TO .... FROM Nelson DATE 12^{th} January 2008 NO 1

Lecture Three: General theory of prolongation REFS⊛

Conformal to Einstein trace-free part of $(\nabla_a\nabla_b\sigma + P_{ab}\sigma) = 0$

$$P_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)}g_{ab}\right)$$

Prolong ① $\nabla_a\sigma = \mu_a$

② $\nabla_a\mu_b = g_{ab}\rho - P_{ab}\sigma$

$$\nabla_b\nabla_a\mu^b = \nabla_a\nabla_b\mu^b + R_{ab}\mu^b = n\nabla_a\rho - \nabla_a(P\sigma) + R_{ab}\mu^b$$
$$= n\nabla_a\rho - \cancel{(\nabla_a P)\sigma} - P\mu_a + R_{ab}\mu^b$$

$$\nabla_b\nabla_a\mu^b = \nabla_a\rho - \cancel{(\nabla^b P_{ab})\sigma} - P_{ab}\mu^b$$

$$\Rightarrow (n-1)\nabla_a\rho = \underbrace{-P_a{}^b\mu_b + P\mu_a - R_a{}^b\mu_b}_{= -(n-2)P_a{}^b\mu_b}, \qquad P = \frac{R}{2(n-1)}$$

$$\Rightarrow ③\ \nabla_a\rho = -P_a{}^b\mu_b$$

⊛ REFS
- ME, Higher symmetries ..., Ann. Math. 161 (2005), 1645-1665
- T.P. Branson, A. Čap, ME, and A.R. Gover, Prolongations ...., Int. Jour. Math. 17 (2006), 641-664.

(and exposition from Srní 2004 on my website).

- ME and T. Leistner, ... square of Laplacian, IMA Volumes 144, Springer Verlag 2007, pp. 319-338.

leads to a connection (assembling ①+②+③ as system)

$$\begin{pmatrix}\sigma \\ \mu_b \\ \rho\end{pmatrix} \xmapsto{D_a} \begin{pmatrix}\nabla_a\sigma - \mu_a \\ \nabla_a\mu_b - g_{ab}\rho + P_{ab}\sigma \\ \nabla_a\rho + P_a{}^b\mu_b\end{pmatrix}$$

tractors

Recall simplest example (in flat space) Killing fields

$$\nabla_{(a}V_{b)} = 0 \iff \begin{aligned}\nabla_a V_b &= F_{ab} \\ \nabla_a F_{bc} &= 0\end{aligned}$$

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Conformal Killing fields

trace-free part of $(\nabla_{(a}V_{b)}) = 0$

① $\nabla_a V_b = F_{ab} + g_{ab}\nu$

$\Rightarrow \nabla_a F_{bc} = -g_{ab}\nabla_c \nu + g_{ac}\nabla_b \nu$

i.e. ② $\nabla_a F_{bc} = g_{ab}\rho_c - g_{ac}\rho_b \qquad \nabla_a \nu = -\rho_a$

$\Downarrow$ (in flat space $\nabla_d \nabla_a F_{bc} = \nabla_a \nabla_d F_{bc}$)

$$g_{ab}\nabla_d \rho_c - g_{ac}\nabla_d \rho_b = g_{db}\nabla_a \rho_c - g_{dc}\nabla_a \rho_b$$

$$\Rightarrow (n-2)\nabla_d \rho_c = -g_{cd}\nabla^a \rho_a$$

$$\Rightarrow (n-2)\nabla^a \rho_a = -n\nabla^a \rho_a$$

$$\Rightarrow \nabla^a \rho_a = 0 \Rightarrow$$

③ $\nabla_a \rho_b = 0.$

$①+②+③ = \left\{\begin{array}{ccc} & * & \\ * & & * \\ & * & \end{array}\right.$

Conformal Killing tensors

trace-free part of $(\nabla_{(a}V_{bc)}) = 0$

0th *

1st * *

2nd * * * *

3rd * *

4th *

Shock Horror!

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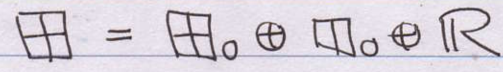
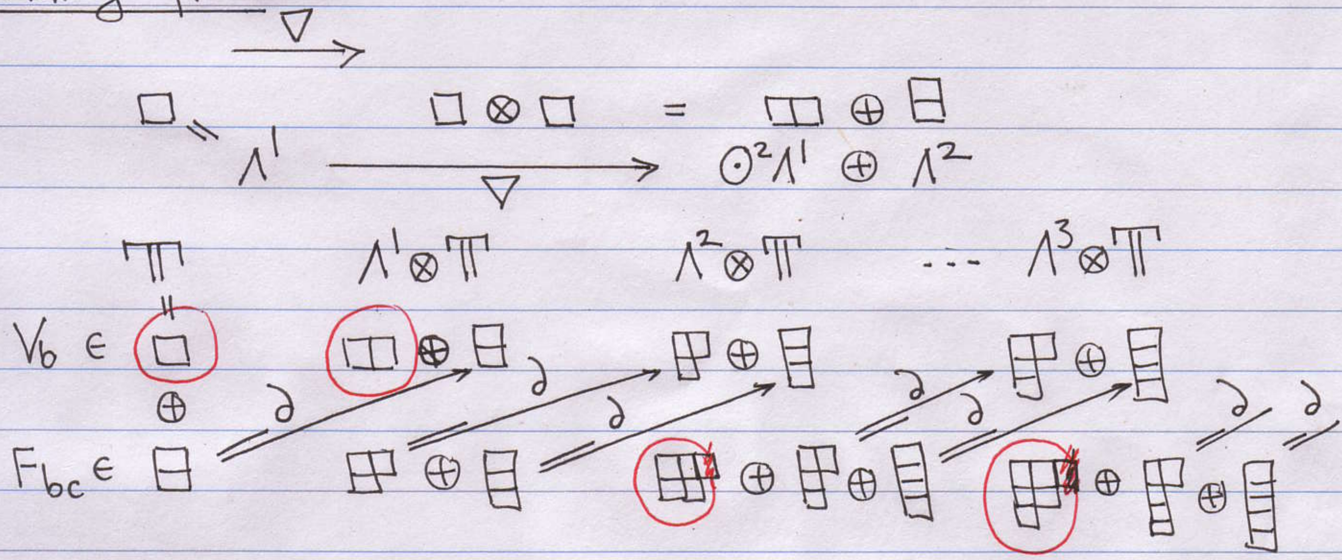
Representation theory!

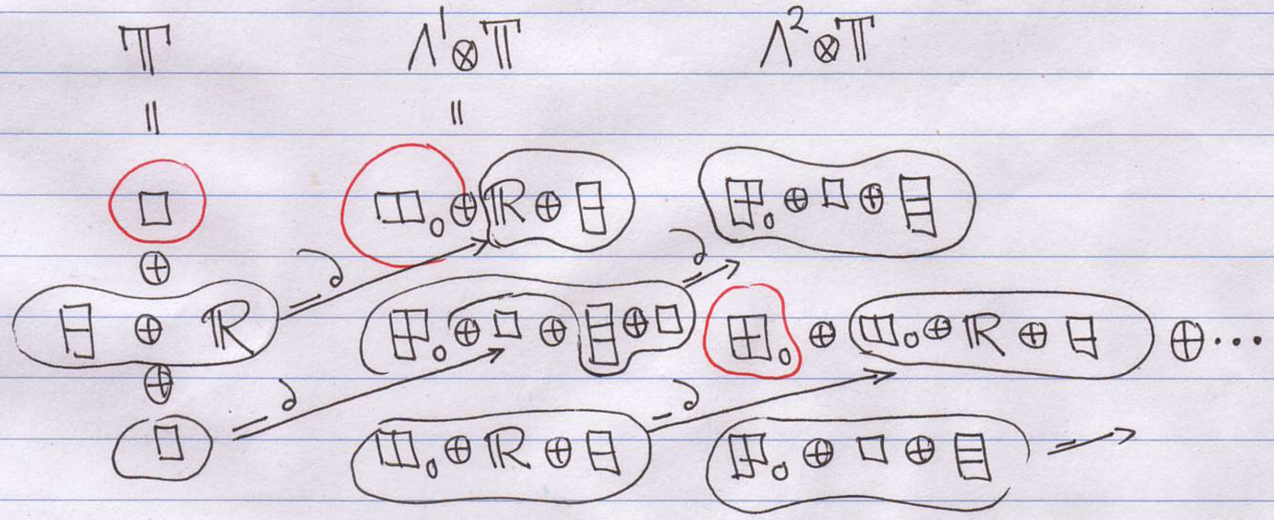
E.g. 

$R = W + \mathring{r} + s$

Killing fields



Conformal Killing fields



Have algebraic mappings ∂ with cohomology in red

What's happening? ↑ Lie algebra cohomology of a very simple type: computable by Kostant

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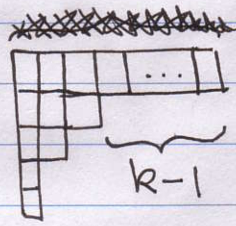
Statement of results

On a manifold with connection (affine)

$$\begin{tikzcd} E \arrow[r, "\nabla^k"] \arrow[rr, bend right, "D"'] & \odot^k\Lambda^1 \otimes E \arrow[r] & \odot^k\Lambda^1 \circledcirc E \end{tikzcd} \quad \text{Cartan product}$$

Can prolong D and the relevant bundle is

$$GL(n,\mathbb{R}) \hookrightarrow GL(n+1,\mathbb{R})$$

 $\leftrightarrow E$ 

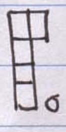
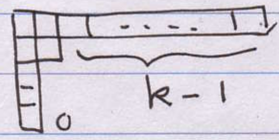
$\leftrightarrow \mathbb{T}$

On a Riemannian manifold

$$\begin{tikzcd} E \arrow[r, "\nabla^k"] \arrow[rr, bend right, "D"'] & \odot^k_0\Lambda^1 \otimes E \arrow[r] & \odot^k_0\Lambda^1 \circledcirc E \end{tikzcd} \quad \text{Cartan}$$

Can prolong D and the relevant bundle is

$$SO(n) \hookrightarrow SO(n+1,1)$$

 $\leftrightarrow E$ 

$\leftrightarrow \mathbb{T}$

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<u>Better (in the flat case)</u> G/P (e.g. $S^n = SO(n+1,1)/P$)

P ← parabolic

BGG resolution — follows the Lie algebra cohomology.

<u>E.g.</u> Killing fields $\rightsquigarrow$ Riemannian deformation complex

Conformal " " $\rightsquigarrow$ conformal deformation complex

Applications to finite element schemes in linear elasticity
(Arnold - Falk - Winther)

Application to symmetries of Laplacian

$$\mathcal{A}_n = \bigoplus_{k=0}^{\infty} \underbrace{\boxed{\;\;|\;\;|\;\;|\;\cdots\cdots\;|\;\;}_{\,0}}$$

representations of $so(n+1,1)$.

As algebra

$$\mathcal{A}_n = \frac{\bigotimes so(n+1,1)}{\left\langle X\otimes Y - X\circledcirc Y - \frac{1}{2}[X,Y] + \frac{n-2}{4n(n+1)}\langle X,Y\rangle \right\rangle}$$

Joseph Ideal $\Leftrightarrow$ in $U(so(n+1,1))$

Cartan Lie Killing