Two Hierarchies of Simple Games

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Abstract

A simple game is a mathematical object that is used in economics and political science to describe the distribution of power among coalitions of players.

An important class of simple games are weighted majority games (von Neumann and Morgenstern(1944), Shapley (1962)). In such a game every player is assigned a real number, his weight. The winning coalitions are the sets of players whose weights total at least q, a certain threshold. It is well-known however that not every simple game has a representation as a weighted majority game(von Neumann and Morgenstern(1944)). The first step in attempt to characterize non-weighted games was the introduction of a class of roughly weighted games (Taylor and Zwicker (1999)). Formally, a simple game G on the set $P = [n] = \{1, 2, ..., n\}$ of players is roughly weighted if there exists a weight function $w: P \to R^+$, not identically equal to zero, and a positive real number q, called *quota*, such that for $X \in 2^P$ the condition $\sum_{i \in X} w_i < q$ implies X is losing, and $\sum_{i \in X} w_i > q$ implies X is winning. Rough weightedness was studied by Gvozdeva and Slinko where a criterion of rough weightedness was given in terms of trading transforms, similar to the characterisation of weightedness by Elgot, and Taylor and Zwicker.

One hierarchy emerges when we allow several thresholds instead of just one. We may say that a game G belongs to the class $\mathcal{B}_k$ if there are k thresholds $0 < q_1 \le q_2 \le \ldots \le q_k$ and any coalition with total weight of players smaller than q_1 is losing, any coalition with total weight greater than q_k is winning, and anything might happen on the thresholds. We also impose an additional condition that, if a coalition X has total weight $w(X)$ which satisfies $q_1 \le w(X) \le q_k$, then $w(X) = q_i$ for some i. We show that $\mathcal{B}$-hierarchy is strict, that is,

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots \subset \mathcal{B}_\ell \subset \ldots$$

with the union of these classes being the class of all games.

Yet another idea is to make the threshold thicker, i.e. use not a number but an interval $[a, b]$ for it. That is, all coalitions with total weight less than a will be losing and all coalitions whose total weight is greater than b winning. This time we do not care how many values weights of coalitions may take if they are not yet classified. We can keep weights normalized so that the lower end of the interval is fixed at 1, then the right end of the interval α becomes a "resource" parameter. Formally, a game G belongs to class $\mathcal{C}_\alpha$ if all coalitions in G with total weight less than 1 are losing and every coalition whose total weight is greater than α is winning. We show that the class of all games is split into a hierarchy of classes of games $\{\mathcal{C}_\alpha\}_{\alpha \in [1,\infty)}$ defined by this parameter. We show that as α increases we get strictly greater descriptive power, i.e., strictly more games can be described, that is, if $\alpha < \beta$, then $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$. In this sense the hierarchy is strict.

The strictness of the latter hierarchy was achieved because we allowed games with unrestricted number of players. The situation will be different if we keep the number of players n fixed. Then there is an interval $[1, s(n)]$ such that all games with n players belong to $\mathcal{C}_{s(n)}$ and $s(n)$ is minimal with this property. There will be also finitely many numbers $q \in [1, s(n)]$ such that the interval $[1, q]$ represents more n-player games that any interval $[1, q']$ with $q' < q$. We call the set of such numbers the nth *spectrum* and denote it $\mathrm{Spec}(n)$. We also call a game with n players *critical* if it belongs to $\mathcal{C}_\alpha$ with $\alpha \in \mathrm{Spec}(n)$ but does not belong to any $\mathcal{C}_\beta$ with $\beta < \alpha$. We calculate the spectrum for $n < 7$ and also a set of critical games, one for each element of the spectrum.

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