{\infty}$, $0 = \lambda_0 < \lambda_1 < \cdots$, be an increasing sequence of non-negative real numbers. Then $\Pi(\Lambda) = \operatorname{span}\{x^{\lambda_k} : k = 0, 1, \ldots\}$, the* **Müntz space associated to** *Λ, is a dense subset of $C[0,1]$ if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty. \tag{2}$$

This is a beautiful theorem because it connects a topological result (the density of a certain subset of a functional space) with an arithmetical one (the divergence of a certain harmonic series). Many people might well have been drawn to this result because of its beauty. Another reason to be interested in Müntz' theorem is that the original result not only solves a nice problem but also opens the door to many new interesting questions. For example, one is tempted to change the space of continuous functions $C[0,1]$ to other function spaces such as $L^p(a,b)$, or to consider the analogous problem in several variables, on complex domains, on intervals away from the origin, for more general exponent sequences, for polynomials with integral coefficients, etc. As a consequence, many proofs (and generalizations) of the theorem have been produced.

In this paper, we concentrate our attention on the Müntz problem in the univariate setting and for the uniform norm. Moreover, we do not include any results about the rate of convergence to zero of the errors of best (uniform) approximation using Müntz polynomials. On the other hand, we do provide proofs in great detail, and we promise to write a second paper where we plan to treat several advanced topics, including the Müntz Theorem for complex domains, Müntz-Jackson theorems, Müntz type theorems for approximation with polynomials with integral coefficients, the p-adic Müntz theorems, and the Müntz Theorem for rational functions.

Let us return to a discussion of this paper. We devote Section 2 to the classical Müntz Theorem. In particular, we give several proofs of this result, showing how the Müntz problem is connected to many apparently different branches of mathematics. In Section 3, we focus our attention on the so called Full Müntz Theorem, i.e., we study the density of $\operatorname{span}\{x^{\lambda_k}\}_{k=0}^{\infty}$ in $C(K)$, where K denotes a compact subset of $[0,\infty)$, for arbitrary sequences of exponents $(\lambda_k)_{k=0}^{\infty}$ and characterize the (uniform) closure of $\operatorname{span}\{x^{\lambda_k}\}_{k=0}^{\infty}$ in the nondense case.

2 The classical Müntz theorem

2.1 Müntz theorem: the original proof with a modification by O. Szász

The original proof by Müntz of Theorem 1, and that which remains essentially the standard proof that you may find in many introductory textbooks on approximation theory, is based on an estimation of the errors $E(x^q,\Pi(\Lambda_n))$, where $\Lambda_n := (\lambda_k)_{k=0}^{n}$ and

$$E(x^q,\Pi(\Lambda_n)) := \inf_{p\in\Pi(\Lambda_n)} \|x^q - p(x)\|_{[0,1]}$$

is the error of best approximation, with respect to the uniform norm in $[0,1]$, to x^q when we take as the set of approximants the space $\Pi(\Lambda_n)$. It is clear that $\Pi(\Lambda)$ is dense in $C[0,1]$ if and only if for all $q\in\mathbb{N}$, $E(x^q,\Pi(\Lambda_n))$ converges to zero as n tends to infinity (this is a consequence of the Weierstrass Approximation Theorem). So, how do we produce a reasonable estimate for $E(x^q,\Pi(\Lambda_n))$?

If we use the $L_2(0,1)$-norm, then we can explicitly compute the errors

$$E(x^q,\Pi(\Lambda_n))_2 = \inf_{p\in\Pi(\Lambda_n)} \|x^q - p(x)\|_2$$

since $L_2(0,1)$ is a Hilbert space. In fact, if we denote by $G(f_1,\dots,f_n)$ the Gram determinant associated with a linearly independent sequence $(f_1,\dots,f_n)$ of elements in a Hilbert space H with inner product $(\cdot,\cdot)$,

$$G(f_1,\dots,f_n) = \det\begin{pmatrix} (f_1,f_1) & \cdots & (f_1,f_n) \\ \vdots & \ddots & \vdots \\ (f_n,f_1) & \cdots & (f_n,f_n)\end{pmatrix},$$

then it is well known [10, Theor. 8.7.4.] that

$$E(g,V_n)_H = \inf_{v\in V_n}\|g-v\|_H = \sqrt{\frac{G(g,f_1,\dots,f_n)}{G(f_1,\dots,f_n)}}$$

holds for all $g\notin V_n = \operatorname{span}\{f_1,\dots,f_n\}$. From this follows (for $\lambda_0 > -1/2$) the formula

$$E(x^q,\Pi(\Lambda_n))_2 = \frac{1}{\sqrt{2q+1}}\prod_{k=0}^{n}\frac{|q-\lambda_k|}{q+\lambda_k+1}$$

for all $q > -1/2$, since

$$G(x^{\lambda_1},\ldots,x^{\lambda_n}) = \det\left(\frac{1}{\lambda_i+\lambda_j+1}\right)_{0\le i,j\le n} = \frac{\prod_{i>j}^n(\lambda_i-\lambda_j)^2}{\prod_{i,j=1}^n(\lambda_i+\lambda_j+1)}.$$

The Cauchy determinant $\det\left(\frac{1}{\lambda_i+\lambda_j+1}\right)_{0\le i,j\le n}$ is a particular case of $\det\left(\frac{1}{a_i+b_j}\right)_{1\le i,j\le n}$ and the argument to compute a closed expression for this determinant is quite similar to the classical argument to compute a Vandermonde determinant. Thus, it consists in considering both sides of the identity

$$\prod_{i=1}^n\prod_{j=1}^n(a_i+b_j)\det\left(\frac{1}{a_i+b_j}\right)_{1\le i,j\le n} = \prod_{i=1}^n\prod_{j=1}^{i-1}(a_i-a_j)(b_i-b_j) \tag{3}$$

as polynomials in the variables a_i, b_j, each time taking into account just one of these variables, and using the zero properties of algebraic polynomials of one variable to prove that both expressions are the same. A detailed proof of (3) can be found in [18, page. 74] or [10, page 268].

From here it is not difficult to prove that

$$\lim_{n\to\infty} E(x^q,\Pi(\Lambda_n))_2 = 0 \text{ for all } q\in\mathbb{N} \quad \text{if and only if} \quad \sum_{k=1}^{\infty}\frac{1}{\lambda_k}=\infty.$$

Hence $\Pi(\Lambda)$ is dense in $L_2(0,1)$ if and only if $\sum_{k=1}^\infty 1/\lambda_k=\infty$. This clearly implies the necessity of the condition $\sum_{k=1}^\infty 1/\lambda_k=\infty$ to guarantee the density of $\Pi(\Lambda)$ in $C[0,1]$.

In order to guarantee the sufficiency of condition (2) to the claim that $\overline{\Pi(\Lambda)}=C[0,1]$, Müntz used Fejér's theorem on summation of Fourier series, but his proof is too complicated to be reproduced here. In 1916 Otto Szász extended Müntz's theorem in the sense that he was able to prove the result also for certain special sequences of complex numbers $(\lambda_k)_{k=0}^\infty$ as exponents (see [33]). Furthermore, he simplified the final step of Müntz's proof, showing that the result in $L^2(0,1)$ implies the same result in $C[0,1]$. This follows from the inequality

$$\begin{aligned}\left|x^q-\sum_{k=1}^n a_kx^{\lambda_k}\right| &= \left|\int_0^x\left(qt^{q-1}-\sum_{k=1}^n a_k\lambda_kt^{\lambda_k-1}\right)\mathrm{d}t\right| \\ &\le \int_0^1\left|qt^{q-1}-\sum_{k=1}^n a_k\lambda_kt^{\lambda_k-1}\right|\mathrm{d}t \\ &\le \left[\int_0^1\left|qt^{q-1}-\sum_{k=1}^n a_k\lambda_kt^{\lambda_k-1}\right|^2\mathrm{d}t\right]^{1/2} \\ &= \left\|qx^{q-1}-\sum_{k=1}^n a_k\lambda_kx^{\lambda_k-1}\right\|_{L^2[0,1]}\end{aligned}$$

which holds for all $x\in[0,1]$. In other words,

$$\left\|x^q-\sum_{k=1}^n a_kx^{\lambda_k}\right\|_{\mathbf{C}[0,1]} \le \left\|qx^{q-1}-\sum_{k=1}^n a_k\lambda_kx^{\lambda_k-1}\right\|_{L^2[0,1]}. \tag{4}$$

□

For historical reasons, we include here (without proof) the precise statement of Szász's theorem.

Theorem 2 (Szász, 1916) *Let*

$$C([0,1],\mathbb{C}) := \{f : [0,1] \to \mathbb{C} : f \text{ continuous}\}$$

be the space of continuous complex-valued functions defined on $[0,1]$ *and assume that the Müntz polynomials have complex coefficients and complex exponents, so that for* $\Lambda = (\lambda_k)_{k=0}^{\infty} \subset \mathbb{C}$ *we set* $\Pi_{\mathbb{C}}(\Lambda) := \mathrm{span}_{\mathbb{C}}\{x^{\lambda_k}\}_{k=0}^{\infty}$. *If* $\lambda_0 = 0$ *and* $\mathrm{Re}(\lambda_k) > 0$ *for all* $k > 0$, *then* $\Pi_{\mathbb{C}}(\Lambda)$ *is a dense subset of* $\mathrm{C}([0,1],\mathbb{C})$ *whenever*

$$\sum_{k=1}^{\infty} \frac{\mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} = \infty. \tag{5}$$

Moreover, if

$$\sum_{k=1}^{\infty} \frac{1+\mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} < \infty,$$

then $\Pi_{\mathbb{C}}(\Lambda)$ *is not a dense subset of* $\mathrm{C}([0,1],\mathbb{C})$. *In particular, if*

$$\liminf_{k\to\infty} \mathrm{Re}(\lambda_k) > 0$$

then $\Pi_{\mathbb{C}}(\Lambda)$ *is a dense subset of* $\mathrm{C}([0,1],\mathbb{C})$ *if and only if* (5) *holds.*

Note that Szász's theorem is not conclusive for all cases. For example, the sequence $\lambda_k = \frac{1}{k} + \mathrm{i}\sqrt{k}$ satisfies

$$\sum_{k=1}^{\infty} \frac{\mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1+\mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} = \infty.$$

2.2 M. von Golitschek's constructive proof of the Müntz theorem

In this subsection, we present another proof of the fact that condition (2) is sufficient if the relation $\lambda_k \to \infty$ holds. To do this, we follow a very nice proof published by M. von Golitschek in [17], which has two distinct advantages with respect to the majority of known proofs of the same result. It is both constructive and short.

The idea is to define, for each $q > 0$, a concrete sequence of approximants to x^q, $(P_n)_{n=0}^{\infty} \subset \Pi(\Lambda)$ and to prove that $Q_n(x) = x^q - P_n(x)$ converges to zero uniformly on $[0,1]$. So, we set $Q_0(x) := x^q$, and, for $n = 1, 2, \ldots$, if we already know that

$$Q_{n-1}(x) = x^q - \sum_{k=1}^{n-1} a_{k,n-1} x^{\lambda_k}$$

with some coefficients $a_{k,n-1}$, then let

$$\begin{aligned} Q_n(x) &:= (\lambda_n - q)x^{\lambda_n} \int_x^1 Q_{n-1}(t)t^{-(1+\lambda_n)}\,\mathrm{d}t \\ &= (\lambda_n - q)x^{\lambda_n} \int_x^1 \left(t^q - \sum_{k=1}^{n-1} a_{k,n-1}t^{\lambda_k}\right) t^{-(1+\lambda_n)}\,\mathrm{d}t \\ &= (\lambda_n - q)x^{\lambda_n} \int_x^1 \left(t^{q-(1+\lambda_n)} - \sum_{k=1}^{n-1} a_{k,n-1}t^{\lambda_k-(1+\lambda_n)}\right) \mathrm{d}t \\ &= (\lambda_n - q)x^{\lambda_n} \left[\frac{t^{q-\lambda_n}}{q-\lambda_n} - \sum_{k=1}^{n-1} a_{k,n-1}\frac{t^{\lambda_k-\lambda_n}}{(\lambda_k-\lambda_n)}\right]_x^1 \\ &=: x^q - \sum_{k=1}^{n} a_{k,n}x^{\lambda_k}, \end{aligned}$$

hence $P_n(x) = \sum_{k=1}^n a_{k,n}x^{\lambda_k}$.

We only need to prove that $\|Q_n\|_{C[0,1]}$ converges to zero as $n \to \infty$. Now $\|Q_0\|_{C[0,1]} = 1$, and for all $n \in \mathbb{N}$ we get from the inequality

$$\lambda x^\lambda(1-x) < 1 \qquad \text{for all } x \in (0,1) \text{ and } \lambda > 0$$

that

$$\|Q_n\|_{C[0,1]} \le \left|1 - \frac{q}{\lambda_n}\right| \|Q_{n-1}\|_{C[0,1]}\,.$$

Hence

$$\|Q_n\|_{C[0,1]} \le \prod_{k=0}^{n} \left|1 - \frac{q}{\lambda_n}\right| \to 0, \qquad n \to \infty.$$

□

2.3 Measure-theoretic focus

The classical Müntz Theorem can be formulated in terms of measures. We explain here the way in which this formulation is attained and we give a proof, due to W. Feller [15], of the 'only if' part of the result based on measure theoretical considerations. For pedagogical reasons, we postpone Feller's proof of the 'if' part to the next subsection.

Let us assume that $\Lambda = (\lambda_k)_{k=1}^\infty$ is an increasing sequence of positive real numbers and, to avoid problems with the origin, let us also assume in this subsection that the functions we want to approximate vanish at the origin. In this case, we can rephrase the classical Müntz Theorem as follows: *The space* $\Pi(\Lambda)$ *is a dense subset of* $C_0[0,1] := C[0,1] \cap \{f : f(0) = 0\}$ *if and only if* $\sum_{k=1}^\infty 1/\lambda_k = \infty$. When dealing with the problem of the density of certain linear subspaces of a Banach space, it is a quite natural to use the Hahn-Banach Theorem in the following way: *if* Y *is* *a closed subspace of the Banach space X, then $Y \neq X$ if and only if there exists a bounded linear functional $L \in X^*$ such that $L \neq 0$ and $L|_Y = 0$.* Thus, taking $X = C_0[0,1]$ and $Y = \overline{\Pi(\Lambda)}$, we have that $\overline{\Pi(\Lambda)} \neq C_0[0,1]$ if and only if there exists an $L \in C_0^*[0,1] \setminus \{0\}$ satisfying $L(x^{\lambda_k}) = 0$ for all $k = 1, 2, \ldots$. The dual of $C_0[0,1]$ is characterized (by the Riesz Representation Theorem) as follows: $L \in C_0^*[0,1]$ if and only if

$$L(f) = \int_0^1 f(t)\,\mathrm{d}\mu(t) \tag{6}$$

for a certain finite signed Borel measure μ on $(0,1]$. Moreover, we know (by the Weierstrass Approximation Theorem) that algebraic polynomials that vanish at 0 form a dense subspace of $C_0[0,1]$. Hence a new formulation of the classical Müntz Theorem is given as follows.

Theorem 3 (Classical Müntz Theorem in terms of Measures) *Let us assume that $(\lambda_k)_{k=1}^\infty$ is an increasing sequence of positive real numbers and let us define, for each finite signed Borel measure μ supported on $(0,1]$, the function*

$$f(z) := \int_0^1 t^z\,\mathrm{d}\mu(t). \tag{7}$$

Then the following claims are equivalent:

(*a*) $\sum_{k=1}^\infty \frac{1}{\lambda_k} < \infty$

(*b*) *There exists a (non-zero) finite signed Borel measure μ on $(0,1]$ such that $f(\lambda_k) = 0$ for all $k \geq 1$ (where f is given by (7)).*

Let us prove that $(a) \Rightarrow (b)$. Given the measure μ we make the change of variable $t = \mathrm{e}^{-s}$ which transforms the interval $(0,1]$ onto the interval $[0,\infty)$, the measure μ into another measure m on $[0,\infty)$ and the expression (7) to the new formula:

$$f(z) = \int_0^\infty \mathrm{e}^{-zs}\,\mathrm{d}m(s). \tag{8}$$

Hence we should prove that under condition (a) there exists a finite signed Borel measure m supported on $[0,\infty)$ such that the function f given by (8) is not identically zero but satisfies $f(\lambda_k) = 0$ for all $k \geq 1$. We present a proof whose order is reversed: we first define a function f that satisfies $f(\lambda_k) = 0$ for all $k \geq 1$ and then we prove that this function admits an expression of the form (8).

Set, with $\eta > 0$,

$$f(t) := \frac{1}{(1+\eta+t)^2} \prod_{k=1}^\infty \frac{\lambda_k - t}{\lambda_k + 2\eta + t}\,.$$

Obviously, (a) guarantees the convergence of the infinite product defining f. This convergence is uniform and absolute on compact subsets of $\mathbb{C} \setminus \{-\lambda_k - 2\eta\}_{k=1}^\infty$. In particular, f is well defined on $[0,\infty)$ and vanishes on the sequence $(\lambda_k)_{k=1}^\infty$.

Let us define

$$f_0(t) := \frac{1}{(1+\eta+t)^2} \qquad \text{and} \qquad f_k(t) := \frac{\lambda_k - t}{\lambda_k + 2\eta + t} f_{k-1}(t), \text{ for } k = 1, 2, \ldots.$$

It is clear that

$$f_0(t) = \int_0^\infty s\mathrm{e}^{-(1+t+\eta)s}\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-ts}u_0(s)\,\mathrm{d}s,$$

where $u_0(s) := s\mathrm{e}^{-(1+\eta)s}$. Let us assume that, for all $k < n$, the function f_k admits an expression of the form

$$f_k(t) = \int_0^\infty \mathrm{e}^{-ts}u_k(s)\,\mathrm{d}s$$

with $u_k(0) = 0$ (which we know to be true for $k = 0$). We will prove that this is then also the case for $k = n$. In fact, taking into account the recursive definition of f_n,

$$f_n(t) = \frac{\lambda_n - t}{\lambda_n + 2\eta + t} f_{n-1}(t) = \frac{\lambda_n - t}{\lambda_n + 2\eta + t}\int_0^\infty \mathrm{e}^{-ts}u_{n-1}(s)\,\mathrm{d}s,$$

and, integrating by parts, we note that

$$tf_{n-1}(t) = \int_0^\infty \mathrm{e}^{-ts}u'_{n-1}(s)\,\mathrm{d}s,$$

and

$$f_n(t) = \int_0^\infty \mathrm{e}^{-ts}u_n(s)\,\mathrm{d}s,$$

where u_n is the solution of the initial value problem

$$\begin{cases} u'_n + u'_{n-1} &= \lambda_n u_{n-1} - (\lambda_n + 2\eta)u_n \\ u_n(0) &= 0. \end{cases}$$

Moreover, it is possible to check that under these conditions the solution u_n satisfies $\lim_{t\to\infty} u_n(t) = 0$. Let us now multiply both sides of the differential equation defining u_n by $(u_n + u_{n-1})$. We get

$$\begin{aligned} \frac{1}{2}[(u_n + u_{n-1})^2]' &= (u'_n + u'_{n-1})(u_n + u_{n-1}) = (u_n + u_{n-1})(\lambda_n u_{n-1} - (\lambda_n + 2\eta)u_n) \\ &= \lambda_n(u_{n-1}^2 - u_n^2) - \eta(2u_n^2 + 2u_n u_{n-1}) \\ &\le (\lambda_n + \eta)(u_{n-1}^2 - u_n^2). \end{aligned}$$

Hence, for all $h \in (0,\infty)$,

$$\frac{1}{2}(u_n(h) + u_{n-1}(h))^2 = \int_0^h \frac{1}{2}[(u_n + u_{n-1})^2]'\,\mathrm{d}t \le (\lambda_n + \eta)\int_0^h (u_{n-1}^2(t) - u_n^2(t))\,\mathrm{d}t,$$

therefore

$$\int_0^\infty u_n^2(t)\,\mathrm{d}t \le \int_0^\infty u_{n-1}^2(t)\,\mathrm{d}t, \qquad n = 1, 2, 3, \ldots.$$

Taking into consideration the convergence of f_n to f and the weak sequential compactness of the unit ball of $L^2(0,\infty)$, we conclude that there exists a function $u \in L^2(0,\infty)$ such that

$$f(t) = \int_0^\infty \mathrm{e}^{-ts}u(s)\,\mathrm{d}s \qquad \text{for all} \quad t \ge 0.$$

Moreover, the same arguments we have used for f should work with $f^*(t) := f(t-\eta)$ (choose $\lambda_k^* = \lambda_k + \eta$ and $\eta^* = 0$ instead of the old values λ_k and η). Hence

$$f^*(t) = \int_0^\infty \mathrm{e}^{-ts} u^*(s)\,\mathrm{d}s \qquad \text{for all} \quad t \geq 0,$$

with $u^*(s) := \mathrm{e}^{\eta s} u(s) \in L^2(0,\infty)$. Of course, this implies that

$$\int_0^\infty |u(t)|\,\mathrm{d}t < \infty,$$

so that f is of the form (8) with m the signed measure that has density u.

2.4 Two proofs of the 'if' part based on the use of divided differences

One of the first things we observe when studying the Müntz Theorem is that the necessity of condition (2) and its sufficiency are two facts of a quite different nature, so that the proofs of the 'if' part and the 'only if' part of the Müntz Theorem are usually independent. Hence it is tempting to present new proofs for each one of these parts in terms of one's own interest in the subject.

In this subsection, we will explain two proofs of the 'if' part of the classical Müntz Theorem, both based on the use of divided differences.

The first proof is due to W. Feller [15]. It is a natural continuation of the proof given in the previous subsection. It uses the strong connection between divided differences and completely monotone functions and certain results from functional analysis and measure theory. The second proof, by Hirschman and Widder [20] and Gelfond [16], uses divided differences to construct an adequate generalization of Bernstein polynomials with the property that the new polynomials only depend on the powers $\{x^{\lambda_k}\}_{k=1}^\infty$.

First, we would like to recall the definition of divided differences. Given a function f and a subset $\{x_k\}_{k=0}^\infty$ of its domain, we define the divided differences of f with respect to the nodes $\{x_k\}_{k=0}^\infty$ recursively:

$$f[x_k] := f(x_k), \text{ and } f[x_{i_0},\dots,x_{i_n}] := \frac{f[x_{i_0},x_{i_1},\dots,x_{i_{n-1}}] - f[x_{i_1},x_{i_2},\dots,x_{i_n}]}{x_{i_0}-x_{i_n}}.$$

These numbers can be characterized in many ways. One of their main properties is that they are the coefficients in the Newton representation of the Lagrange interpolation polynomial of f at the nodes $\{x_0,x_1,\dots,x_n\}$. More precisely, if $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$ is the unique polynomial of degree $\leq n$ that satisfies $P(x_k) = f(x_k)$, $k = 0,1,\dots n$, then

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + \cdots + f[x_0,x_1,\dots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1}), \tag{9}$$

and this characterizes the values

$$f[x_0], f[x_0,x_1], \dots, f[x_0,\dots,x_n].$$

An easy consequence of (9) is that, for all x, the error $R_n(x) := f(x) - P_n(x)$ is given by

$$R_n(x) = f[x,x_0,x_1,\dots,x_n](x-x_0)(x-x_1)\cdots(x-x_n). \tag{10}$$

Moreover, for functions f in $C^{(n)}(I)$, where $I=[\min\{x_i\}_{i=0}^n,\max\{x_i\}_{i=0}^n]$, taking into account that $R_n(x_i)=0$ for $i=0,1,\cdots,n$, we conclude that $R_n'(x)$ has at least n zeros in the interval I, $R_n''(x)$ has at least $n-1$ zeros therein, etc., so that $R_n^{(n)}(\tau)=0$ for a certain value $\tau\in I$. Now,

$$R_n^{(n)}(\tau)=f^{(n)}(\tau)-n!f[x_0,x_1,\cdots,x_n],$$

so that for a function f that is sufficiently many times differentiable, the divided differences satisfy

$$f[x_0,x_1,\ldots,x_n]=\frac{1}{n!}f^{(n)}(\tau) \tag{11}$$

for a certain $\tau\in I=[\min\{x_i\}_{i=0}^n,\max\{x_i\}_{i=0}^n]$.

Finally, it is also useful to note that:

$$f[x_0,x_1,\ldots,x_n]=\sum_{k=0}^n\frac{f(x_k)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}. \tag{12}$$

This follows from the fact (see (9)) that $f[x_0,\ldots,x_n]$ is the coefficient of x^n in the power form of the interpolating polynomial, and the Lagrange expression of this polynomial.

Feller's Proof of the 'if' part of Classical Müntz Theorem.

Taking into account the decomposition properties of signed measures, we see that in order to prove $(b)\Rightarrow(a)$ in Theorem 3 (which corresponds to the 'if' part of the classical Müntz Theorem) it suffices to prove the assertion for nonnegative measures μ. Now, we note that functions $f:(0,\infty)\to\mathbb{R}$ admitting an expression of the form (7) for a certain nonnegative measure μ are completely monotone on $(0,\infty)$. This means that they satisfy the inequalities

$$(-1)^nf^{(n)}(t)\geq 0 \text{ for all } t>0 \text{ and all } n=0,1,2,\ldots.$$

(Indeed, it is a well known result by S.N. Bernstein [4] that f being completely monotone on $(0,\infty)$ and of the form (7) for a certain nonnegative measure μ are equivalent claims). Let us now assume that (a) is not true, and suppose first that $(\lambda_k)\uparrow\infty$. Under these conditions we can use the following theorem:

Theorem 4 (Feller, 1968) *Let us assume that* $0<\lambda_0<\lambda_1<\lambda_2<\cdots$ *with* $\lambda_n\to\infty$, *and* $\sum_{n=0}^\infty 1/\lambda_n=\infty$, *and let* $f:(0,\infty)\to\mathbb{R}$ *be a completely monotone function. Then*

$$f(t)=\sum_{n=0}^\infty f[\lambda_0,\lambda_1,\ldots,\lambda_n](t-\lambda_0)(t-\lambda_1)\cdots(t-\lambda_{n-1}), \tag{13}$$

where the series is absolutely convergent for all $t>0$.

This proves that if $f(\lambda_k)=0$ for all k, then $f(k)=0$ for all k, and this means that the integral of any polynomial against μ is zero, hence μ is zero, so (b) is also false.

Finally, we can use Morera's theorem to prove that $f(z)=\int_0^1 t^z\,\mathrm{d}\mu(t)$ is holomorphic on the half plane $\{z:\mathrm{Re}z>0\}$, so that the well known principle of identity shows that f is completely determined by its values on any increasing bounded sequence $(\lambda_k)_{k=0}^\infty$. This ends the proof of $(b)\Rightarrow(a)$ in Theorem 3.

Proof of of Theorem 4. It follows from the fact that f is completely monotone and (11) that

$$(-1)^n f[\lambda_0, \lambda_1, \ldots, \lambda_n] \geq 0 \text{ for all } n \geq 0.$$

Let us now assume that $t \in [0, \lambda_0)$. Then all terms of the series

$$\sum_{n=0}^{\infty} f[\lambda_0, \lambda_1, \ldots, \lambda_n](t-\lambda_0)(t-\lambda_1)\cdots(t-\lambda_{n-1}) \tag{14}$$

are positive. Setting

$$P_n(t) := \sum_{k=0}^{n} f[\lambda_0, \lambda_1, \ldots, \lambda_k](t-\lambda_0)(t-\lambda_1)\cdots(t-\lambda_{k-1})$$

and $R_n(t) := f(t) - P_n(t)$, we obtain

$$R_n(t) = f[t, \lambda_0, \lambda_1, \ldots, \lambda_n](t-\lambda_0)(t-\lambda_1)\cdots(t-\lambda_n),$$

so that

$$P_0(t) \leq P_1(t) \leq \cdots \leq f(t) \tag{15}$$

and there exists a function $\alpha(t)$ such that

$$R_n(t) \downarrow \alpha(t).$$

We want to show that $\alpha(t) = 0$. Now, for $0 < s < \lambda_0$, we have that

$$0 \leq R_n(s) \leq f[s, \lambda_0, \lambda_1, \ldots, \lambda_n](-1)^{n+1}\lambda_0\lambda_1\cdots\lambda_n,$$

so that

$$\frac{s}{\lambda_n}\alpha(s) \leq \frac{s}{\lambda_n}R_n(s) \leq f[s, \lambda_0, \lambda_1, \ldots, \lambda_n](-1)^{n+1}s\lambda_0\lambda_1\cdots\lambda_{n-1}. \tag{16}$$

Now, the right side of (16) is the nth term of the series (13) evaluated at $t = 0$ when the point s is added to the sequence $(\lambda_k)_{k=0}^{\infty}$. It follows that the series $\sum_{k=0}^{\infty} \alpha(s)/\lambda_k$ is convergent, which is consistent with our hypotheses on the sequence $(\lambda_k)_{k=0}^{\infty}$ only if $\alpha(s) = 0$. This proves (13) for all $t \in (0, \lambda_0)$. The same argument works for $t \in (\lambda_{2k-1}, \lambda_{2k})$ except that the inequalities (15) can be asserted only for $n \geq 2k$. On the intervals $(\lambda_{2k}, \lambda_{2k+1})$ the inequalities are reversed. This ends the proof, since $\lim_{k\to\infty} \lambda_k = \infty$ implies that all points $t > 0$ have been already considered. $\Box$

Hirschman-Widder's and Gelfond's proof of the 'if' part of the Müntz Theorem

The most famous proof of the Weierstrass Approximation Theorem is based on the use of the Bernstein polynomials:

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Thus, it was an interesting (and difficult!) problem to find out whether a suitable generalization of the Bernstein polynomials would give a new proof of the Müntz Theorem. This question was solved in the positive by Hirschman and Widder [20] in 1949. Moreover, their proof was modified and extended by A. O. Gelfond [16] in 1958 and included by G. G. Lorentz in his book on Bernstein polynomials [23, pp. 46–47]. In this subsubsection, we follow the discussion in Lorentz's monograph.

The polynomials that will play the role of Bernstein polynomials are defined in terms of the sequence of exponents $(\lambda_k)_{k=0}^{\infty}$ as follows: Given $n, k \in \mathbb{N}$ such that $k \leq n$, we set

$$g_{n,k}(x) := (-1)^{n-k}\lambda_{k+1}\cdots\lambda_n \sum_{i=k}^{n} \frac{x^{\lambda_i}}{(\lambda_i-\lambda_k)\cdots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_n)} \tag{17}$$

and, given $f \in C[0,1]$, we set

$$\eta_{n,k} := \left[(1-\frac{\lambda_1}{\lambda_{k+1}})\cdots(1-\frac{\lambda_1}{\lambda_n})\right]^{\frac{1}{\lambda_1}} \quad \text{for } \ 0 \leq k < n, \quad \text{and} \quad \eta_{n,n} := 1,$$

and

$$B_n^{\Lambda}(f)(x) := \sum_{k=0}^{n} f(\eta_{n,k}) g_{n,k}(x). \tag{18}$$

We can now state and prove the main result:

Theorem 5 (Hirschman-Widder [20], and Gelfond [16]) *Let $f \in C[0,1]$ and assume that*

$$0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

Then

$$\lim_{n\to\infty} \|B_n^{\Lambda}(f) - f\|_{[0,1]} = 0.$$

Proof. Let us consider, for the function $f(z) = x^z$, its divided differences with respect to the nodes $(\lambda_k)_{k=1}^{\infty}$. It is clear that

$$g_{n,k}(x) = (-1)^{n-k}\lambda_{k+1}\cdots\lambda_n f[\lambda_k, \lambda_{k+1}, \ldots, \lambda_n]. \tag{19}$$

In particular, this implies that

$$g_{n,k}(x) = (-1)^{n-k}\lambda_{k+1}\cdots\lambda_n \frac{1}{2\pi \mathrm{i}} \int_C \frac{x^z \, \mathrm{d}z}{(z-\lambda_k)\cdots(z-\lambda_n)},$$

where C is any simple closed curve that contains the nodes $(\lambda_i)_{i=k}^{n}$ in its interior $\mathrm{Int}(C)$, and such that $f(z) = x^z$ is holomorphic in a neighborhood of $\mathrm{Int}(C) \cup C$. Now we prove a few technical results:

Lemma 6 *The polynomials $\{g_{n,k}\}_{k=0}^{n}$ form a partition of unity on $[0,1]$.*

Proof. Taking into account (19) and (11), we get

$$g_{n,k}(x) = \frac{\lambda_{k+1}\cdots\lambda_n}{(n-k)!} x^{\tau}(-\log x)^{n-k} \geq 0.$$

Moreover, taking into account the identity (easily checked by induction on n)

$$\frac{1}{z} = \frac{1}{z-\lambda_n} - \frac{\lambda_n}{(z-\lambda_{n-1})(z-\lambda_n)} + \cdots + (-1)^n \frac{\lambda_1 \cdots \lambda_n}{z(z-\lambda_1)\cdots(z-\lambda_n)}$$

and multiplying by $x^z/(2\pi\mathrm{i})$ and integrating along C, we get

$$1 = \frac{1}{2\pi\mathrm{i}} \int_C \frac{x^z}{z}\,\mathrm{d}z = \sum_{k=0}^{n} g_{n,k}(x),$$

which is what we wanted to prove. □

Lemma 7 *The following identities hold:*

$$x^{\lambda_1} = \sum_{k=0}^{n} \eta_{n,k}^{\lambda_1} g_{n,k}(x) \tag{20}$$

and

$$x^{2\lambda_1} = \sum_{k=1}^{n} \eta_{n,k}^{*} g_{n,k} + \frac{1}{2}\eta_{n,0}^{*} g_n^{*}(x), \tag{21}$$

where

$$\eta_{n,k}^{*} := \left(1 - \frac{2\lambda_1}{\lambda_{k+1}}\right)\cdots\left(1 - \frac{2\lambda_1}{\lambda_n}\right)$$

and g_n^{*} is the polynomial $g_{n+1,1}$ associated with the nodes $\lambda_0 = 0, \lambda_1, 2\lambda_1, \lambda_2, \ldots, \lambda_n$ (taken in increasing order).

Proof. The idea is analogous to that in the previous lemma. We write $1/(z-\lambda_1)$ in a different way (this is again easy to check by induction on n):

$$\frac{1}{z-\lambda_1} = \frac{1}{z-\lambda_n} - \frac{\lambda_n - \lambda_1}{(z-\lambda_{n-1})(z-\lambda_n)} + \cdots + (-1)^{n-1} \frac{(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1)}{(z-\lambda_1)\cdots(z-\lambda_n)}$$

and multiplying by $x^z/(2\pi\mathrm{i})$ and integrating over C, we get (20). To prove (21), we use the same arguments but based on the formula

$$\frac{1}{z-2\lambda_1} = \frac{1}{z-\lambda_n} - \frac{\lambda_n - 2\lambda_1}{(z-\lambda_{n-1})(z-\lambda_n)} + \cdots + (-1)^n \frac{\lambda_1(\lambda_2-2\lambda_1)\cdots(\lambda_n-2\lambda_1)}{(z-\lambda_1)(z-2\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)}.$$

□

Let us continue with the proof of Theorem 5. Consider the functions T_n given by

$$T_n(x) := \sum_{k=0}^{n} (x^{\lambda_1} - \eta_{n,k}^{\lambda_1})^2 g_{n,k}(x).$$

Then

$$\begin{aligned}T_n(x) &= x^{2\lambda_1} - 2x^{\lambda_1}\sum_{k=0}^{n}\eta_{n,k}^{\lambda_1}g_{n,k}(x) + \sum_{k=0}^{n}\eta_{n,k}^{2\lambda_1}g_{n,k} \\ &= \sum_{k=1}^{n}(\eta_{n,k}^{2\lambda_1} - \eta_{n,k}^{*})g_{n,k}(x) - \frac{1}{2}\eta_{n,0}^{*}g_n^{*}(x).\end{aligned}$$

Now, the sequence $(\eta_{n,0}^*)$ converges to zero since the product $\prod_{n=1}^{\infty}(1-2\lambda_1/\lambda_n)$ diverges to zero because of our hypothesis on the sequence $(\lambda_k)_{k=1}^{\infty}$. This means that $-\frac{1}{2}\eta_{n,0}^*g_n^*(x)$ converges uniformly to zero, since we know that $0 \le g_n^*(x) \le 1$ for all $x \in [0,1]$.

Let us now show that

$$\eta_{n,k}^{2\lambda_1} - \eta_{n,k}^{*} \to 0 \tag{22}$$

uniformly in $k \ge 1$. To prove this, let $\varepsilon > 0$ be arbitrary and fix n_0 such that, for all $k \ge n_0$, $2\lambda_1 < \lambda_k$ and

$$(1+\varepsilon)\log(1-\frac{\lambda_1}{\lambda_k})^2 \le \log(1-\frac{2\lambda_1}{\lambda_k}) \le \log(1-\frac{\lambda_1}{\lambda_k})^2.$$

(This is possible since $\lambda_k \to \infty$, log is an increasing function and $\log t < 0$ for $t \in (0,1)$.) Then

$$(1-\frac{\lambda_1}{\lambda_k})^{2(1+\varepsilon)} \le (1-\frac{2\lambda_1}{\lambda_k}) \le (1-\frac{\lambda_1}{\lambda_k})^2. \tag{23}$$

Since the products $\prod_{k=1}^{\infty}(1-\frac{\lambda_1}{\lambda_k})$ and $\prod_{k=1}^{\infty}(1-\frac{2\lambda_1}{\lambda_k})$ are both divergent to zero, we can consider only the values $k \ge n_0$ in (22), so that we can use (23) for the factors representing $\eta_{n,k}^{2\lambda_1}$ and $\eta_{n,k}^*$. If $\eta_{n,k}^{2\lambda_1} \le \varepsilon$ then also $\eta_{n,k}^* \le \varepsilon$ and $|\eta_{n,k}^{2\lambda_1} - \eta_{n,k}^*| \le 2\varepsilon$. On the other hand, if $\eta_{n,k}^{2\lambda_1} \ge \varepsilon$ then

$$0 \le \eta_{n,k}^{2\lambda_1} - \eta_{n,k}^* \le \eta_{n,k}^{2\lambda_1} - (\eta_{n,k}^{2\lambda_1})^{1+\varepsilon} \le 1 - (\eta_{n,k}^{2\lambda_1})^{\varepsilon} \le 1 - \varepsilon^{\varepsilon},$$

which is arbitrarily small for $\varepsilon \to 0$. This proves (22) and, consequently, that $T_n(x) \to 0$ uniformly in $x \in [0,1]$.

Let us take $f \in C[0,1]$ and let $M \ge \|f\|_{[0,1]}$. Obviously, f is uniformly continuous so that we may assume that for a given $\varepsilon > 0$ we have chosen $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x^{1/\lambda_1}) - f(y^{1/\lambda_1})| < \varepsilon$. Then

$$B_n^{\Lambda}(f)(x) - f(x) = \sum_{k=0}^{n}(f(\eta_{n,k}) - f(x))g_{n,k}(x)$$

(recall that the polynomials $\{g_{n,k}\}_{k=0}^n$ are a partition of unity in $[0,1]$). Hence we can decompose the summation formula into two parts: the first one containing those indices k such that $|\eta_{n,k}^{\lambda_1} - x^{\lambda_1}| \le \delta$ and the second one, where this inequality does not hold. The first part of the summation formula is $\le \varepsilon$ and for the second part we use that $|f(\eta_{n,k}) - f(x)| \le 2M$ and $(\eta_{n,k}^{\lambda_1} - x^{\lambda_1})^2/\delta^2 \ge 1$ to conclude that $2MT_n(x)/\delta^2$ is an upper bound of this part. This obviously implies that $B_n^{\Lambda}(f)(x)$ converges to $f(x)$ uniformly on $[0,1]$. □

2.5 Proof of Müntz theorem via complex analysis

In this subsection, we will use some basic results from complex analysis to give another proof of the Müntz Theorem. It is because of our use of a complex variable that we introduce a minor modification in the space of functions we want to approximate. Concretely, we assume that our space of functions is $C([0,1],\mathbb{C})$ and Müntz polynomials have complex coefficients. Clearly, the Müntz Theorem corresponding to this context is the following one (which is equivalent to the classical Müntz Theorem since the variable z runs on $[0,1]$ which is a subset of $\mathbb{R}$, so that in order to approximate a continuous function $f(z)=u(z)+\mathrm{i}v(z)$ with a complex polynomial $p(z)=\sum_{i=0}^n \alpha_i z^{\lambda_i}$ we only need to choose the coefficients $\alpha_i=a_i+\mathrm{i}b_i$ in such a form that $\sum_{i=0}^n a_i z^{\lambda_i}$ approximates $u(z)$ and $\sum_{i=0}^n b_i z^{\lambda_i}$ approximates $v(z)$):

Theorem 8 (Müntz Theorem for Complex-Valued Functions) *Let $\Lambda=(\lambda_k)_{k=0}^\infty$, $0=\lambda_0<\lambda_1<\cdots$ be an increasing sequence of non-negative real numbers. Then $\Pi_{\mathbb{C}}(\Lambda)$ is a dense subset of $C([0,1],\mathbb{C})$ if and only if $\sum_{k=1}^\infty 1/\lambda_k=\infty$.*

Let $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ be the unit disc, and

$$H^\infty(\mathbb{D}):=\{f : f \text{ is holomorphic on } \mathbb{D} \text{ and } \|f\|_{H^\infty(\mathbb{D})}=\sup_{z\in\mathbb{D}}|f(z)|<\infty\}$$

be the algebra of bounded analytic functions defined on $\mathbb{D}$. The proof we present of Theorem 8 (which is due to Feinerman and Newman [14]) is based on the following lemmas.

Lemma 9 (Blaschke Products) *The function $f:\mathbb{D}\to\mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ if and only if it can be decomposed as*

$$f(z)=z^p\prod_{k=0}^\infty (z-\lambda_k)h(z)$$

for a certain choice of a natural number $p\geq 0$, a sequence of points $(\lambda_k)\subset\mathbb{D}$ such that $\sum_{k=0}^\infty(1-|\lambda_k|)<\infty$, and a function $h\in H^\infty(\mathbb{D})$ without zeros on $\mathbb{D}$.

Proof. See [21, pages 63–67]. □

Lemma 10 *If $\sum_{k=0}^\infty 1/\lambda_k=\infty$ and η is a complex Borel measure on $[0,1]$ such that*

$$\int_0^1 t^{\lambda_k}\,\mathrm{d}\eta(t)=0, \qquad k=0,1,\ldots,$$

then

$$\int_0^1 t^{k}\,\mathrm{d}\eta(t)=0, \qquad k=0,1,\ldots.$$

Proof. We may assume, without loss of generality, that our measure is concentrated on $(0,1]$. Then we use Morera's theorem to prove that $h(z):=\int_0^1 t^z\,\mathrm{d}\eta(t)$ is holomorphic on the half plane $\{z:\mathrm{Re}z>0\}$ and $h(\lambda_k)=0$, $k=0,1,\ldots$. Moreover, h is bounded on the half plane, since if we decompose $z=x+\mathrm{i}y$, then $|t^z|=t^x\leq 1$ for all $t\in[0,1]$. It follows that $g(z):=h((1+z)/(1-z))\in H^\infty(\mathbb{D})$ and $g(\alpha_k)=0$, $k=0,1,\ldots$, where $\alpha_k:=(\lambda_k-1)/(\lambda_k+1)$, for all k.

Now, it is clear that $\sum 1/\lambda_k=\infty$ (which is our hypothesis), implies that $\sum(1-|\alpha_k|)=\infty$, so that $g=0$. This of course implies that $h(k)=0$ for all $k\in\mathbb{N}$. □

Proof of Theorem 8. Let $Y := \overline{\Pi(\Lambda)}$ be the closure of $\Pi(\Lambda)$ in $X = C[0,1]$. It follows from the Hahn-Banach Theorem that $f \in X \setminus Y$ if and only if there exists a bounded linear functional $L \in X^*$ such that $L|_Y = 0$ but $Lf \neq 0$. Now, if $L \in C^*[0,1]$ then $Lf = \int_0^1 f(t)\,\mathrm{d}\eta$ for all f and a certain finite complex Borel measure η defined on $[0,1]$. It follows from Lemma 10 (and the Weierstrass Approximation Theorem) that $Lf = 0$ for all f, which is in contradiction to our hypotheses. This proves that $\sum 1/\lambda_k = \infty$ is a sufficient condition for the density of $\Pi(\Lambda)$ in $C[0,1]$.

We now prove that the condition is also necessary. Let us assume that $\sum 1/\lambda_k < \infty$, and let us define the function:

$$f(z) := \frac{z}{(2+z)^3} \prod_{k=1}^{\infty} \frac{\lambda_k - z}{2 + \lambda_k + z}\,.$$

Now,

$$1 - \frac{\lambda_k - z}{2 + \lambda_k + z} = \frac{2 + 2z}{2 + \lambda_k + z},$$

so that the infinite product that appears in the definition of f converges uniformly on compact subsets of $\mathbb{C} \setminus (\{-2\} \cup \{-2 - \lambda_k\}_{k=0}^{\infty})$. Hence f is a meromorphic function on $\mathbb{C}$ with poles $\{-2\} \cup \{-2 - \lambda_k\}_{k=0}^{\infty}$ and zeros $\{0\} \cup \{\lambda_k\}_{k=0}^{\infty}$. Furthermore, each factor of our infinite product is (in absolute value) less than 1 for all z such that $\operatorname{Re} z > -1$. On the other hand, the restriction of f to the line $\operatorname{Re} z = -1$ is an absolutely integrable function (this follows from the fact that we have divided by $(2+z)^3$). Let us fix z with $\operatorname{Re} z > -1$ and consider the Cauchy formula for f taking as path of integration the circle centered at -1 of radius $R > 1 + |z|$, from $-1 - \mathrm{i}R$ to $-1 + \mathrm{i}R$, plus the interval $[-1 + \mathrm{i}R, -1 - \mathrm{i}R]$. If we let $R \to \infty$ then we can eliminate the part of the formula associated with the semicircle (note that $|2+z|^3 > R^3$ there), and we obtain

$$\begin{aligned} f(z) &= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(-1 + \mathrm{i}s)\,\mathrm{d}s}{-1 + \mathrm{i}s - z} \qquad (24) \\ &= \int_0^1 t^z \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(-1 + \mathrm{i}s) \exp(-\mathrm{i}s \log t)\,\mathrm{d}s \right\} \mathrm{d}t, \end{aligned}$$

since

$$\frac{1}{-1 + \mathrm{i}s - z} = \int_0^1 t^{z - \mathrm{i}s}\,\mathrm{d}t = \int_0^1 t^z \exp(-\mathrm{i}s \log t)\,\mathrm{d}t.$$

If we define $g(s) := f(-1 + \mathrm{i}s)$ then the inner integral which appears in formula (24) is $\widehat{g}(\log t)$, where $\widehat{g}$ denotes the Fourier transform of g which is clearly a continuous bounded function defined on $(0,1]$, so that if $\mathrm{d}\eta(t) = \widehat{g}(\log t)\,\mathrm{d}t$ then η is a complex Borel measure. Therefore, the bounded linear functional $h \mapsto \int h\,\mathrm{d}\eta$ annihilates Y but it is not identically zero, hence $Y \neq X$ whenever $\sum_{k=0}^{\infty} 1/\lambda_k < \infty$, which is what we wanted to prove. □

3 The Full Müntz theorem and polynomial inequalities

The Classical Müntz Theorem was only stated for increasing sequences of nonnegative real numbers $0 = \lambda_0 < \lambda_1 < \cdots$. It would be interesting to know if a general result, dealing with arbitrary sequences of exponents, is possible. We call such a result a **Full Müntz Theorem**. Moreover, it would be interesting to know such a result not only for the space $C[0,1]$ but also for $C(K)$ with K

a compact subset of $\mathbb{R}$ or $\mathbb{C}$ and for other function spaces such as $L^p[a,b]$, etc. As we have already noted, Szász' proof of the Müntz Theorem in 1916 included sequences of exponents more general than those treated by Müntz in his paper of 1914. In particular, for real exponents, he proved a result that works whenever $\liminf_{k\to\infty}\lambda_k > 0$, so that Szász' theorem was not properly a Full Müntz theorem although it was near to it. It is remarkable that, after Szász' work, the search for a Full Müntz theorem was not immediate. Moreover, since the appearance of the Müntz and Szász theorems it was clear that the role of the origin in these results was very important and to extend the results to spaces of functions defined away from the origin was a nontrivial task. Moreover, nobody had answered the main question of characterizing the elements in the closure of a Müntz space $\Pi(\Lambda)$ when this space is not a dense subset of $C[a,b]$.

It was Fields medalist Laurent Schwartz [30] who proved a Full Müntz theorem for the space of square-integrable functions $L^2[a,b]$ on general intervals $[a,b]$. Schwartz also characterized the density of Müntz spaces in $C[a,b]$ for $0\notin[a,b]$ and conjectured a necessary and sufficient condition for the density in $C[0,1]$ of $\Pi(\Lambda)$ with general sequences, but he did not prove the result.

It was only a few years later when Siegel [31], in a beautiful paper where he included a difficult generalization of Szász' theorem, proved Schwartz' conjecture for the first time using complex variable techniques. The deepest work related to the Full Müntz Theorem has only recently been done by P. Borwein and T. Erdélyi, sometimes in collaboration with several other authors. They proved that there is a strong connection between density results for Müntz spaces and the study of some special inequalities for these polynomials. In fact, their book "Polynomials and Polynomial Inequalities" [5] constitutes a deep contribution to this subject and contains a guided investigation of the Full Müntz Theorem. The reader should attempt the solution of the exercises in Chapter 4 of that book or, avoiding much effort, read this section where we will concentrate almost all our attention on the study of the Full Müntz Theorem for the spaces $C[0,1]$, $L^p[0,1]$ and $C[a,b]$ (with $0<a<b$). We also include some ideas related to the proof by Borwein and Erdélyi of a Full Müntz Theorem for the space $C(K)$, where K is a compact set with positive Lebesgue measure, and a recent result by the author where a Full Müntz Theorem is proved for the space of continuous functions on quite general countable compact sets.

3.1 Full Müntz theorem on $[0,1]$

We start with the precise statement of the main results of this section.

Theorem 11 (Full Müntz Theorem for $C[0,1]$) *Let us assume that $\Lambda=(\lambda_k)_{k=1}^{\infty}$ is a sequence of distinct real positive numbers. Then $\Pi(\Lambda\cup\{0\})$ is dense in $C[0,1]$ if and only if*

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k^2+1}=\infty. \tag{25}$$

Theorem 12 (Full Müntz Theorem for $L_2[0,1]$) *Let us assume that $\Lambda=(\lambda_k)_{k=1}^{\infty}$ is a sequence of distinct real numbers greater than $-1/2$. Then $\Pi(\Lambda)$ is dense in $L_2[0,1]$ if and only if*

$$\sum_{k=1}^{\infty}\frac{2\lambda_k+1}{(2\lambda_k+1)^2+1}=\infty. \tag{26}$$

Although we will not prove it in this paper, we would like to include here the corresponding result for $L^p[0,1]$:

Theorem 13 (Full Müntz Theorem for $L_p[0,1]$) *Let $p \in (0,\infty)$ and let us assume that $\Lambda = (\lambda_k)_{k=1}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$. Then $\Pi(\Lambda)$ is dense in $L_p[0,1]$ if and only if*

$$\sum_{k=1}^{\infty} \frac{\lambda_k + 1/p}{(\lambda_k + 1/p)^2 + 1} = \infty. \tag{27}$$

Theorem 11 was conjectured by Schwartz, proved by A. R. Siegel [31] for the first time and, following the ideas introduced by Szász in his famous 1916 paper, reproved by Borwein and Erdélyi. Theorem 12 was proved by Szász [34]. Theorem 13 was proved for $p = 1$ and conjectured for $p > 0$ by Borwein and Erdélyi [5]. Moreover, it was proved by Operstein [27] for the case $1 < p < \infty$, by Erdélyi and Johnson [11] (using quasi-Banach space theory) for $0 < p < \infty$ and, quite recently, by Erdélyi [13] for $0 < p < \infty$ with an "elementary" proof.

In this section, we will prove Theorem 11. As a first step, we prove the easier Theorem 12 and we use it to prove some particular cases of Theorem 11. Then we introduce several polynomial inequalities that will be needed to complete the proof of this theorem. These inequalities are proved in the second part of this section. The third and fourth part are devoted to a characterization of the closure of nondense Müntz subspaces of $C[0,1]$ and the statement and proof of the Full Müntz Theorem for intervals $[a,b]$ away from the origin.

Finally, the fifth and sixth part are devoted to the Full Müntz theorem for $C(K)$ for compact sets $K \subset [0,\infty)$ more general than intervals.

Proof of Theorem 12. We have already proved the formula:

$$E(x^q, \Pi(\Lambda_n))_{L^2(0,1)} = \frac{1}{\sqrt{2q+1}} \prod_{k=1}^{n} \frac{|q-\lambda_k|}{|q+\lambda_k+1|}.$$

Hence $x^q \in \overline{\Pi(\Lambda)}^{L^2(0,1)}$ if and only if

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left|\frac{q-\lambda_k}{q+\lambda_k+1}\right| \ \left(= \lim_{n\to\infty} \prod_{k=1}^{n} \left|1 - \frac{2q+1}{q+\lambda_k+1}\right|\right) = 0.$$

We decompose the above product making a distinction between the cases $\lambda_k \in (-1/2, q]$ and $\lambda_k \in (q,\infty)$, which leads us to the following reformulation of the above condition:

$$\lim_{n\to\infty} \prod_{k\le n;\lambda_k \in(-1/2,q]} \left|1 - \frac{2q+1}{q+\lambda_k+1}\right| \prod_{k\le n;\ \lambda_k\in(q,\infty)} \left|1 - \frac{2q+1}{q+\lambda_k+1}\right| = 0,$$

which is clearly equivalent to stating that

$$\sum_{k\ge1;\ \lambda_k\in(q,\infty)} \frac{1}{2\lambda_k+1} = \infty \qquad \text{or} \qquad \sum_{k\ge1;\ \lambda_k\in(-1/2,q]} (2\lambda_k+1) = \infty,$$

and this can be rewritten as

$$\sum_{k=1}^{\infty} \frac{2\lambda_k+1}{(2\lambda_k+1)^2+1} = \infty,$$

which is what we wanted to prove. □

We subdivide the proof of Theorem 11 into several cases, depending on which of the following three conditions are satisfied by the sequence (λ_k):

H1 $\inf_{k\in\mathbb{N}} \lambda_k > 0$.

H2 $\lim_{k\to\infty} \lambda_k = 0$. (In this case, the identities $\sum \frac{\lambda_k}{\lambda_k^2+1} = \infty$ and $\sum \lambda_k = \infty$ are equivalent.)

H3 $(\lambda_k) = (\alpha_k) \cup (\beta_k)$, with $\alpha_k \to 0$ and $\beta_k \to \infty$. (In this case, the identities $\sum \frac{\lambda_k}{\lambda_k^2+1} = \infty$ and $\sum \alpha_k + \sum \frac{1}{\beta_k} = \infty$ are equivalent.)

At first glance it seems that condition H3 is not significant since, taking subsequences if necessary, H1 and H2 produce all possibilities. Theorem 11 has been stated in a unified form that depends on the convergence character of the series $\sum \frac{\lambda_k}{\lambda_k^2+1}$, and this depends on the knowledge of the boundedness character of the set $\{\lambda_k\}_{k=0}^{\infty}$. In particular, for the case $\inf_{k\geq 0} \lambda_k = 0$ and in order to characterize for which sequences we have $\sum \frac{\lambda_k}{\lambda_k^2+1} = \infty$, we must take into account both possibilities H2 and H3. We will see that the study of the case described by H3 is precisely the most difficult to handle.

In fact, H1 – H3 do not cover all cases (e.g. if there is a subsequence converging to 0 and another one converging to 1) but in the missing cases there is a subsequence that is bounded away from 0 and infinity, and in this case (25) is automatically true, and so is the denseness by H1 applied to this subsequence.

Proof of Theorem 11, using Theorem 12, and assuming that $\inf_{k\in\mathbb{N}} \lambda_k > 0$**.** Let $0 < \delta \leq \inf_{k\in\mathbb{N}} \lambda_k$. We make the change of variable $x \to x^{\frac{1}{\delta}}$ and solve the problem for exponents $\lambda_k^* = \lambda_k/\delta$ that satisfy $\inf_{k\in\mathbb{N}} \lambda_k^* \geq 1$. This means that we may assume, without loss of generality, that $\inf_{k\in\mathbb{N}} \lambda_k \geq 1$. Then

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k^2+1} = \infty \qquad \text{if and only if} \qquad \sum_{k=1}^{\infty} \frac{(\lambda_k - 1)}{(\lambda_k-1)^2+1} = \sum_{k=1}^{\infty} \frac{2(\lambda_k-1)+1}{(2(\lambda_k-1)+1)^2+1} = \infty.$$

Let us first assume that

$$\sum \frac{2(\lambda_k-1)+1}{(2(\lambda_k-1)+1)^2+1} = \infty.$$

It then follows from Theorem 12 that $\Pi((\lambda_k-1)_{k=1}^{\infty})$ is dense in $L^2[0,1]$. Let now $q \in \mathbb{N}$ be arbitrarily chosen. We can use the Szász trick as described by the inequalities (4) (see Section 2 of this paper) to prove that

$$E(x^q, \Pi(\Lambda_n))_{C[0,1]} \leq qE(x^{q-1}, \Pi(\Lambda_n^*))_{L^2[0,1]},$$

where $\Lambda_n = (\lambda_k)_{k=1}^n$ and $\Lambda_n^* = (\lambda_k - 1)_{k=1}^n$. This error goes to zero for all choices $q > 0$, which proves that condition $\sum \lambda_k/(\lambda_k^2+1) = \infty$ is sufficient for the density of $\Pi(\Lambda)$ in $C[0,1]$.

On the other hand, if $\Pi((\lambda_k)_{k=1}^{\infty})$ is dense in $C[0,1]$ then, taking into consideration that $C[0,1]$ is dense in $L^2[0,1]$ and $\|\cdot\|_{L^2[0,1]} \leq \|\cdot\|_{C[0,1]}$, we have that $\Pi((\lambda_k)_{k=1}^{\infty})$ is also dense in $L^2[0,1]$. Hence, using Theorem 12, $\sum \lambda_k/(\lambda_k^2+1) = \infty$. □

Proof of Theorem 11 when $\lambda_k \to 0$. We start by noting that in this case the identities $\sum \lambda_k/(\lambda_k^2+1) = \infty$ and $\sum \lambda_k = \infty$ are equivalent. It follows from the Hahn-Banach and Riesz Representation Theorems that span $\{1, x^{\lambda_k}\}_{k=1}^{\infty}$ is dense in $C[0,1]$ if and only if $\{\lambda_k\}$ is not a subset of the set of zeros of any nontrivial function of the form

$$f_\mu(z) = \int t^z \,\mathrm{d}\mu(t)$$

for some finite Borel measure μ. This condition is equivalent to saying that $\{(\lambda_k-1)/(\lambda_k+1)\}$ is not the zero set of any function of the form

$$g(z) := f_\mu((1+z)/(1-z)) \in H^\infty(\mathbb{D}).$$

Now, if $\lambda_k \to 0$, then the equation $\sum \lambda_k = \infty$ implies that

$$\sum_{k=1}^{\infty}\left(1-\left|\frac{\lambda_k-1}{\lambda_k+1}\right|\right) = \infty,$$

so that, using Lemma 9, it is clear that $\{(\lambda_k-1)/(\lambda_k+1)\}$ is not the zero set of any $g(z) \in H^\infty(\mathbb{D})$. This means that $\sum \lambda_k = \infty$ is a sufficient condition (whenever $\lambda_k \to 0$) for the density of span $\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $C[0,1]$.

In order to prove that condition (25) is also necessary, we need to introduce the following theorem:

Theorem 14 (Newman's Inequality) *Assume that $\Lambda = (\lambda_k)_{k=1}^{\infty}$ is a sequence of distinct positive real numbers. Then the inequality*

$$\left\|xp'(x)\right\|_{[0,1]} \le 11\left(\sum_{k=0}^{n}\lambda_k\right)\|p(x)\|_{[0,1]}$$

holds for all $p \in \Pi(\Lambda_n)$ and all $n \in \mathbb{N}$.

If $M := \sum_{k=1}^{\infty}\lambda_k < \infty$, then we have that

$$\left\|xp'(x)\right\|_{[0,1]} \le 11M\,\|p(x)\|_{[0,1]}$$

for all $p \in \Pi(\Lambda)$, which contradicts the density of $\Pi(\Lambda)$ in $C[0,1]$. For let us assume that $\Pi(\Lambda)$ is dense in $C[0,1]$. If, for example, we set $f(x) = (1-x)^{1/2}$ then for every natural number m there exists a $p \in \Pi(\Lambda)$ such that $||p-f|| \le 1/m^2$. Hence

$$\begin{aligned} |p(1-1/m^2)-p(1)| &\ge |f(1-1/m^2) - 1/m^2 - (f(1)+1/m^2)| \\ &= 1/m - 2/m^2, \end{aligned}$$

and it then follows from the Mean Value Theorem that

$$\begin{aligned} |\xi p'(\xi)| &= \xi\frac{|p(1-1/m^2)-p(1)|}{1/m^2} \ge (1-1/m^2)\frac{1/m-2/m^2}{1/m^2} \\ &= (1-1/m^2)(m-2) \ge \frac{m-2}{2} \end{aligned}$$

for a certain $\xi \in (1-1/m^2, 1)$. This clearly is in contradiction with

$$\left\|xp'(x)\right\|_{[0,1]} \le 11M\,\|p(x)\|_{[0,1]},$$

since m is arbitrary. □

Proof of Newman's Inequality. We may assume, without loss of generality, that $\sum_{k=0}^{n} \lambda_k = 1$ since we may make the change of variable

$$x \to x^{1/(\sum_{k=0}^{n} \lambda_k)}.$$

Set $x = \mathrm{e}^{-t}$. If $p(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}$ and $q(t) = p(\mathrm{e}^{-t}) = \sum_{k=0}^{n} a_k \mathrm{e}^{-\lambda_k t}$ then

$$xp'(x) = \sum_{k=0}^{n} \lambda_k a_k x^{\lambda_k} = \sum_{k=0}^{n} \lambda_k a_k \mathrm{e}^{-\lambda_k t} = q'(t),$$

so that we have changed our problem to one of estimating the uniform norm, on the interval $[0, \infty)$, of the derivatives of functions of the form

$$\sum_{k=0}^{n} a_k \mathrm{e}^{-\lambda_k t} \tag{28}$$

in terms of their uniform norms in the same interval.

Let

$$B(z) := \prod_{k=0}^{n} \frac{z - \lambda_k}{z + \lambda_k}$$

and define

$$T(t) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{-zt}}{B(z)}\, \mathrm{d}z, \qquad \text{where} \qquad \Gamma := \{z : |z - 1| = 1\}.$$

It follows from the residue theorem that T is of the form (28). To prove Newman's inequality we first prove the following estimate:

$$|B(z)| \geq 1/3 \quad \text{ for all } \quad z \in \Gamma. \tag{29}$$

It is easy to check that the Möbius transform $z \mapsto (z-\lambda)(z+\lambda)$ sends the circle Γ onto the circle that contains the interval $[-1, (2 - \lambda)/(2 + \lambda)]$ as a diameter, so that the inequality

$$\left| \frac{z - \lambda}{z + \lambda} \right| \geq \frac{2 - \lambda}{2 + \lambda} = \frac{1 - \lambda/2}{1 + \lambda/2}$$

holds for all $z \in \Gamma$, and

$$|B(z)| \geq \prod_{k=0}^{n} \frac{1 - \lambda_k/2}{1 + \lambda_k/2}.$$

To estimate the above product, we take into consideration the fact that for all $x, y \geq 0$, the inequality

$$\frac{1 - x}{1 + x} \cdot \frac{1 - y}{1 + y} \geq \frac{1 - (x + y)}{1 + x + y}$$

holds. This leads us to the inequality

$$\prod_{k=0}^{n} \frac{1 - \lambda_k/2}{1 + \lambda_k/2} \geq \frac{1 - \frac{1}{2}\sum_{k=0}^{n} \lambda_k}{1 + \frac{1}{2}\sum_{k=0}^{n} \lambda_k} = \frac{1 - 1/2}{1 + 1/2} = 1/3,$$

which proves (29).

It follows from the definition of T, that

$$T''(t) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{z^2 \mathrm{e}^{-zt}}{B(z)} \, \mathrm{d}z$$

and, using $\alpha(\theta) = 1 + \mathrm{e}^{\mathrm{i}\theta}$ as a parametrization of Γ, we have (taking into account Fubini's Theorem) that

$$\begin{aligned} \int_0^\infty |T''(t)| \, \mathrm{d}t &\leq (3/2\pi) \int_0^\infty \int_0^{2\pi} 2(1+\cos\theta)\mathrm{e}^{-(1+\cos\theta)t} \, \mathrm{d}\theta \, \mathrm{d}t \\ &= (3/2\pi) \int_0^{2\pi} 2(1+\cos\theta) \frac{1}{(1+\cos\theta)} \, \mathrm{d}\theta = 6. \end{aligned}$$

Now we will compute integrals of the form $\int_0^\infty \mathrm{e}^{-\lambda_k t} T''(t) \, \mathrm{d}t$ in terms of the scalars λ_k. To do this, we note that

$$\int_0^\infty \mathrm{e}^{-\lambda_k t} T''(t) \, \mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{z^2}{B(z)(z+\lambda_k)} \, \mathrm{d}z \tag{30}$$

and, taking into consideration the fact that

$$\frac{z^2}{B(z)(z+\lambda_k)}$$

has no poles in the exterior of Γ, the above integral depends only on its residue at ∞. Now

$$\frac{1}{z+\lambda_k} = \frac{1}{z} \sum_{j=0}^{\infty} (-1)^j (\lambda_k/z)^j = 1/z - \lambda_k/z^2 + \lambda_k^2/z^3 - \cdots$$

and

$$\begin{aligned} \frac{1}{B(z)} &= \prod_{k=0}^{\infty} \frac{1+\lambda_k/2}{1-\lambda_k/2} = 1 + \frac{2\sum_{k=0}^{\infty} \lambda_k}{z} + \frac{2\left(\sum_{k=0}^{\infty} \lambda_k\right)^2}{z^2} + \cdots \\ &= 1 + \frac{2}{z} + \frac{2}{z^2} + \cdots \end{aligned}$$

so that

$$\frac{z^2}{B(z)(z+\lambda_k)} = z + (2 - \lambda_k) + \frac{\lambda_k^2 - 2\lambda_k + 2}{z} + \cdots .$$

This, in conjunction with (30), leads us to the formula

$$\int_0^\infty \mathrm{e}^{-\lambda_k t} T''(t) \, \mathrm{d}t = \lambda_k^2 - 2\lambda_k + 2. \tag{31}$$

Now, let q be an exponential polynomial of the form (28). Then, if we take into consideration (31), we conclude that

$$\begin{aligned}\int_0^{\infty} q(t+a)T''(t)\,\mathrm{d}t &= \int_0^{\infty}\left(\sum_{k=0}^{n} a_k \mathrm{e}^{-\lambda_k t}\mathrm{e}^{-\lambda_k a}\right)T''(t)\,\mathrm{d}t \\ &= \sum_{k=0}^{n} a_k \mathrm{e}^{-\lambda_k a}\int_0^{\infty}\mathrm{e}^{-\lambda_k t}T''(t)\,\mathrm{d}t \\ &= \sum_{k=0}^{n} a_k \mathrm{e}^{-\lambda_k a}(\lambda_k^2 - 2\lambda_k + 2) \\ &= q''(a) - 2q'(a) + 2q(a).\end{aligned}$$

Hence,

$$\begin{aligned}\left|q''(a) - 2q'(a) + 2q(a)\right| &= \left|\int_0^{\infty} q(t+a)T''(t)\,\mathrm{d}t\right| \\ &\leq \int_0^{\infty}\left|q(t+a)T''(t)\right|\,\mathrm{d}t \leq \|q\|_{[0,\infty)}\int_0^{\infty}\left|T''(t)\right|\,\mathrm{d}t \\ &\leq 6\|q\|_{[0,\infty)} \text{ (for all } a \geq 0),\end{aligned}$$

so that

$$\|q''\|_{[0,\infty)} \leq 2\|q'\|_{[0,\infty)} + 8\|q\|_{[0,\infty)}.$$

It is well known that the inequality

$$\|f'\|^2_{[0,\infty)} \leq 4\|f\|_{[0,\infty)}\|f''\|_{[0,\infty)}$$

holds for all functions $f \in C^{(2)}[0,\infty)$ (see [22]), so that

$$\|q'\|^2_{[0,\infty)} \leq 4\|q\|_{[0,\infty)}\|q''\|_{[0,\infty)} \leq \|q\|_{[0,\infty)}(8\|q'\|_{[0,\infty)} + 16\|q\|_{[0,\infty)})$$

and

$$\left(\frac{\|q'\|_{[0,\infty)}}{\|q\|_{[0,\infty)}}\right)^2 \leq 8\frac{\|q'\|_{[0,\infty)}}{\|q\|_{[0,\infty)}} + 16$$

which clearly implies that

$$\frac{\|q'\|_{[0,\infty)}}{\|q\|_{[0,\infty)}} \leq 11$$

for all expressions of the form (28). □

Theorem 14 is a nice generalization of the classical Markov inequality, which states that algebraic polynomials of degree $\leq n$ (i.e., polynomials of the form $p(x) = a_0 + a_1x + \cdots + a_nx^n$) satisfy

$$\|p'\|_{[-1,1]} \leq n^2\|p\|_{[-1,1]}.$$

Markov's inequality is indeed related to another classical inequality, due to Bernstein, which states that algebraic polynomials of degree $\leq n$ satisfy

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}}\|p\|_{[-1,1]}, \qquad \text{for all } x \in (-1,1).$$

It is not by chance that Theorem 14 appeared in the middle of our proof. In fact, the study of certain classical polynomial inequalities and how they can be extended (and adapted) from the usual spaces of algebraic polynomials to Müntz spaces, has proven to be a deep tool for studying the density of these spaces. Concretely, the following generalization of the classical Bernstein inequality will be of fundamental importance for the main objectives of this section.

Theorem 15 (Bounded Bernstein's and Chebyshev's inequalities) *Let us assume that* $0 \leq \lambda_0 < \lambda_1 < \cdots$ *and* $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$. *Then for each* $\varepsilon > 0$ *there are constants* $c_\varepsilon, c_\varepsilon^* > 0$ *such that*

$$\|p\|_{[0,1]} \leq c_\varepsilon \|p\|_{[1-\varepsilon,1]}$$

and

$$\|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon^* \|p\|_{[1-\varepsilon,1]} \leq c_\varepsilon^* \|p\|_{[0,1]}$$

for all $p \in \Pi(\Lambda)$.

The proof of this theorem is especially tricky, so that we postpone it to part 2 of this section. We prefer, at present, to explain how a clever use of this theorem helps us answer several questions related to the study of the density of Müntz spaces. In particular, we close this subsection by concluding the proof of the Full Müntz Theorem for the space $C[0,1]$.

Proof of Theorem 11 when $(\lambda_k) = (\alpha_k) \cup (\beta_k)$**, where** $\alpha_k \to 0$ **and** $\beta_k \to \infty$**.** In this case, the relations $\sum_{k=1}^{\infty}(\lambda_k)/(\lambda_k^2+1) = \infty$ and

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \frac{1}{\beta_k} = \infty \tag{32}$$

are equivalent. If (32) holds then we have already proved that $\Pi(\Lambda)$ is dense in $C[0,1]$. Let us now assume that $\sum_{k=1}^{\infty} \alpha_k < \infty$ and $\sum_{k=1}^{\infty} 1/\beta_k < \infty$.

Recall that a sequence of functions $(f_k)_{k=0}^n \subseteq C(K)$ is called a Haar system on K if

$$\dim \operatorname{span} \{f_k\}_{k=0}^n = n+1$$

and the only element $f \in \operatorname{span}\{f_k\}_{k=0}^n$ that vanishes at $n+1$ points is the zero function. A special type of Haar systems are Chebyshev systems, which are those given by a sequence of functions $(f_k)_{k=0}^n \subseteq C(K)$ such that

$$\det \begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n) \end{pmatrix} > 0$$

holds whenever $x_0 < x_1 < \cdots < x_n$, $\{x_i\}_{i=0}^n \subseteq K$.

Proposition 16 *Let us assume that* $\lambda_0 < \lambda_1 < \cdots < \lambda_n$. *Then* $(x^{\lambda_k})_{k=0}^n$ *is a Chebyshev system on* $(0,\infty)$.

Proof. Let $\Delta = \{(\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} : \exists i \neq j, \alpha_i = \alpha_j\}$. Then it is not difficult to prove that

$$D(\rho_0, \dots, \rho_n) := \det \begin{pmatrix} x_0^{\rho_0} & x_0^{\rho_1} & \cdots & x_0^{\rho_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\rho_0} & x_n^{\rho_1} & \cdots & x_n^{\rho_n} \end{pmatrix} \neq 0$$

whenever $(\rho_0, \dots, \rho_n) \in \mathbb{R}^{n+1} \setminus \Delta$ and $0 < x_0 < x_1 < \cdots < x_n$. The difficult thing is to show that this determinant must be positive. Now, we take $\tau = (\tau_0, \dots, \tau_n) : [0,1] \to \mathbb{R}^{n+1} \setminus \Delta$ a continuous path such that $\tau(0) = (\lambda_0, \dots, \lambda_n)$ and $\tau(1) = (0, 1, \dots, n)$ (this is possible since $\lambda_0 < \cdots < \lambda_n$). The continuity of τ implies that

$$\mathrm{sign}(D(\tau(0))) = \mathrm{sign}(D(\tau(1))) = +1,$$

since the last determinant is the well known Vandermonde determinant (see, e.g., [10]). □

Let us assume that $(f_k)_{k=0}^n$ is a Chebyshev system. Under these conditions, it is possible to prove some interesting results about uniqueness and characterization of best approximants from the space $\mathrm{span}\,\{f_k\}_{k=0}^n$. In particular, the existence of a unique best approximation to f_n by elements of $\mathrm{span}\,\{f_k\}_{k=0}^{n-1}$ is guaranteed. If P_n is such an approximant, then the function $T_n := (f_n - P_n)/\|f_n - P_n\|$ is, by definition, the Chebyshev polynomial associated with the Chebyshev system $(f_k)_{k=0}^n$. As we shall see, these polynomials play a main role in our theory. In particular, they satisfy the following nice interlacing property (for a proof, see [5], page 116):

Theorem 17 (Zeros of Chebyshev Polynomials) *Let us assume that $\mathcal{T} = (f_0, \dots, f_{n-1}, g)$ and $\mathcal{S} = (f_0, \dots, f_{n-1}, h)$ are Chebyshev systems on $[a,b]$ and that $T_n = T_{n,\mathcal{T}}$ and $S_n = S_{n,\mathcal{S}}$ denote the associated Chebyshev polynomials. If $(f_0, \dots, f_{n-1}, g, h)$ is also a Chebyshev system then the zeros of T_n and S_n interlace (i.e., there exists exactly one zero of S_n between any two consecutive zeros of T_n).*

Moreover, the following theorem also holds:

Theorem 18 (Alternation property of Chebyshev Polynomials) *Let us assume that $\mathcal{T} = (f_0, \dots, f_{n-1}, f_n)$ is a Chebyshev system on $[a,b]$ and that $T_n = T_{n,\mathcal{T}}$ denotes the associated Chebyshev polynomial. Then there are $n+1$ points $x_0 < x_1 < \cdots < x_n$ in $[a,b]$ such that*

$$|T_n(x_i)| = \varepsilon(-1)^i, i = 0, 1, \dots, n,$$

where $\varepsilon \in \{1, -1\}$ is the same for all i.

Let us use the following notation:

- $T_{n,\alpha}$ denotes the Chebyshev polynomial associated to the system $(1, x^{\alpha_k})_{k=1}^n$.
- $T_{n,\beta}$ denotes the Chebyshev polynomial associated to the system $(1, x^{\beta_k})_{k=1}^n$.
- $T_{2n,\alpha,\beta}$ denotes the Chebyshev polynomial associated to the system

$$(1, x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_n}, x^{\beta_1}, x^{\beta_2}, \cdots, x^{\beta_n}).$$

It follows from Newman's inequality

$$\left\|xT'_{n,\alpha}(x)\right\|_{[0,1]} \leq 11M(\{\alpha_k\})\,\|T_{n,\alpha}\|_{[0,1]} = \text{const} < \infty \ (n \in \mathbb{N}),$$

that for each $\varepsilon > 0$ there exists a constant $k_1(\varepsilon)$ which only depends on ε and $\{\alpha_k\}$ (but does not depend on n) such that $T_{n,\alpha}$ has at most $k_1(\varepsilon)$ zeros in $[\varepsilon, 1)$ and at least $n - k_1(\varepsilon)$ zeros in $[0, \varepsilon)$. (It is not possible to increase the number of zeros of $T_{n,\alpha}$ in $[\varepsilon, 1)$ without increasing the modulus of the derivative of $T_{n,\alpha}$ at least in some points of the same interval.) On the other hand, and due to similar reasons, it follows from the bounded Bernstein's inequality (Theorem 15), applied to $T_{n,\beta}$, that

$$\|T'_{n,\beta}\|_{[0,1-\varepsilon]} \leq c^*_\varepsilon \|T_{n,\beta}\|_{[0,1]} = c^*_\varepsilon < \infty.$$

Hence $T_{n,\beta}$ has at most $k_2(\varepsilon)$ zeros in $[0, 1-\varepsilon)$ and at least $n - k_2(\varepsilon)$ zeros in $[1-\varepsilon, 1)$.

Now, if we take into account the interlacing properties of the Chebyshev's polynomials (Theorem 17), and the fact that the system $(1, x^{\alpha_k}, x^{\beta_k})_{k=1}^n$ is an extension of both systems $(1, x^{\alpha_k})_{k=1}^n$ and $(1, x^{\beta_k})_{k=1}^n$, it follows that, for n big enough, $T_{2n,\alpha,\beta}$ has at least $n - k_1(\varepsilon) - 1$ zeros on $[0, \varepsilon]$ and at least $n - k_2(\varepsilon) - 1$ zeros on $[1-\varepsilon, 1]$. Hence we conclude that there exists a certain constant $k = k(\varepsilon)$ (which only depends on the sequence (λ_k)) such that $T_{2n,\alpha,\beta}$ has at most $k(\varepsilon)$ zeros in the interval $(\varepsilon, 1-\varepsilon)$.

Set $k = k(1/4)$ and let us take a set of points

$$1/4 < t_0 < t_1 < \cdots < t_{k+3} < 3/4$$

and a function $f \in C[0,1]$ such that $f(x) = 0$ for all $x \in [0, 1/4] \cup [3/4, 1]$ and $f(t_i) = (-1)^i 2$ for all $0 \leq i \leq k+3$. Let us assume that there exists a polynomial $p \in \Pi(\Lambda)$ such that $\|f - p\|_{[0,1]} < 1$. Then $p - T_{2n,\alpha,\beta}$ has at least $2n+1$ zeros in the interval $(0,1)$ (where we have used that p dominates in $[1/4, 3/4]$ and $T_{2n,\alpha,\beta}$ dominates outside this interval). This is in contradiction to the fact that $p - T_{2n,\alpha,\beta} \in \Pi_{2n}(\Lambda)$ for all n large enough (which implies that $p - T_{2n,\alpha,\beta}$ has at most $2n$ zeros). This ends the proof whenever Λ has no accumulation points in $(0, \infty)$. □

Proof of Theorem 11 for the case in which Λ has some accumulation point in $(0,\infty)$. In this case there exists an infinite subsequence $(\alpha_k) \subset \Lambda$ such that $\inf\{\alpha_k\} > 0$ and, in such a case, we already know that $\Pi((\alpha_k)) \subset \Pi(\Lambda)$ is a dense subset of $C[0,1]$. □

3.2 Proof of the bounded Bernstein and Chebyshev inequalities

We devote this subsection to the proof of Theorem 15. The proof is long, so we divide it into several steps:

Step 1: Bernstein's and Chebyshev's exponents satisfying a jump condition

In this step, we prove Bernstein's and Chebyshev's inequalities for sequences of exponents that satisfy the following jump condition: $\inf_{k\in\mathbb{N}}(\lambda_k - \lambda_{k-1}) > 0$. In particular, we start with Bernstein's inequality in this special case:

Theorem 19 (Bounded Bernstein Inequality for Special Sequences) *Let us assume that $\Lambda = (\lambda_k)_{k=0}^\infty$ is a sequence of nonnegative real numbers that satisfies the jump condition* $\inf_{k\in\mathbb{N}}(\lambda_k -$

$\lambda_{k-1}) > 0$, and $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$, $\lambda_0 = 0$, $\lambda_1 \geq 1$. Then for all $\varepsilon \in (0,1)$ there exists a constant $c_\varepsilon = c(\varepsilon, \Lambda) > 0$ such that the inequalities

$$\|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon \|p\|_{L^2(0,1)}, \quad \|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon \|p\|_{[0,1]}$$

hold for all $p \in \Pi(\Lambda)$.

Proof. It follows from direct computation of the errors $E(x^\alpha, \Pi((\lambda_k)_{k=0}^n))_2$ that, for all $m \in \mathbb{N}$ and all $p \in \Pi(\Lambda \setminus \{\lambda_m\})$, the inequality

$$\begin{aligned} \|x^{\lambda_m} - p(x)\|_{L^2(0,1)} &\geq \frac{1}{\sqrt{2\lambda_m + 1}} \prod_{k\geq 0;\ k\neq m} \left| \frac{\lambda_m - \lambda_k}{\lambda_m + \lambda_k + 1} \right| \\ &= \frac{1}{\sqrt{2\lambda_m + 1}} \prod_{k\geq 0;\ k\neq m} \left| \frac{(\lambda_k + 1/2) - (\lambda_m + 1/2)}{(\lambda_k + 1/2) + (\lambda_m + 1/2)} \right| \end{aligned} \tag{33}$$

holds. Hence it is of interest to study products of the form:

$$\prod_{k\geq 0;\ k\neq m} \left| \frac{\alpha_k + \alpha_m}{\alpha_k - \alpha_m} \right|$$

for sequences $(\alpha_k)_{k=0}^{\infty}$ such that $\inf_{k\in\mathbb{N}}(\alpha_k - \alpha_{k-1}) > 0$, and $\sum_{k=0}^{\infty} 1/\alpha_k < \infty$. (Note that we have, for ease of exposition, reversed the quotients.)

We decompose:

$$\begin{aligned} \prod_{k\geq 0;\ k\neq m} \left| \frac{\alpha_k + \alpha_m}{\alpha_k - \alpha_m} \right| &= \prod_{k\geq 0;\ \alpha_k < \alpha_m} \left| 1 + \frac{2\alpha_m}{\alpha_k - \alpha_m} \right| \times \\ &\times \prod_{k\geq 0;\ \alpha_m < \alpha_k < 2\alpha_m} \left| 1 + \frac{2\alpha_m}{\alpha_k - \alpha_m} \right| \prod_{k\geq 0;\ \alpha_k \geq 2\alpha_m} \left| 1 + \frac{2\alpha_m}{\alpha_k - \alpha_m} \right|. \end{aligned}$$

Clearly, for all k such that $\alpha_k \geq 2\alpha_m$ we have that $\alpha_k/2 \geq \alpha_m$ so that $\alpha_k - \alpha_m \geq \alpha_k - \alpha_k/2 = \alpha_k/2$. This means that

$$\begin{aligned} \prod_{k\geq 0;\ \alpha_k \geq 2\alpha_m} \left| 1 + \frac{2\alpha_m}{\alpha_k - \alpha_m} \right| &\leq \exp\left(\sum_{k\geq 0;\ \alpha_k \geq 2\alpha_m} \left| \frac{2\alpha_m}{\alpha_k - \alpha_m} \right| \right) \\ &\leq \exp\left(4\alpha_m \sum_{k\geq 0;\ \alpha_k \geq 2\alpha_m} \frac{1}{\alpha_k} \right), \end{aligned}$$

which implies that there exists a constant $\xi_m > 0$ such that $\xi_m \leq 4\sum_{k\geq 0;\ \alpha_k \geq 2\alpha_m} 1/\alpha_k$ and

$$\prod_{k\geq 0;\ \alpha_k \geq 2\alpha_m} \left| 1 + \frac{2\alpha_m}{\alpha_k - \alpha_m} \right| = \exp(\alpha_m \xi_m).$$

The products

$$\prod_{k\geq 0;\ \alpha_k < \alpha_m} \left| 1 + \frac{2\alpha_m}{\alpha_k - \alpha_m} \right|$$

are bounded. Moreover, we can use that in between α_m and $2\alpha_m$ there are $o(\alpha_m)$ terms (since $\sum_k \frac{1}{\alpha_k} < \infty$) to estimate the product $\prod_{k\geq 0;\ \alpha_m<\alpha_k<2\alpha_m} \left|1+\frac{2\alpha_m}{\alpha_k-\alpha_m}\right|$ and prove that there are constants γ_m such that

$$\prod_{k\geq 0;\ k\neq m} \left|\frac{\alpha_k+\alpha_m}{\alpha_k-\alpha_m}\right| \leq \exp(\alpha_m\gamma_m), \qquad \lim_{m\to\infty}\gamma_m = 0.$$

Hence

$$\prod_{k\geq 0;\ k\neq m} \left|\frac{\alpha_k-\alpha_m}{\alpha_k+\alpha_m}\right| \geq \exp(-\alpha_m\gamma_m), \qquad \lim_{m\to\infty}\gamma_m = 0$$

and, taking into consideration the formula (33), we obtain that

$$\|x^{\lambda_m} - p(x)\|_{L^2(0,1)} \geq \exp(-\gamma_m\lambda_m)$$

where $\lim_{m\to\infty}\gamma_m = 0$ and $p \in \Pi((\Lambda\backslash\{\lambda_m\})$. This clearly implies that for every polynomial $p = \sum a_k x^{\lambda_k} \in \Pi(\Lambda)$, the inequality

$$\|p\|_{L^2(0,1)} = \|a_k x^{\lambda_k} - (a_k x^{\lambda_k} - p(x))\|_{L^2(0,1)} \geq |a_k| \exp(-\gamma_k\lambda_k)$$

holds. Hence

$$|a_k| \leq \exp(\gamma_k)^{\lambda_k}\|p\|_{L^2(0,1)} \leq c_\varepsilon(1+\varepsilon)^{\lambda_k}\|p\|_{L^2(0,1)} \tag{34}$$

for a certain constant c_ε, since only a finite number of values γ_k satisfy $\exp(\gamma_k) > 1+\varepsilon$ (the constant c_ε only depends on the behaviour of the other values γ_k). Another proof – indeed the original one – of this inequality was given by Clarkson and Erdös [9] in 1943.

Taking into consideration that $\lambda_1 \geq 1$ and $c = \inf_{k\in\mathbb{N}}(\lambda_k - \lambda_{k-1}) > 0$, we have that there exists a strictly increasing sequence of natural numbers m_j, $j \geq 0$, such that $\{\lfloor\lambda_k\rfloor\}_{k=0}^\infty = \{m_j\}_{j=0}^\infty$, where $\lfloor\lambda_k\rfloor$ denotes the integer part of λ_k for each $k \in \mathbb{N}$. Furthermore,

$$M := M(\Lambda) := \max_{j\geq 0} \#\{k : \lfloor\lambda_k\rfloor = m_j\} < \infty,$$

so that

$$\begin{aligned}
\sum_{k=0}^n \lambda_k \alpha^{\lambda_k - 1} &\leq \sum_{k=0}^n (\lfloor\lambda_k\rfloor + 1)\alpha^{\lfloor\lambda_k\rfloor - 1} \\
&= \sum_{k=0}^n \lfloor\lambda_k\rfloor \alpha^{\lfloor\lambda_k\rfloor - 1} + \sum_{k=0}^n \alpha^{\lfloor\lambda_k\rfloor - 1} \\
&\leq M\left(\sum_{k=0}^n m_k\alpha^{m_k-1} + \sum_{k=0}^n \alpha^{m_k-1}\right) \\
&\leq M\left(\sum_{k=0}^\infty m_k\alpha^{m_k-1} + \sum_{k=0}^\infty \alpha^{m_k-1}\right) \\
&\leq C(\alpha, M) := M\left(\sum_{t=1}^\infty t\alpha^{t-1} + \sum_{t=1}^\infty \alpha^{t-1}\right) < \infty
\end{aligned}$$

for all $\alpha \in (0,1)$.

We can use the above inequality to estimate the norm $\|p'\|_{[0,1-\varepsilon]}$ as follows:

$$\begin{aligned}\|p'\|_{[0,1-\varepsilon]} &\leq \sum_{k=0}^{n} \lambda_k |a_k| (1-\varepsilon)^{\lambda_k - 1} \\ &\leq c_\varepsilon (1+\varepsilon) \sum_{k=0}^{n} \lambda_k [(1+\varepsilon)(1-\varepsilon)]^{\lambda_k - 1} \|p\|_{L^2(0,1)} \\ &= c_\varepsilon (1+\varepsilon) \sum_{k=0}^{n} \lambda_k [(1-\varepsilon^2)]^{\lambda_k - 1} \|p\|_{L^2(0,1)} \\ &\leq c_\varepsilon (1+\varepsilon) C(1-\varepsilon^2, M) \|p\|_{L^2(0,1)} \\ &\leq c(\varepsilon, \Lambda) \|p\|_{L^2(0,1)} \leq c(\varepsilon, \Lambda) \|p\|_{C[0,1]},\end{aligned}$$

which is what we wanted to prove. □

The next theorem, proved by L. Schwartz [30], is an important consequence of the inequality (34).

Theorem 20 (Closure of Nondense Müntz Spaces for Special Sequences) *Let us assume that $\Lambda = (\lambda_k)_{k=0}^{\infty}$ is a sequence of nonnegative real numbers such that $\inf_{k\in\mathbb{N}}(\lambda_k - \lambda_{k-1}) > 0$ and $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$, $\lambda_0 = 0$, $\lambda_1 \geq 1$. Then the functions that belong to the closure of $\Pi(\Lambda)$ can be analytically extended to $\mathbb{D} \setminus [-1,0]$. If λ_k is an integer for all k, then the functions of the closure of $\Pi(\Lambda)$ can be analytically extended to the unit disc. Finally, if Λ is lacunary (i.e., $\inf\{\lambda_k/\lambda_{k-1}\}_{k=2}^{\infty} > 1$) then the closure of $\Pi(\Lambda)$ is precisely the set*

$$\left\{ f \in C[0,1] : f(x) = \sum_{k=0}^{\infty} a_k x^{\lambda_k},\ x \in [0,1] \right\}.$$

Proof. Let us assume that $\lim_{n\to\infty} \|f - q_n\|_{C[0,1]} = 0$, where $q_n(x) := \sum_{k=0}^{k_n} a_{n,k} x^{\lambda_k}$. Then the sequence of polynomials $(q_n)_{n=0}^{\infty}$ is a Cauchy sequence in $C[0,1]$. It follows that for each $\delta > 0$ and all $n \in \mathbb{N}$,

$$|a_{n,k} - a_{m,k}| \leq c_\delta (1+\delta)^{\lambda_k} \|q_n - q_m\|_{C[0,1]} \to 0 \ (\ n, m \to \infty).$$

This means that there are numbers $a_k \in \mathbb{R}$ such that $\lim_{n\to\infty} a_{n,k} = a_k$ $(k \in \mathbb{N})$. Let $h(x) := \sum_{k=0}^{\infty} a_k x^{\lambda_k}$. Then for all $\delta > 0$ we can write

$$|a_k| = \lim_{n\to\infty} |a_{n,k}| \leq \lim_{n\to\infty} c_\delta (1+\delta)^{\lambda_k} \|q_n\|_{C[0,1]} = c_\delta (1+\delta)^{\lambda_k} \|f\|_{C[0,1]}.$$

It follows that the series $h(x) = \sum_{k=0}^{\infty} a_k x^{\lambda_k}$ is absolutely convergent for all $x < 1$. To prove this claim we take into account that $\lambda_k/k \geq c$ for all k, so that the inequality

$$|a_k x^{\lambda_k}|^{1/k} \leq (c_\delta \|f\|_{C[0,1]})^{1/k} ((1+\delta)x)^{\lambda_k/k} \leq (c_\delta \|f\|_{C[0,1]})^{1/k} ((1+\delta)x)^{c} < 1$$

holds for k sufficiently large and δ such that $(1+\delta)x < 1$. Now, it is clear that h coincides with the function f. Consider the branch of logarithm that is defined on the complex plane cut along

$(-\infty, 0]$ and that is positive for values > 1. For any λ this defines a branch of $z^\lambda = \exp(\lambda \log z)$. Now if $z \in \mathbb{D} \setminus [-1, 0]$ then

$$\sum_{k=0}^{\infty} \left| a_k z^{\lambda_k} \right| \leq f(|z|) < \infty.$$

This proves that $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ is analytic on $\mathbb{D} \setminus [-1, 0]$. If λ_k is an integer for all k, then it is clear that the above arguments are also valid for all $z \in (-1, 0)$ so that f is analytic in the unit disc. If the sequence is lacunary then we can use a theorem by Hardy and Littlewood [19] that claims that if the power series $\sum_k a_k x_k^\lambda$, with radius of convergence 1, is lacunary and $\lim_{x \to 1^-} \sum_k a_k x^{\lambda_k} = a$, then $\sum_k a_k = a$, to conclude that the series $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$ also converges for $z = 1$. In the other cases there are counterexamples, i.e., there are series of the form $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$ that belong to the closure of $\Pi(\Lambda)$ in $C[0, 1]$ and they do not converge for $z = 1$ (see [9]). □

We prove, for the special case we are considering in this step, Chebyshev's inequality which claims that the norms of the elements of (nondense) Müntz spaces essentially depend on the behaviour of the elements near $x = 1$.

Corollary 21 (Bounded Chebyshev Inequality for Special Sequences) *Under the hypotheses of Theorem* 19*, for each* $\varepsilon \in (0, 1)$ *there exists a constant* $c_\varepsilon = c(\varepsilon, \Lambda)$ *such that* $\|p\|_{C[0,1]} \leq c_\varepsilon \|p\|_{C[1-\varepsilon,1]}$ *for all* $p \in \Pi(\Lambda)$.

Proof. Making (if necessary) the change of variable $y = x^{1/\lambda_1}$ we may assume, without loss of generality, that $\lambda_1 = 1$. Let us now assume that there exists a sequence of polynomials $(p_n) \subset \Pi(\Lambda)$ such that $\lim_{n\to\infty} \|p_n\|_{C[0,1]} = \infty$ but $\|p_n\|_{C[1-\varepsilon,1]} = 1$ for all n. Then $q_n := p_n / \|p_n\|_{C[0,1]}$ satisfies $\|q_n\|_{C[0,1]} = 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|q_n\|_{C[1-\varepsilon,1]} = 0$. It follows from the bounded Bernstein inequality that for each $\delta \in (0, 1)$ there exists a constant c_δ such that $\|q_n'\|_{[0,1-\delta]} \leq c_\delta$ for all n. We may use the Arzelà-Ascoli theorem in the interval $[0, 1 - \varepsilon/2]$ to obtain from (q_n) a subsequence that converges uniformly to a certain $f \in C[0, 1 - \varepsilon/2]$. Using the same arguments as in the proof of Theorem 20, we get more information: f must be analytic on $(0, 1 - \varepsilon/2)$. But $\lim_{n\to\infty} \|q_n\|_{C[1-\varepsilon,1]} = 0$ implies that $f|_{(1-\varepsilon,1-\varepsilon/2)} = 0$, which clearly implies that f is the null function (just apply the well known Identity Principle of complex analysis). The fact that $f = 0$ and $\|q_n\|_{C[0,1]} = 1$ for all n are simultaneously impossible. □

Step 2. Comparison results

The main goal of this step is to introduce a few results that will be useful for the proof, the next step, of Bernstein's and Chebyshev's inequalities for general sequences of exponents $(\lambda_k)_{k=0}^{\infty}$. These results are expressed in terms of the Chebyshev polynomials associated with the Müntz system $(x^{\lambda_k})_{k=0}^{\infty}$.

Let us proceed by stages. We first introduce some notation. We say that $(f_0, \ldots, f_n)$ is a Descartes system on $[a, b]$ if

$$\det \begin{pmatrix} f_{i_0}(x_0) & f_{i_1}(x_0) & \cdots & f_{i_m}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_0}(x_m) & f_{i_1}(x_m) & \cdots & f_{i_m}(x_m) \end{pmatrix} > 0$$

holds whenever $0 \leq i_0 < i_1 < \cdots < i_m \leq n$ and $a \leq x_0 < x_1 < \cdots < x_m \leq b$. We say that $(f_0, \ldots, f_n)$ is a Markov system in $C[a,b]$ if for all $f \in C[a,b]$ and all $k \leq n$ there exists a unique best approximation to f by elements of $M_k := \text{span}\,\{f_i\}_{i=0}^k$. The nth Chebyshev polynomial associated with the Markov system $(f_i)_{i=0}^n$ is given by $T_n := (f_n - p_{n-1})/\|f_n - p_{n-1}\|_{C[a,b]}$, where p_{n-1} is the unique best approximation to f_n by elements of M_{n-1}. Sometimes, by a misuse of notation, we also say that the Chebyshev polynomial T_n is associated with M_n.

Now, one of the main properties of Müntz spaces is that

$$(x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}),$$

where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$, is a Descartes system on each interval $[a,b] \subset [0,\infty)$ (see Proposition 16). The following technical lemma, whose proof is quite involved, is a nice refinement of the classical Descartes rule of signs and was proved by Pinkus and (independently) by P. W. Smith. It is the key for the proof of the comparison results we will need (see [5, p. 103], or [32] for a proof).

Lemma 22 (Pinkus-Smith) *Let us assume that $(f_0, \ldots, f_n)$ is a Descartes system on $[a,b]$, and let*

$$p = f_k + \sum_{i=1}^{r} a_i f_{k_i};\ q = f_k + \sum_{i=1}^{r} b_i f_{t_i}; \quad \text{with } a_i, b_i \in \mathbb{R}$$

be chosen such that $0 \leq t_i \leq k_i < k$ for all $i \in \{1, \ldots, m\}$ and $k < t_i \leq k_i \leq n$ for all $i \in \{m+1, \ldots, r\}$, with strict inequality for at least one of the indices $i \in \{1, \ldots, r\}$.
If $p(x_i) = q(x_i) = 0$ for the distinct points $x_i \in [a,b]$, $i = 1, \ldots, r$, then

$$|p(x)| \leq |q(x)|, \qquad x \in [a,b].$$

Furthermore, the inequality is strict for all $x \in [a,b] \setminus \{x_i\}_{i=1}^r$.

We use this result with the Müntz spaces $M_n(\Lambda) = \Pi((\lambda_k)_{k=0}^n)$ and $M_n(\Gamma) = \Pi((\gamma_k)_{k=0}^n)$, where we assume that $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$, $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n$, and $\lambda_k \geq \gamma_k$ for all k. With this idea in mind, we take $s \in (0,1)$ and denote by $T_{n,\lambda}$ and $T_{n,\gamma}$ the Chebyshev polynomials associated with $M_n(\Lambda)$ and $M_n(\Gamma)$, respectively, on the interval $[1-s, 1]$.

Lemma 23 *With the hypotheses and notation just introduced, the following claims hold:*

(*a*) *Let $y \in [0, 1-s)$. Then the maximum values of the expressions*

$$\max_{0 \neq p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_{[1-s,1]}} \qquad \text{and} \qquad \max_{0 \neq p \in M_n(\Lambda)} \frac{|p'(y)|}{\|p\|_{[1-s,1]}}$$

are both attained by $p = T_{n,\lambda}$. (In the second case we assume that $\lambda_1 \geq 1$ whenever $y = 0$.)

(*b*) $|T_{n,\lambda}(0)| \leq |T_{n,\gamma}(0)|$. *Furthermore, if $\lambda_1 = \gamma_1 = 1$ then also $|T'_{n,\lambda}(0)| \leq |T'_{n,\gamma}(0)|$.*

Proof. We propose the proof of (a) as an exercise. Let us prove (b). Let $p \in M_n(\Gamma)$ be such that it interpolates $T_{n,\lambda}$ at its zeros (which are all of them simple zeros), and in $(0,1)$. It follows from the Pinkus-Smith Lemma that $|p(x)| \leq |T_{n,\lambda}(x)|$ for all $x \in [0,1]$. In particular, $\|p\|_{[1-s,1]} \leq \|T_{n,\lambda}\|_{[1-s,1]} = 1$ and, taking into account part (a) of this lemma, we get

$$|T_{n,\lambda}(0)| = |p(0)| \leq \frac{|p(0)|}{\|p\|_{[1-s,1]}} \leq \frac{|T_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} = |T_{n,\gamma}(0)|,$$

which proves the first part of (b). To prove the second claim, the argument is similar. We take $0 \neq p \in M_n(\Gamma)$ such that it interpolates $T_{n,\lambda}$ at its zeros in $[1-s,1]$ (there are n zeros), and we normalize by imposing the additional condition $p'(0) = T'_{n,\lambda}(0)$. (Note that $p'(0) \neq 0$, since otherwise we would have that $p \in \operatorname{span}\{x^{\gamma_k} : k = 0,2,3,\ldots,n\}$ has n zeros in $[1-s,1]$, which is impossible since $(x^{\gamma_k} : k = 0,2,3,\ldots,n)$ is a Descartes system.) Then $|p(x)| \leq |T_{n,\lambda}(x)|$ for all $x \in [0,1]$. Hence $\|p\|_{[1-s,1]} \leq \|T_{n,\lambda}\|_{[1-s,1]} = 1$ and it follows again from part (a) of this lemma that

$$|T'_{n,\lambda}(0)| = |p'(0)| \leq \frac{|p'(0)|}{\|p\|_{[1-s,1]}} \leq \frac{|T'_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} = |T'_{n,\gamma}(0)|,$$

which proves the second part of (b). □

Lemma 24 *$|T_{n,\lambda}(x)|$ and $|T_{n,\gamma}(x)|$ are monotone decreasing functions on the interval $[0,1-s]$. Furthermore, if $\lambda_1 = \gamma_1 = 1$, then also $|T'_{n,\lambda}(x)|$ and $|T'_{n,\gamma}(x)|$ are monotone decreasing on the interval $[0,1-s]$.*

Proof. Let us assume that $|T_{n,\lambda}(x)|$ is not monotone decreasing on $[0,1-s]$. Then $T'_{n,\lambda}(x) \in \operatorname{span}\{x^{\lambda_k - 1} : k \in \{1,2,3,\ldots,n\}\}$ has at least n zeros in $(0,1)$, which is impossible. The second claim can be proved by similar arguments. □

Theorem 25 (Comparison Theorem) *The inequality*

$$\max_{0 \neq p \in M_n(\Lambda)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}}$$

holds. Furthermore, if $\lambda_1 = \gamma_1 = 1$ then

$$\max_{0 \neq p \in M_n(\Lambda)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1-s,1]}}.$$

Proof. Let $y \in [0,1-s)$. Then

$$\begin{aligned}
\max_{0 \neq p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_{[1-s,1]}} &= \frac{|T_{n,\lambda}(y)|}{\|T_{n,\lambda}\|_{[1-s,1]}} = |T_{n,\lambda}(y)| \leq |T_{n,\gamma}(0)| \\
&= \frac{|T_{n,\gamma}(0)|}{\|T_{n,\lambda}\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p\|_{[0,1-s]}}{\|p\|_{[1-s,1]}} \\
&\leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}}.
\end{aligned}$$

On the other hand, if $y \in [1-s,1]$, then

$$\max_{0\neq p\in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_{[1-s,1]}} \leq 1 \leq \max_{0\neq p\in M_n(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}}.$$

Hence

$$\max_{0\neq p\in M_n(\Lambda)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}} \leq \max_{0\neq p\in M_n(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}},$$

which is what we wanted to prove. By analogous arguments, we have that

$$\begin{aligned}\max_{0\neq p\in M_n(\Lambda)} \frac{|p'(y)|}{\|p\|_{[1-s,1]}} &= \frac{|T'_{n,\lambda}(y)|}{\|T_{n,\lambda}\|_{[1-s,1]}} = |T'_{n,\lambda}(y)| \leq |T'_{n,\lambda}(0)| \leq |T'_{n,\gamma}(0)| \\ &= \frac{|T'_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} \leq \max_{0\neq p\in M_n(\Gamma)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1-s,1]}},\end{aligned}$$

which is the second claim of the theorem. □

Remark 26 *It is possible (with similar arguments) to extend Theorem 25 to include Müntz polynomials with arbitrary real exponents (i.e., we can also consider negative powers of x).*

Step 3. The General Bernstein and Chebyshev Inequalities

We now complete the proof of Theorem 15.

We know that $\lim_{k\to\infty} \lambda_k/k = \infty$, since $\sum_{k=1}^{\infty} 1/\lambda_k$ converges and (λ_k) is monotone. Let $m \in \mathbb{N}$ be such that $\lambda_k > 2k$ for all $k \geq m$, and let us take $\Gamma := (\gamma_k)_{k=0}^{\infty}$ defined by:

$$\gamma_k := \begin{cases} \min\{\lambda_k, k\}, & \text{if } k \in \{0,1,\ldots,m\}, \\ \frac{1}{2}\lambda_k + k, & \text{if } k > m. \end{cases}$$

Then $\sum_{k=1}^{\infty} 1/\gamma_k < \infty$, $0 \leq \gamma_0 < \gamma_1 < \cdots$, and

$$\gamma_k - \gamma_{k-1} = \begin{cases} \min\{\lambda_k, k\} - \min\{\lambda_{k-1}, k-1\}, & \text{if } k \in \{0,1,\ldots,m\} \\ \frac{1}{2}\lambda_{m+1} + m + 1 - \min\{\lambda_m, m\}, & \text{if } i = m+1 \\ \frac{1}{2}(\lambda_k - \lambda_{k-1}) + 1, & \text{if } k > m+1 \end{cases}$$

satisfies $\gamma_k - \gamma_{k-1} \geq 1$ for all $k \in \mathbb{N}$. Furthermore $\gamma_k \leq \lambda_k$ for all k. This implies (using Theorems 19 and 25, Corollary 21 and Remark 26) that the inequalities

$$\max_{0\neq p\in M_n(\Lambda)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}} \leq \max_{0\neq p\in M_n(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}} = c_\varepsilon < \infty$$

and

$$\max_{0\neq p\in M_n(\Lambda)} \frac{\|p'\|_{[0,1-\varepsilon]}}{\|p\|_{[1-s,1]}} \leq \max_{0\neq p\in M_n(\Gamma)} \frac{\|p'\|_{[0,1-\varepsilon]}}{\|p\|_{[1-s,1]}} = c^*_\varepsilon < \infty,$$

both hold. □

Corollary 27 *Let $(p_n)_{n=0}^{\infty}$ be a sequence of polynomials in $\Pi(\Lambda)$ uniformly bounded in $[0,1]$, and let us assume that $0 \leq \lambda_0 < \lambda_1 < \cdots$ and $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$. Then for each $a \in (0,1)$, the sequence $(p_n)_{n=0}^{\infty}$ is a relatively compact subset of $C[0,a]$.*

Proof. Set $\varepsilon = 1-a$. Then $\|p_n'\|_{[0,a]} \leq c_\varepsilon^* \|p_n\|_{[a,1]} \leq c_\varepsilon^* M$ for all n, where $M = \sup \|p_n\|_{[0,1]} < \infty$. This implies that $\{p_n\}_{n=0}^{\infty}$ is equicontinuous in $C[0,a]$. The corollary follows from the well known Arzelà-Ascoli Theorem. □

3.3 Description of the closure of nondense Müntz spaces: the C[0,1] case

If $\Pi(\Lambda)$ is not a dense subspace of $C[0,1]$, it is natural to ask what is its topological closure. Since the density of $\Pi(\Lambda)$ depends on the convergence character of a certain series associated with the sequence of exponents Λ, it is clear that given two nondense Müntz spaces $\Pi(\Lambda_1)$, $\Pi(\Lambda_2)$, their sum $\Pi(\Lambda_1) + \Pi(\Lambda_2) = \Pi(\Lambda_1 \cup \Lambda_2)$ is also nondense in $C[0,1]$. This means that the closures of nondense Müntz spaces $\Pi(\Lambda)$ are, in a certain sense, of small dimension, when viewed as subspaces of $C[0,1]$.

This observation was first made in a famous paper by Clarkson and Erdős [9] published in 1943 in the Duke Math. Journal. They proved, for the case of integer exponents $\Lambda = (n_k)_{k=0}^{\infty} \subset \mathbb{N}$, that $\sum_{k=0}^{\infty} 1/n_k < \infty$ implies that the elements in the closure of $\Pi(\Lambda)$ are analytic functions defined inside the unit circle and that their Maclaurin series involves only the powers x^{n_k} and may diverge at the point $z=1$. Moreover, if the sequence of exponents is lacunary (which means that $\inf_{k\geq 0} n_{k+1}/n_k = c > 1$), this series converges for $z=1$. Finally, they used this result to prove, in the particular case where the exponents are nonnegative integers and for intervals away from the origin (i.e., intervals $[a,b]$ with $0 \notin [a,b]$), the natural extension of Müntz' theorem (i.e., they proved that $\sum_{k=0}^{\infty} 1/n_k = \infty$ is the necessary and sufficient condition for density of the Müntz space $\Pi((n_k)_{k=0}^{\infty})$ independently of the appearance or not of the zero power in the exponents sequence $(n_k)_{k=0}^{\infty}$).

This same question was tackled by L. Schwartz [30] for certain strictly increasing sequences of exponents (he assumed $\inf_{k\in\mathbb{Z}}(\lambda_k - \lambda_{k-1}) > 0$ and proved Theorem 20 of the previous subsection) and by Borwein and Erdélyi [5] and Erdélyi [12] for general sequences. In all cases the conclusion is that the elements in the closure of a nondense Müntz space are analytic functions. The most general result is the following one, proved by Erdélyi [12] in 2003.

Theorem 28 (Full Clarkson-Erdős-Schwartz Theorem) *Let $\Lambda = (\lambda_k)_{k=1}^{\infty} \subset (0,\infty)$ be a sequence of positive real numbers such that $M := \Pi(\Lambda \cup \{1\})$ is a nondense Müntz subspace of $C[0,1]$. Then every function that belongs to the closure of M in the uniform norm can be represented as an analytic function defined on the set $\{z : z \in \mathbb{C} \setminus (-\infty, 0], |z| < 1\}$.*

3.4 Full Müntz theorem away from the origin

It is remarkable that the extension of the Müntz Theorem to intervals away from the origin is a nontrivial task. Of course, a linear change of variable of the form $x = bt$ allows to extend the Müntz Theorem to the interval $[0,b]$. If $\Pi(\Lambda)$ is dense in $C[0,b]$, then given $f \in C[a,b]$ with $0 < a < b$ one can extend f with continuity to a function $\overline{f} \in C[0,b]$ that vanishes at the origin. This function can of course be approximated uniformly on $[0,b]$ by elements of $\Pi(\Lambda \setminus \{0\})$, so that f also belongs to the closure of $\Pi(\Lambda \setminus \{0\})$ in $C[a,b]$. This means that if the Müntz condition is satisfied, then the Müntz polynomials are dense in $C[a,b]$.

The difficult part is to prove that the Müntz condition is also necessary for intervals away from the origin and, as we have already noted, this was proved for the first time and for the particular case of nonnegative integer exponents by Clarkson and Erdős. Their work was continued by L. Schwartz who proved a full Müntz theorem for intervals away from the origin and general sequences of exponents. In particular, he noticed that if we assume $0 < a < b$, then the monomials x^λ are continuous functions for all $\lambda \in \mathbb{R}$ so that it makes sense to ask for necessary and sufficient conditions for arbitrary sequences of real numbers $\Lambda = (\lambda_k)_{k=0}^\infty \subset \mathbb{R}$ in order to make $\Pi(\Lambda)$ a dense subspace of $C[a,b]$, and proved the following nice result.

Theorem 29 (Full Müntz Theorem away from the Origin) *Let $\Lambda = (\lambda_k)_{k=0}^\infty \subset \mathbb{R}$ be a sequence of distinct real numbers, and let $0 < a < b$. Then $\Pi(\Lambda)$ is dense in $C[a,b]$ if and only if $\sum_{\lambda_k \neq 0} 1/|\lambda_k| = \infty$.*

We devote this subsection to providing a proof of this result. Clearly, there is no loss of generality if we assume that $0 < a < b = 1$. We start by assuming that the exponents can be rearranged in such a way that they form a biinfinite sequence $(\lambda_k)_{k=-\infty}^\infty$ satisfying the following restrictions:

- $\lambda_k > 0$ for all $k > 0$,
- $\lambda_k < 0$ for all $k < 0$,
- $\inf_{k\in\mathbb{Z}}(\lambda_k - \lambda_{k-1}) > 0$.

We define, for each polynomial $p(z) = \sum_{|k|\leq n} a_k z^{\lambda_k}$, the associated polynomials

$$p^+(z) := \sum_{0\leq k\leq n} a_k z^{\lambda_k} \qquad \text{and} \qquad p^-(z) := \sum_{-n\leq k<0} a_k z^{\lambda_k}.$$

Under these restrictions, it is possible to prove the following relations between the uniform norms of the polynomials p^+, p^- and p:

Lemma 30 *Let $\Lambda = (\lambda_k)_{k=-\infty}^\infty$ satisfy the conditions we have just described and let us also assume that $\sum_{k\in\mathbb{Z}\setminus\{0\}} 1/|\lambda_k| < \infty$. Then there exists a constant $c = c(\Lambda)$ such that*

$$\|p^+\|_{C[a,b]} \leq c\|p\|_{C[a,b]} \qquad \textit{and} \qquad \|p^-\|_{C[a,b]} \leq c\|p\|_{C[a,b]}$$

hold for all $p \in \Pi(\Lambda)$.

Proof. We assume that $0 < a < b = 1$. It is sufficient to prove the first inequality of the lemma since the other one is obtained from the first via the change of variable $y = x^{-1}$. If we see the map $p \mapsto p^-$ as a linear projector $L : \Pi(\Lambda) \to \Pi((\lambda_k)_{k=-\infty}^{-1})$, the inequality we want to prove can be reformulated as: *L is bounded whenever we use the uniform norm in the interval $[a,1]$ for both spaces $\Pi(\Lambda)$ and $\Pi((\lambda_k)_{k=-\infty}^{-1})$.*

If L is unbounded then there exists a sequence $(p_n)_{n=0}^\infty \subset \Pi(\Lambda)$ such that $\|p_n^-\|_{[a,1]} = 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \|p_n\|_{[a,1]} = 0$. This clearly implies that $\{p_n^+\}_{n=0}^\infty$ is a bounded subset of $C[a,1]$ (just take into consideration that $p_n = p_n^+ + p_n^-$ for all n), so that it is also a bounded subset of $C[0,1]$, since $\|p_n^+\|_{[0,1]} \leq c_a \|p_n^+\|_{[a,1]}$ holds for all n. We can use Theorem 20 and Corollary 21 to prove that there exists a sequence of natural numbers $(n_i)_{i=0}^\infty$ such that $(p_{n_i}^+)_{i=0}^\infty$ converges uniformly on compact subsets of $[0,1)$ to a certain function $f^+ = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ analytic on $\mathbb{D}_1 := (\mathbb{C}\setminus(-\infty,0])\cap\mathbb{D}(0,1)$, and the sequence $(p_{n_i}^-)_{i=0}^{\infty}$ converges uniformly on compact subsets of (a,∞) to a certain function $f^- = \sum_{k=-\infty}^{-1} a_k z^{\lambda_k}$ which is analytic on $\mathbb{E}_a := (\mathbb{C}\setminus(-\infty,0])\cap(\mathbb{C}\setminus\overline{\mathbb{D}}(0,a))$. The last claim can be proved by just making the change of variable $t = a/x$, since then $r_n^+(t) = p_n^-(a/x) \in \Pi((-\lambda_k)_{k=-\infty}^{-1})$ is a Cauchy sequence in $C[a,1]$, so that we can assume that $(r_{n_i}^+)_{i=0}^{\infty}$ converges uniformly on compact subsets of $[0,1)$ to a certain $h^+ = \sum_{k=-\infty}^{-1} h_k z^{-\lambda_k}$ which is analytic in $\mathbb{D}_1 := (\mathbb{C}\setminus(-\infty,0])\cap\mathbb{D}(0,1)$, for an adequate choice of $(n_k)_{k=0}^{\infty}$. We then go back with the change of variable, obtaining that $f^- = h^+(a/z) = \sum_{k=-\infty}^{-1} a_k z^{\lambda_k}$ (where $a_k = h_k a^{-\lambda_k}$ for all k) is analytic in $\mathbb{E}_a$ and $(p_{n_k}^-)_{k=0}^{\infty}$ converges uniformly to f^- on compact subsets of (a,∞). Now, $\lim_{n\to\infty} \|p_{n_k}\|_{[a,1]} = 0$ and $p_{n_k} = p_{n_k}^+ + p_{n_k}^-$ for all i, so that $f^+ + f^- = 0$ in $(a,1)$. This implies that

$$g(z) := \begin{cases} f^+(\mathrm{e}^z), & \mathrm{Re}(z) < 0, \\ -f^-(\mathrm{e}^z), & \mathrm{Re}(z) > \log a, \end{cases}$$

can be extended as a bounded entire function [5, page 181]. It follows from Liouville's theorem that $g = \text{const}$, so that $g = 0$ since $\lim_{t\to\infty} f^-(t) = 0$. Hence $f^+ = 0$ in $[0,1)$ and $f^- = 0$ in (a,∞), which implies $\lim_{k\to\infty} \|p_{n_k}^-\|_{[a,1]} = 0$, a contradiction. □

We now characterize the closure of $\Pi(\Lambda)$ whenever $\Lambda = (\lambda_k)_{k=-\infty}^{\infty}$ satisfies the additional condition $\sum_{\lambda_k\neq 0} 1/|\lambda_k| < \infty$.

Theorem 31 *Assume that $\Lambda = (\lambda_k)_{k=-\infty}^{\infty}$ satisfies $\lambda_k > 0$ for all $k > 0$, $\lambda_k < 0$ for all $k < 0$, $\inf_{k\in\mathbb{Z}}(\lambda_k - \lambda_{k-1}) > 0$, and $\sum_{\lambda_k\neq 0} 1/|\lambda_k| < \infty$. Then the elements of the closure of $\Pi(\Lambda)$ in $C[a,b]$ can be extended as analytic functions to the domain*

$$\{z : z \in \mathbb{C}\setminus(-\infty,0],\ a < |z| < b\}.$$

Proof. Let us assume, without loss of generality, that $0 < a < b = 1$. We have already proved in Lemma 30, under the hypotheses we have imposed on Λ, that if f belongs to the closure of $\Pi(\Lambda)$ in $C[a,b]$ then $f = f^+ + f^-$, where f^+ is analytic in $\mathbb{D}_1$ and f^- is analytic in $\mathbb{E}_a$. Hence f is analytic in

$$\mathbb{D}_1 \cap \mathbb{E}_a = \{z : z \in \mathbb{C}\setminus(-\infty,0],\ a < |z| < 1\},$$

and the proof is complete. □

Proof of the Full Müntz theorem away from the Origin. Let us decompose the proof into the following four cases:

Case 1. $\{\lambda_k\}_{k=0}^{\infty}$ has some accumulation point $\lambda \neq 0$.

We can assume without loss of generality that $\lambda > 0$. (Otherwise, consider the map $S : C[a,b] \to C[a,b]$ given by $S(f)(x) = x^{-\lambda+1} f(x)$ and take into account that S is a linear isomorphism of Banach spaces, so that it transforms dense subspaces into dense subspaces and *vice versa*.) Hence this case is an easy corollary of the Full Müntz Theorem for the interval $[0,b]$.

Case 2. 0 is an accumulation point of $\{\lambda_k\}_{k=0}^{\infty}$.

This case is reduced to Case 1 as follows: first we consider the sequence $(\lambda_k + 1)_{k=0}^{\infty}$, which is in Case 1, so that $\Pi((\lambda_k + 1)_{k=0}^{\infty})$ is dense in $C[a,b]$. Now we consider the isomorphism of Banach spaces given by $T : C[a,b] \to C[a,b]$, $(Tf)(x) := x^{-1}f(x)$ and conclude that $\Pi((\lambda_k)_{k=0}^\infty)$ is dense in $C[a,b]$, since $T(\Pi((\lambda_k+1)_{k=0}^\infty)) = \Pi((\lambda_k)_{k=0}^\infty)$.

Case 3. $\{\lambda_k\}_{k=0}^\infty$ has no accumulation points and $\sum_{\lambda_k>0}\frac{1}{\lambda_k} = \infty$ or $\sum_{\lambda_k<0}\frac{1}{|\lambda_k|} = \infty$.

In this case, we may assume, without loss of generality, that $0 \notin \{\lambda_k\}_{k=0}^\infty$ since otherwise we can take $\varepsilon > 0$ such that $0 \notin \{\lambda_k + \varepsilon\}_{k=0}^\infty$ and (by using the same kind of arguments as in Case 2) we have that the density of $\Pi((\lambda_k)_{k=0}^\infty)$ and the density of $\Pi((\lambda_k+\varepsilon)_{k=0}^\infty)$ are equivalent claims.

If $\sum_{\lambda_k>0}\frac{1}{\lambda_k} = \infty$, then we can use the Full Müntz Theorem on $C[0,b]$ to conclude the proof. If $\sum_{\lambda_k<0}\frac{1}{|\lambda_k|} = \infty$ but $\sum_{\lambda_k>0}\frac{1}{\lambda_k} < \infty$, then we use the change of variable $t = 1/x$, and that $S : C[a,b] \to C[1/b,1/a]$, $S(f)(x) := f(1/x)$ is a linear isometry ($\|S(f)\|_{C[1/b,1/a]} = \|f\|_{C[a,b]}$ is clear), to prove that $\Pi((\lambda_k)_{k=0}^\infty)$ is dense in $C[a,b]$ if and only if $S(\Pi((\lambda_k)_{k=0}^\infty)) = \Pi((-\lambda_k)_{k=0}^\infty)$ is dense in $C[1/b,1/a]$, which puts us once more in the case $\sum_{\lambda_k>0}\frac{1}{\lambda_k} = \infty$.

Case 4. $\sum_{\lambda_k\neq 0} 1/|\lambda_k| < \infty$.

We rearrange the sequence $(\lambda_k)_{k=0}^\infty$ as $(\lambda_k^*)_{k=-\infty}^\infty = (\lambda_k)_{k=0}^\infty$, with $\lambda_k^* < \lambda_{k+1}^*$ for all $k \in \mathbb{Z}$, $\lambda_k^* < 0$ if $k < 0$ and $\lambda_k^* > 0$ if $k > 0$. Then there exists a sequence $\Gamma := (\gamma_k)_{k=-\infty}^\infty$ such that $\inf_{k\in\mathbb{Z}}(\gamma_k - \gamma_{k-1}) > 0$, $\sum_{k\in\mathbb{Z}} 1/|\gamma_k| < \infty$, and $\gamma_k < \gamma_{k+1}$, $|\gamma_k| < |\lambda_k^*|$ for all $k \in \mathbb{Z}$, $\gamma_k < 0$ if $k < 0$ and $\gamma_k > 0$ if $k \geq 0$. Now, it follows from Theorem 31 that there is an m such that $x^m \notin \overline{\Pi(\Gamma)}$ and from the comparison theorem for real exponents (see also Remark 26) that $x^m \notin \overline{\Pi(\Lambda)}$.

This completes the proof of the Full Müntz Theorem away from the origin. □

3.5 Full Müntz theorem for measurable sets

Borwein and Erdélyi have recently published several papers in which they prove a Full Müntz Theorem for the spaces $C(A)$ and

$$L_w^q(A) = \{f : \left(\int_A |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} < \infty\},$$

for sets A with positive Lebesgue measure and weight functions w (i.e., $w > 0$ is measurable in the sense of Lebesgue). To be more precise, we state here one of their main results (see [6],[7], and [8]).

Theorem 32 (Borwein-Erdélyi) *If $\Lambda = (\lambda_k)_{k=-\infty}^\infty \subset \mathbb{R}$ is a sequence of distinct real numbers with $\lambda_k < 0$ for all $k < 0$, $\lambda_k \geq 0$ for all $k \geq 0$ such that $\sum_{\lambda_k\neq 0} 1/|\lambda_k| < \infty$ and $A \subset (0,\infty)$ is a set with positive Lebesgue measure such that $\inf A > 0$, then $\Pi(\Lambda)$ is not dense in $L_w^q(A)$ for all weight functions $w : A \to [0,\infty)$ with $\int_A w > 0$ and all $q \in (0,\infty)$. Moreover, every function that belongs to the closure of $\Pi(\Lambda)$ in $L_w^q(A)$ can be analytically extended to the domain $\{z : z \in \mathbb{C} \setminus (-\infty, 0],\ a_w < |z| < b_w\}$, where*

$$a_w := \inf\{y \in [0,\infty) : \int_{A\cap(0,y)} w > 0\},$$

$$b_w := \sup\{y \in [0,\infty) : \int_{A\cap(y,\infty)} w > 0\}.$$

Finally, if

$$\inf\{\lambda_k - \lambda_{k-1} : k \in \mathbb{Z}\} > 0$$

then all functions that belong to the closure of $\Pi(\Lambda)$ *in* $L^q_w(A)$ *admit a representation of the form*

$$f(x) = \sum_{k=-\infty}^{\infty} a_k x^{\lambda_k}, \qquad x \in A \cap (a_w, b_w).$$

The key idea in proving Theorem 32 is to use Egorov's theorem and the following important polynomial inequality (see [7]):

Theorem 33 (Remez Inequality for Müntz Polynomials) *Let* $\Lambda = (\lambda_k)_{k=-\infty}^{\infty} \subset \mathbb{R}$ *be an arbitrary sequence of real numbers. If* $\sum_{\lambda_k \neq 0} 1/|\lambda_k| < \infty$, *then for all sets* $A \subset [0, \infty)$ *with Lebesgue measure* $m(A) > 0$ *and all intervals* $[\alpha, \beta] \subset (\operatorname{ess\,inf}(A), \operatorname{ess\,sup}(A))$, *there exists a constant* $c = c(\Lambda, A, \alpha, \beta)$ *such that*

$$\|p\|_{C[\alpha,\beta]} \leq c \, \|p\|_{C(A)}$$

for all $p \in \Pi(\Lambda)$.

In fact, it follows from this theorem that we can easily prove the following result, which is a main step in the proof of the corresponding Full Müntz Theorem for sets of positive Lebesgue measure:

Corollary 34 *Let* $\Lambda = (\lambda_k)_{k=-\infty}^{\infty} \subset \mathbb{R}$ *be an arbitrary sequence of real numbers such that* $\sum_{\lambda_k \neq 0} 1/|\lambda_k| < \infty$. *Then, for any set* $A \subset [0, \infty)$ *with positive Lebesgue measure* $m(A) > 0$, *we have that if the sequence of polynomials* $(p_n)_{n=0}^{\infty} \subset \Pi(\Lambda)$ *converges pointwise to* $f \in C(A)$, *then for all* $[\alpha, \beta] \subset (a, b) := (\operatorname{ess\,inf}(A), \operatorname{ess\,sup}(A))$, $(p_n)_{n=0}^{\infty}$ *is a Cauchy sequence in* $C[\alpha, \beta]$.

Proof. First of all, we would like to recall that Egorov's theorem guarantees that if (f_n) is a sequence of measurable functions on A (where $0 < m(A) < \infty$) that converges almost everywhere to a certain function f (that is finite almost everywhere on A), then for all $\varepsilon > 0$ there exists a measurable set $B \subset A$ such that $m(A \setminus B) < \varepsilon$ and (f_n) converges uniformly on B to f.

Let $(p_n)_{n=0}^{\infty}$ and f satisfy the hypotheses of this corollary and let $[\alpha, \beta] \subset (a, b)$. It follows from the definition of (a, b) and from Egorov's theorem that there are sets of positive Lebesgue measure

$$B_1 \subset A \cap (0, \alpha) \text{ and } B_2 \subset A \cap (\beta, \infty)$$

such that $(p_n)_{n=0}^{\infty}$ converges uniformly to f on $B = B_1 \cup B_2$.

Now, the application of the Remez inequality for Müntz polynomials on $[\alpha, \beta] \subset (\rho, \sigma)$ (where $\rho := \operatorname{ess\,inf}(B)$ and $\sigma := \operatorname{ess\,sup}(B)$),

$$\|p_i - p_j\|_{[\alpha,\beta]} \leq C(B, [\alpha, \beta], \Lambda) \, \|p_i - p_j\|_B \, ,$$

proves that $(p_n)_{n=0}^{\infty}$ is a Cauchy sequence in $C[\alpha, \beta]$. $\square$

3.6 Full Müntz theorem for countable compact sets

It is quite surprising that for a long period of time the Müntz Theorem has been studied in many cases but not for the space $C(K)$ with K a countable compact set. This is surprising because, in principle, this case should be the easiest one. This question has been addressed quite recently by the author [2], and it transpires that in many cases the Müntz condition can be weakened in a sensible way when dealing with countable compact sets. In particular, the following result holds.

Theorem 35 (Almira, 2006) *Let $K \subset [0,\infty)$ be an infinite countable compact set and let $\Lambda = (\lambda_k)_{k=0}^{\infty} \subset \mathbb{R}$ be a fixed sequence of exponents, satisfying $\lambda_0 = 0$. Then the following holds:*

i) *If $\Lambda \subset [0,\infty)$ is an infinite bounded sequence and $K \setminus \{0\}$ is compact then $\Pi(\Lambda)$ is dense in $C(K)$.*

ii) *If $\Lambda \subset [0,\infty)$ and K does not contain strictly increasing infinite sequences then $\Pi(\Lambda)$ is dense in $C(K)$ if and only if $\#\Lambda = \infty$. Moreover, if $\Lambda \subset (-\infty,0]$ and K does not contain strictly decreasing infinite sequences then $\Pi(\Lambda)$ is dense in $C(K)$ if and only if $\#\Lambda = \infty$.*

Proof. The main idea in the proof is to use the Riesz Representation Theorem. Clearly, the unique measures that exist for countable compact sets are atomic. Thus, if $K = \{0\} \cup \{t_i\}_{i=1}^{\infty}$ then $L \in C^*(K)$ if and only if $L(f) = \alpha_0 f(0) + \sum_{i=1}^{\infty} \alpha_i f(t_i)$ for a certain sequence $(\alpha_i)_{i=0}^{\infty}$ such that $\sum_{i=0}^{\infty} |\alpha_i| < \infty$. Thus, as a consequence of the Hahn-Banach Theorem, span$\{x^{\lambda_k}\}_{k=0}^{\infty}$ is dense in $C(K)$ if and only if the following holds: if $\sum_{i=0}^{\infty} \alpha_i = 0$,

$$\sum_{i=1}^{\infty} \alpha_i t_i^{\lambda_k} = 0, \quad k = 1,2,\dots, \quad \text{and} \quad \sum_{i=0}^{\infty} |\alpha_i| < \infty,$$

then $\alpha_i = 0$ for all $i \geq 0$.

Thus, let us assume that

$$\sum_{i=0}^{\infty} \alpha_i = 0, \quad \sum_{i=1}^{\infty} \alpha_i t_i^{\lambda_k} = 0, \quad k = 1,2,\dots, \quad \text{and} \quad \sum_{i=0}^{\infty} |\alpha_i| < \infty.$$

Then we set $\Gamma := \{t_i : i \geq 1, \alpha_i \neq 0\}$ and we take $\gamma := \sup \Gamma$. Clearly, $\gamma \in K$ since K is compact. If $\Gamma = \emptyset$ then $L(f) = \alpha_0 f(0)$ and $L(1) = 0$ implies $\alpha_0 = 0$, which ends the proof. If $\Gamma \neq \emptyset$ then $\gamma > 0$ and there exists $t_s \in K$ such that $\gamma = t_s$. Thus, we take $t_a \in K$ such that $t_a < t_s$ and we set $z_\lambda := (t_a/t_s)^{\lambda}$. Clearly, the equation $z_\lambda^{p_j} = (t_j/t_s)^{\lambda}$ is uniquely solved by $p_j = (\ln(t_j/t_s))/\ln(t_a/t_s)$, which is a positive real number for all $j \neq s$. Hence $L(x^{\lambda_k}) = 0$, $k = 0,1,2,\dots$, can be written in the following equivalent way:

$$0 = \sum_{i=0}^{\infty} \alpha_i \quad \text{and} \quad 0 = (t_s)^{\lambda_k} \sum_{t_i \in \Gamma} \alpha_i \left(\frac{t_i}{t_s}\right)^{\lambda_k}, \quad k = 1,2,\dots.$$

Hence $\varphi(z_{\lambda_k}) = 0$ for all $k \geq 1$, where

$$\varphi(z) := \sum_{t_i \in \Gamma} \alpha_i z^{p_i}.$$

We decompose the proof into several steps, according to the boundedness properties of the sequence of exponents Λ.

Step 1. $\Lambda \subset [0,\infty)$ and $\lim_{k\to\infty}\lambda_k = \infty$ and K does not contain strictly increasing infinite sequences.

Under these conditions, it is clear that $t_s \in \Gamma$ and $\lim_{k\to\infty} z_{\lambda_k} = 0$. Thus $\varphi(0) = \lim_{k\to\infty}\varphi(z_{\lambda_k}) = 0$ since $\varphi(z)$ is continuous at the origin. On the other hand, $t_s \in \Gamma$ implies that $\alpha_s \neq 0$. Hence we can use the fact that $\varphi(z) = \sum_{t_i\in\Gamma\setminus\{t_s\}} \alpha_i z^{p_i} + \alpha_s$ (since $p_s = (\ln 1)/\ln(t_a/t_s) = 0$) to claim that $\varphi(0) = \alpha_s \neq 0$, a contradiction.

Step 2. $\Lambda = (\lambda_k)_{k=0}^{\infty} \subset [0,\infty)$ is bounded, $0 \neq \lim_{k\to\infty}\lambda_k$.

Clearly, we can assume without loss of generality that Λ is itself a convergent sequence. We note that $\varphi(z) = \sum_{t_i\in\Gamma}\alpha_i z^{p_i}$ is analytic in the open set $\Omega = \{z : |z| < 1, |1-z| < 1\}$. If $\lim_{k\to\infty}\lambda_k = \lambda^* \neq 0$ then $\lim_{k\to\infty} z_{\lambda_k} = z_{\lambda^*} \in (0,1) \subset \Omega$. Hence φ vanishes on a set with accumulation points inside Ω, so that $\varphi(z)$ vanishes identically on Ω and $\alpha_i = 0$ for all $i > 0$. If $0 \notin K$ the proof is complete. On the other hand, if $0 \in K$ then $0 = L(1) = \sum_{t_i\in\Gamma}\alpha_i + \alpha_0 = \alpha_0$ and the proof is also complete.

Step 3. $\lim_{k\to\infty}\lambda_k = 0$ and $K \setminus \{0\}$ is compact.

In this case, we can use the following trick: the equations

$$0 = \sum_{t_i\in\Gamma}\alpha_i t_i^{\lambda_k}, \quad k = 1,\dots$$

can be rewritten as

$$0 = \sum_{t_i\in\Gamma}\beta_i t_i^{\lambda_k^*}, \quad k = 1,\dots,$$

where $\beta_i := \alpha_i/t_i$ for all i and $\lambda_k^* := \lambda_k + 1$ for all k (taking into account that $\sum_{t_i\in\Gamma}|\beta_i| < \infty$ since $K\setminus\{0\}$ is compact). Thus $\lim_{k\to\infty}\lambda_k^* = 1$ and we conclude that $\alpha_j/t_j = 0$ for all j. The proof follows.

Step 4. $\Lambda \subset \mathbb{R}$ and $K\setminus\{0\}$ is compact.

Clearly, if Λ is an infinite set then it contains either infinitely many positive elements or infinitely many negative elements. Thus, we may assume that either $\Lambda \subset [0,\infty)$ or $\Lambda \subset (-\infty,0]$. The first case has been already studied in Steps 1 and 2. Thus, let us assume that $\Lambda \subset (-\infty,0]$ and $L(f) = \alpha_0 f(0) + \sum_{j=1}^{\infty}\alpha_j f(t_j) \in C^*(K)$. Then the equations $L(x^{\lambda_k}) = 0$, $k = 0,1,\dots$, can be rewritten as

$$\sum_{i=0}^{\infty}\alpha_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{\infty}\alpha_i(1/t_i)^{\lambda_k} = 0, \quad k = 1,2,\dots.$$

This means that the functional given by

$$S(f) := \alpha_0 f(0) + \sum_{j=1}^{\infty}\alpha_j f(1/t_j),$$

which belongs to $C^*(E)$, where $E := \{0\}\cup\{1/t_j\}_{j=1}^{\infty}$, which is a countable compact subset of $[0,\infty)$ since $K\setminus\{0\}$ is compact, satisfies $S(x^{-\lambda_k}) = 0$ for all $k \geq 0$. Moreover, if K does not contain decreasing sequences then E does not contain increasing sequences. Now, we use the results proved in Steps 1, 2 and 3 to conclude that $\alpha_i = 0$ for all i. $\square$

Remark 36 There is another proof of step 2. Taking into consideration that $t_i^{\lambda_k} = \exp(\lambda_k \log t_i)$ for all i, we have that the relations

$$\sum_{t_i \in \Gamma} \alpha_i t_i^{\lambda_k} = 0, \qquad k = 1, 2, \dots$$

are equivalent to the relations

$$\Psi(\lambda_k) = 0, \qquad k = 1, 2, \dots$$

where

$$\Psi(z) := \sum_{t_i \in \Gamma} \alpha_i \exp((\log t_i) z)$$

is an entire function of exponential type. This means, in particular, that Λ cannot be an infinite bounded sequence (otherwise, Ψ should vanish everywhere).

Remark 37 Clearly, if Λ is bounded then $|\lambda_k|^{-1} \geq 1/\sup \Lambda$ for all $\lambda_k \neq 0$. Hence $\sum_{k=1}^{\infty} |\lambda_k|^{-1} = \infty$ and case (i) of Theorem 35 follows from the Müntz Theorem away from the origin (Theorem 29) whenever $0 \notin K$. This proof uses a very difficult result in order to prove a simpler one. This is the reason we gave our own elementary proof of this fact.

Remark 38 There are many countable compact sets with the property that they do not have (strictly) increasing sequences. An interesting example is given by:

$$K = \{0\} \cup \{1/n\}_{n=1}^{\infty} \cup \{1/n + 1/m\}_{n,m=1}^{\infty} .$$

Obviously, this compact set has infinitely many accumulation points and it has no increasing sequences! These cases are covered by Theorem 35 above.

Open question. We have already shown that in order to give a Full Müntz Theorem for the general case (i.e., for arbitrary countable compact sets $K \subset [0, \infty)$), it is a good idea to study the zero sets of the Müntz type series

$$\varphi(z) = \sum_{i=1}^{\infty} \alpha_j z^{p_j},$$

where $(p_j)_{j=1}^{\infty}$ decreases to zero, $\sum_{j=1}^{\infty} |\alpha_j| < \infty$ and, for the case in which $K \setminus \{0\} = \{t_i\}_{i=1}^{\infty}$ is compact, the zero sets of the entire functions of exponential type given by

$$\Psi(z) = \sum_{t_i \in \Gamma} \alpha_i \exp((\log t_i) z) \qquad \text{where} \qquad \sum_{j=1}^{\infty} |\alpha_j| < \infty.$$

Is it possible to find a series $\varphi(z)$ with a sequence of infinitely many zeros $(z_k)_{k=0}^{\infty}$ that converges to zero? What about a function $\Psi(z)$ with infinitely many zeros? These questions seem to be still open and not easy to solve.

Acknowledgement. The author is infinitely grateful to the referee of a previous version of this paper. With his (her) help the manuscript improved not only its readability but also the details in many proofs. Thanks.

References

[1] N. I. Achieser, *Theory of Approximation*, Ungar, New-York, 1956.

[2] J. M. Almira, On Müntz Theorem for countable compact sets, *Bull. Belg. Math. Soc. Simon Stevin* **13** (2006) 69–73.

[3] S. N. Bernstein, Sur les recherches récentes relatives à la meilleure approximation des fonctions continues par les polynômes, in *Proc. of 5th Inter. Math. Congress* Vol. 1, 1912, 256-266.

[4] S. N. Bernstein, Sur les fonctions absolument monotones, *Acta Mathematica*, 1928, 1-66.

[5] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1996.

[6] P. Borwein, T. Erdélyi, The Full Müntz Theorem in $C[0,1]$ and $L_1(0,1)$, *J. London Math. Soc.* **54** (1996) 102–110.

[7] P. Borwein, T. Erdélyi, Generalizations of Müntz Theorem via a Remez-type inequality for Müntz spaces, *J. Amer. Math. Soc.* **10** (1997) 327–349.

[8] P. Borwein, T. Erdélyi, Müntz's Theorem on Compact Subsets of Positive Measure, in *Approximation Theory*, eds. Govil et. al., Marcel Dekker, (1998) 115–131.

[9] J. A. Clarkson, P. Erdős, Approximation by polynomials, *Duke Math. J.* **10** (1943) 5–11.

[10] P. J. Davis, *Interpolation and Approximation*, Dover, 1975.

[11] T. Erdélyi, W. B. Johnson, The full Müntz Theorem in $L_p(0,1)$ for $0 < p < \infty$, *J. Anal. Math.* **84** (2001) 145–172.

[12] T. Erdélyi, The "Full Clarkson-Erdős-Schwartz Theorem" on the closure of non-dense Müntz spaces, *Studia Math.* **155** (2003) 145–152.

[13] T. Erdélyi, The "Full Müntz Theorem" revisited, *Constr. Approx.* **21** (2005) 319–335.

[14] R. P. Feinerman, D. J. Newman, *Polynomial Approximation*, Williams and Wilkins, Baltimore, 1974.

[15] W. Feller, On Müntz' theorem and completely monotone functions, *Amer. Math. Monthly* **75** (1968) 342–350.

[16] A. O. Gelfond, *Differenzenrechnung*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1958.

[17] M. v. Golitschek, A short proof of Müntz Theorem, *J. Approx. Theory* **39** (1983) 394–395.

[18] V. I. Gurariy, W. Lusky, *Geometry of Müntz Spaces and Related Questions*, Lecture Notes in Mathematics Vol. **1870**, Springer, 2005.

[19] G. H. Hardy, J. E. Littlewood, A further note on the converse of Abel's theorem, *Proc. London Math. Soc.* **25** (1926) 219-236.

[20] I. I. Hirschman Jr., D. V. Widder, Generalized Bernstein Polynomials, *Duke Math. J.* **16** (1949) 433–438.

[21] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, 1962.

[22] A. N. Kolmogorov, Une généralisation de l'inégalité de M. J. Hadamard entre les bornes supérieures des dérivées successives d'une fonction, *C. R. Acad. Sci. Paris* **207** (1938) 764–765.

[23] G. G. Lorentz, *Bernstein Polynomials,* University of Toronto, Toronto, 1953.

[24] W. A. J. Luxemburg, J. Korevaar, Entire functions and Müntz-Szász type approximation, *Trans. Amer. Math. Soc.* **157** (1971) 23–37.

[25] Ch. H. Müntz, Über den Approximationssatz von Weierstrass, in H. A. Schwarz's *Festschrift*, Berlin, 1914, pp. 303-312.

[26] D. J. Newman, Derivative Bounds for Müntz Polynomials, *J. Approx. Theory* **18** (1976) 360–362.

[27] V. Operstein, Full Müntz Theorem in $L_p(0,1)$, *J. Approx. Theory* **85** (1996) 233–235.

[28] A. Pinkus, Weierstrass and Approximation Theory, *J. Approx. Theory* **107** (2000), 1–66.

[29] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New-York, 1966.

[30] L. Schwartz, *Étude des Sommes D'Exponentielles*, Hermann, Paris, 1959.

[31] A. R. Siegel, On the Müntz-Szász Theorem for $C[0,1]$, *Proc. Amer. Math. Soc.* **36** (1972) 161–166.

[32] P. W. Smith, An improvement theorem for Descartes Systems, *Proc. Amer. Math. Soc.* **70** (1978) 26–30.

[33] O. Szász, Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen, *Math. Ann.* **77** (1916) 482–496.

[34] O. Szász, On closed sets of rational functions, *Ann. Mat. Pura. Appl.* **34** (1953) 195–218.

[35] K. Yosida, *Functional Analysis*, Grundlehren, Springer-Verlag, 1965.

J. M. Almira.
Departamento de Matemáticas. Universidad de Jaén.
E.U.P. Linares 23700 Linares (Jaén) Spain
email: jmalmira@ujaen.es