Introduction to

Conformal Geometry

Lecture 3

Ambient Metric

Tractor Bundle & Connection

Today I will describe two objects associated to a general conformal manifold $(M,[g])$: the ambient metric and the tractor bundle & connection. Both constructions were introduced classically, forgotten, and later rediscovered.

Ambient metric: Haantjes-Schouten 1936-7
Rediscovered by Fefferman-Graham 1985

Tractor bundle/connection: T. Thomas 1920's
Rediscovered by Bailey-Eastwood-Gover 1994

Both constructions have been at the center of much recent progress in conformal geometry.

Begin with ambient metric. It is motivated by the fundamental role played by the Lorentzian metric $\widetilde{g} = \sum_{\alpha=0}^{n} (dx^\alpha)^2 - (dx^{n+1})^2$ in the conformal geometry of S^n, as we saw in Lecture 1. The idea is to construct an analogue of $\widetilde{g}$ associated to any conformal manifold $(M,[g])$.

Recall that S^n was realized as the quadric $\mathcal{Q} = \mathbb{P}(\mathcal{N}) \subset \mathbb{P}^{n+1}$, where $\mathcal{N}$ is the null cone

$$\mathcal{N} = \left\{ x \in \mathbb{R}^{n+2} \setminus \{0\} : \sum_{\alpha=0}^{n} (x^\alpha)^2 - (x^{n+1})^2 = 0 \right\}$$

Each $x \in \mathcal{N}$ defined an inner product $g^{(x)}$ at $\ell = [x] \in \mathcal{Q}$ by:

$$g^{(x)}(v, w) = \tilde{g}(V, W), \qquad \text{for } v,\ w \in T_\ell \mathcal{Q},$$

where V, $W \in T_x\mathcal{N}$ satisfy $\pi_*(V) = v$, $\pi_*(W) = w$.

Had $g^{(sx)} = s^2 g^{(x)}$. In particular, $g^{(-x)} = g^{(x)}$, and the map $x \to g^{(x)}$ parametrizes the set of all inner products in the conformal class at $\ell \in \mathcal{Q}$. The set $\mathcal{N}/\{\pm 1\}$ can be identified with the union over all points ℓ of $\mathcal{Q}$ of the possible inner products in the conformal class at ℓ.

In this realization, there is a natural analogue of the space $\mathcal{N}/\{\pm 1\}$ associated to any manifold M with a conformal class of metrics $[g]$. Recall: $[g]$ denotes an equivalence class of metrics on M, where $\hat{g} \sim g$ if $\hat{g} = e^{2\omega} g$ for some $\omega \in C^\infty(M)$. Define the metric bundle $\mathcal{G} \subset S^2T^*M$ of $(M, [g])$ by:

$$\mathcal{G} = \{(p, g(p)) : \ p \in M,\ g \in [g]\}.$$

There is a projection π: $\mathcal{G} \to M$. $\mathcal{G}$ is the curved version of the null cone $\mathcal{N}$ (really $\mathcal{G}$ corresponds to $\mathcal{N}/\{\pm 1\}$).

$\mathcal{G}$ has natural dilations: for $s > 0$, define

δ_s: $\mathcal{G} \to \mathcal{G}$ by $\delta_s(p, g) = \Big(p, s^2 g\Big)$.

Define also the infinitesimal dilation vector field

$T = \frac{d}{ds}\delta_s|_{s=1}$.

Then $\pi_*(T) = 0$ and $\delta_{s*}T = T$. T is the curved version of the position vector field $X = x^I\partial_I$.

$\mathcal{G}$ carries a tautological symmetric 2-tensor $\mathbf{g}_0 \in S^2T^*\mathcal{G}$ defined as follows. If $(p, g) \in \mathcal{G}$ and Y, $Z \in T_{(p,g)}\mathcal{G}$, define

$\mathbf{g}_0(Y, Z) = g\,(\pi_*Y, \pi_*Z)$.

Easy to check that $\delta_s^*\mathbf{g}_0 = s^2\mathbf{g}_0$. As a quadratic form on $\mathcal{G}$, $\mathbf{g}_0$ is degenerate:

$\mathbf{g}_0(T, Y) = 0$ for all $Y \in T\mathcal{G}$.

$\mathbf{g}_0$ is the curved version of $\widetilde{g}|_{T\mathcal{N}}$.

So far, we have constructed the curved analog $\mathcal{G}$ of $\mathcal{N}$, which comes with dilations and an analogue $\mathbf{g}_0$ of $\widetilde{g}|_{T\mathcal{N}}$.

Next we "fatten up" $\mathcal{G}$ to a larger space $\widetilde{\mathcal{G}}$ on which $\widetilde{g}$ will live.

Define the ambient space $\widetilde{\mathcal{G}} = \mathcal{G} \times (-1,1)$. $\widetilde{\mathcal{G}}$ is analogous to a homogeneous neighborhood of $\mathcal{N}$ in the flat case. Write ρ for the variable in $(-1,1)$.

Dilations extend to $\widetilde{\mathcal{G}}$: $\delta_s(z,\rho) = (\delta_s z, \rho)$ for $z \in \mathcal{G}$. View $\mathcal{G}$ as a hypersurface in $\widetilde{\mathcal{G}}$ via the inclusion $\iota : \mathcal{G} \to \widetilde{\mathcal{G}}$ given by $\iota(z) = (z, 0)$.

Definition: An ambient metric (of order r) for $(M,[g])$ is a smooth metric $\widetilde{g}$ of signature $(n+1,1)$ on $\widetilde{\mathcal{G}}$ such that:

1. $\delta_s^* \widetilde{g} = s^2 \widetilde{g}$

2. $\iota^* \widetilde{g} = \mathbf{g}_0$

3. $Ric(\widetilde{g}) = 0$ to order r along $\mathcal{G}$.

Actually, the precise condition of vanishing to order r is shifted by 1 for certain components. We will ignore this subtlety.

These conditions generalize properties satisfied by the flat metric on $\mathbb{R}^{n+2}$.

If $M = S^n$ with its usual conformal structure, and we identify $\tilde{\mathcal{G}}$ with a neighborhood of $\mathcal{N}$ in $\mathbb{R}^{n+2}$, then $\tilde{g} = \sum_{\alpha=0}^{n} (dx^\alpha)^2 - (dx^{n+1})^2$ is an ambient metric. In this case, $Ric(\tilde{g}) \equiv 0$.

If $\Phi : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ is a diffeomorphism satisfying

1. Φ is homogeneous, i.e. $\delta_s \circ \Phi = \Phi \circ \delta_s$

2. $\Phi|_{\mathcal{G}} = I$,

then $\Phi^*\tilde{g}$ is an ambient metric whenever $\tilde{g}$ is. So an ambient metric is determined at most up to such a diffeomorphism. The main result concerning the existence and uniqueness of ambient metrics is the following:

n **odd:** There exists an infinite order ambient metric, uniquely determined to infinite order up to diffeomorphism. If there is a real-analytic metric in the conformal class $[g]$, then the formal series for $\tilde{g}$ converges so that an ambient metric exists satisfying $Ric(\tilde{g}) = 0$ in a neighborhood of $\mathcal{G}$ in $\tilde{\mathcal{G}}$.

n **even:** There is an ambient metric of order $n/2-1$ uniquely determined to order $n/2$ up to diffeomorphism. There is a local obstruction to existence at order $n/2$: the ambient obstruction tensor.

Ingredients of the proof:

- Normal form for $\tilde{g}$ relative to a choice of metric g in conformal class to break diffeomorphism invariance.

- Formal power series analysis of equation $Ric(\tilde{g}) = 0$ for $\tilde{g}$ in normal form.

An extension of the ambient metric to infinite order for n even has recently been derived by Graham and Hirachi.

There have been many applications of the ambient metric. I will describe one: a construction of the tractor bundle and tractor connection associated to a conformal manifold in terms of the ambient metric.

Levi-Civita connection ∇ is the most basic object in Riemannian geometry. What is the conformal analogue?

Saw in Lecture 2 that ∇ changes under $g \to \widehat{g} = e^{2\omega}g$. Can't take a "piece" of ∇ like with R.

Solution: change the bundle. The conformal connection is a connection on an "enlargement" of the tangent bundle–the tractor bundle $\mathcal{T}$.

The tractor bundle and connection are determined by the conformal class $[g]$ of metrics alone, not by a choice of metric in the class.

Understand flat model first. For Riemannian geometry: flat model is Euclidean space. Levi-Civita connection for Euclidean metric is usual directional derivative of the components of the vector field:

If $V(x) = \sum_{j=1}^{n} V^j(x)\partial_j$, then

$\nabla_W V = \sum W(V^j)\partial_j$.

For the conformal flat model, recall the ambient realization: $\mathcal{N} \subset \mathbb{R}^{n+2}$ and $\pi : \mathcal{N} \to \mathcal{Q}$. The conformal group $O(n+1,1)$ acts on the whole picture and induces conformal tranformations of $\mathcal{Q}$, as we saw in Lecture 1. Would like an $O(n+1,1)$-invariant bundle and connection on $\mathcal{Q}$. The background $\mathbb{R}^{n+2}$ has its canonical flat connection $\widetilde{\nabla}$, which is invariant under all of $GL(n+2)$. The idea is to pass this connection down to a connection on a bundle $\mathcal{T}$ on $\mathcal{Q}$.

First step is to pass to a bundle and connection on $\mathcal{N}$: no problem. Can consider $T\mathbb{R}^{n+2}\Big|_{\mathcal{N}}$ with the induced connection $\widetilde{\nabla}|_{\mathcal{N}}$. But not immediately clear how to pass this to $\mathcal{Q}$, since for $\ell \in \mathcal{Q}$, there are many possible $x \in \ell$ to choose from. Need a way to identify the tangent spaces $T_x\mathbb{R}^{n+2}$ and $T_{x'}\mathbb{R}^{n+2}$ for x, $x' \in \ell$. Do this via the dilations δ_s.

First recall how to interpret $T\mathcal{Q}$ in this manner. As discussed in Lecture 1, if $\ell \in \mathcal{Q}$ and $x \in \ell$ then we can consider $v \in T_\ell\mathcal{Q}$ as the equivalence class modulo X_x of those $V \in T_x\mathcal{N}$ satisfying $\pi_*(V) = v$. As $x \in \ell$ varies, such V may be chosen to satisfy $\delta_{s*}V_x = V_{sx}$. Therefore we can identify

$$T_\ell \mathcal{Q} \cong \{V \in \Gamma(T\mathcal{N}|_{\,\ell}) : \delta_{s*}V = V\} \,/\, \mathrm{span}\,\{X\}\,.$$

The definition of $\mathcal{T}$ is similar, except we change the homogeneity degree along the lines. For $\ell \in \mathcal{Q}$, define

$$\mathcal{T}_\ell \equiv \left\{U \in \Gamma(T\mathbb{R}^{n+2}|_{\,\ell}) : \delta_{s*}U = s\,U\right\}.$$

The $\mathcal{T}_\ell$ are the fibers of a vector bundle $\mathcal{T}$ on $\mathcal{Q}$.

One reason for the choice of homogeneity in the definition of $\mathcal{T}$ is so that the metric $\widetilde{g}$ will pass to a metric (of Lorentz signature) on $\mathcal{T}$. Since $\delta_s^*\widetilde{g} = s^2\widetilde{g}$, it follows that if U, $V \in \mathcal{T}_\ell$, then $\widetilde{g}(U,V)$ is homogeneous of degree 0 in s, i.e. $\widetilde{g}(U,V) \in \mathbb{R}$. So $\mathcal{T}$ has a Lorentz metric, also denoted $\widetilde{g}$.

Similarly, the homogeneities are right so that the connection $\widetilde{\nabla}$ on $T\mathbb{R}^{n+2}|_{\mathcal{N}}$ passes to a connection ∇ on $\mathcal{T}$ as follows. Let $\ell \in \mathcal{Q}$, let U be a section of $\mathcal{T}$ near ℓ, and let $v \in T_\ell\mathcal{Q}$. Choose V as above which represents v and define $\nabla_v U \in \mathcal{T}_\ell$ by:

$$\nabla_v U = \widetilde{\nabla}_V U\,.$$

It must be shown that this is independent of the ambiguity of X in V, and that it has the correct homogeneity under δ_{s*} so as to define an element of $\mathcal{T}_\ell$. To see this, reinterpret the homogeneity condition $\delta_{s*}U = sU$ on $U \in \mathcal{T}_\ell$. Write

$$U = U^I(x)\partial_I, \qquad x \in \ell.$$

The homogeneity condition is then equivalent to $U^I(sx) = U^I(x)$. So the coefficients of the vector field U are homogeneous of degree 0, i.e. they are independent of x. Thus

$$\widetilde{\nabla}_X U = X(U^I)\partial_I = 0,$$

so $\widetilde{\nabla}_V U$ is independent of the X ambiguity in V. Similarly, the condition $\delta_{s*}V = V$ is equivalent to $V^I(sx) = sV^I(x)$. Thus

$$\left(\widetilde{\nabla}_V U\right)^I = V^J \partial_J U^I$$

is homogeneous of degree 0, and $\widetilde{\nabla}_V U$ defines an element of $\mathcal{T}_\ell$.

Conclusion: We have defined a bundle $\mathcal{T}$ on $\mathcal{Q}$, the tractor bundle, whose fiber at ℓ can be identified with the tangent space of $\mathbb{R}^{n+2}$ at a point $x \in \ell$. The bundle $\mathcal{T}$ has a natural fiber metric $\widetilde{g}$ of Lorentz signature. We have also defined a connection ∇ on $\mathcal{T}$, the tractor connection, from the canonical flat connection $\widetilde{\nabla}$ on $T\mathbb{R}^{n+2}$.

Directly from the properties of $\widetilde{\nabla}$ on $T\mathbb{R}^{n+2}$, one sees that:

1. ∇ is flat (i.e. it has curvature 0)

2. $(\mathcal{T}, \nabla)$ is conformally invariant
   (i.e. it is invariant under $O(n+1,1)$)

3. $\nabla \widetilde{g} = 0$ ($\widetilde{g}$ is parallel)

$\mathcal{T}$ also has a conformally invariant filtration. This can be defined as follows.

At $x \in \mathcal{N}$, there are two subspaces in $T_x\mathbb{R}^{n+2} \cong \mathbb{R}^{n+2}$ which are invariant under $O(n+1,1)$, namely ℓ itself, and $T_x\mathcal{N}$, of dimensions 1 and $n+1$, resp. These subspaces are dilation-invariant, so pass to subspaces of $\mathcal{T}_\ell$. This gives the filtration

$$0 \subset \ell \subset T\mathcal{N} \subset \mathcal{T}_\ell.$$

$T\mathcal{N}$ can be identified with $\ell^\perp$, where the orthogonal is with respect to $\tilde{g}$. The subquotient $T\mathcal{N}/\ell$ can be identified with $T\mathcal{Q}$ (modulo a shift of homogeneity). The filtration is not preserved by the tractor connection ∇: the direction X moves in $\mathbb{R}^{n+2}$ as ℓ moves in $\mathcal{Q}$.

This construction of the tractor bundle and connection is useful because the conformal invariance is clear. One could alternately define $\mathcal{T}$ directly to be the trivial $\mathbb{R}^{n+2}$-bundle on $\mathcal{Q}$ and ∇ to be the canonical flat connection on the trivial bundle, but then some work has to be done to realize the conformal invariance.

Another advantage of this construction of the tractor bundle and connection in the flat case is that it generalizes virtually without change to a general conformal manifold once one has the ambient metric.

All of the structure that was used in this construction is present in the ambient metric construction in general: the metric bundle $\mathcal{G} \subset \widetilde{\mathcal{G}}$, the dilations δ_s, the infinitesimal dilation T, and the Levi-Civita connection $\widetilde{\nabla}$ of the ambient metric $\widetilde{g}$. In fact, all that is used of $\widetilde{\nabla}$ is its action on $T\widetilde{\mathcal{G}}|_{\mathcal{G}}$. This depends only on the connection coefficients of $\widetilde{\nabla}$ evaluated on $\mathcal{G}$, which depend only on first derivatives of $\widetilde{g}$ on $\mathcal{G}$. As $\widetilde{g}$ is well-defined to first order in all dimensions $n \geq 3$, even or odd, this derivation of the tractor connection works just fine for general conformal structures in all dimensions.

One obtains a rank $n+2$ bundle $\mathcal{T}$ over any conformal manifold $(M, [g])$ together with a connection ∇ on $\mathcal{T}$. $\mathcal{T}$ has a Lorentzian metric $\widetilde{g}$ induced from the ambient metric, which is parallel with respect to ∇. $\mathcal{T}$ also has a natural filtration $0 \subset \mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}$, with $\mathcal{T}_1/\mathcal{T}_0 \cong TM$ (modulo a homogeneity shift). The bundle $\mathcal{T}$, connection ∇, metric $\widetilde{g}$, and the filtration are conformally invariant in the sense that they depend only on the conformal structure $[g]$ and so are preserved by any conformal diffeomorphism between conformal manifolds.

In the curved case, ∇ is of course not flat. Its curvature can be identified with the Weyl and Cotton tensors of a metric in the conformal class in a natural way. An alternate proof of the fact that a metric with vanishing Weyl and Cotton tensors is a conformal multiple of a metric isometric to Euclidean can be based on the flatness of the tractor connection associated to such a metric.