Maths 362 Lecture 1

Topics for today:
Partial derivatives and Taylor series (review of material from Maths 253)

Reading for this lecture: Greenberg Sections 13.3, 13.5

Suggested exercises: Greenberg Section 13.5: 1, 2, 9, 11

Reading for next lecture: Greenberg Sections 14.2-14.4

Today's handout: Course guide
Why study vector calculus?

The Navier-Stokes equations model fluid flow:

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Here \( \mathbf{v} \) is velocity of fluid (vector), \( p \) is pressure (scalar). First equation is Newton’s law for fluid, second equation is conservation of mass.
Revision of some calculus for functions of several variables

Partial derivatives: Let \( f(x,y) \) be a function of variables \( x \) and \( y \) defined for \( (x,y) \) near a point \( (x_0, y_0) \).

Fix \( y=y_0 \). Then \( f(x,y)=f(x,y_0) \) is a function of \( x \) alone.

The \( x \)-derivative of this function at \( x_0 \) (if it exists) is called the partial derivative of \( f \) with respect to \( x \) at \( (x_0,y_0) \), and is written

\[
\frac{\partial f}{\partial x} \quad \text{or} \quad f_x
\]

There is a similar definition for the partial derivative w.r.t. \( y \).
Example 1: Find partial derivatives w.r.t. $x$ and $y$ for $f(x,y)=x^3y^5$.

Example 2: Find the first partial derivatives of $f(x,y,z)=xy^2z^3$. 
Partial derivatives may themselves be functions of the variables and we can take partial derivatives of these functions to get **second partial derivatives**:

\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \text{etc.}
\]

**Example 3**: Find the second partial derivatives of \( f(x, y) = x^3 y^5 \).
The order of differentiation may matter. For example, maybe

\[ f_{xy} \neq f_{yx} \]

for a particular function \( f \).

However, if all the first and second partial derivatives exist and are continuous near \((x_0, y_0)\) then

\[ f_{xy} = f_{yx} \]
Taylor’s formula and Taylor series

Let $f(x)$ be a function of one variable $x$, with $f'(x)$, $f''(x)$, … etc all existing. Then Taylor’s formula is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^{n-1} + R_n(x).$$

$R_n(x)$ is the remainder term:

$$R_n(x) = \frac{f^{(n)}(\xi)(x-a)^n}{n!}$$

with $\xi$ being a point in $[x,a]$.

Taylor’s formula tells us we can approximate $f(x)$ by a polynomial of degree $(n-1)$ with error bounded by $R_n(x)$. 

Example 4: Find the Taylor formula up to terms of order two, for the expansion about $a = -1$ of

$$f(x) = \frac{1}{1 + x^2}.$$
If the function $f$ is infinitely differentiable then we can let $n \to \infty$ in the Taylor formula to get the **Taylor series**:

$$\text{TS } f|_a = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x - a)^j.$$  

The Taylor series represents $f$ if the series converges in some interval of $x$ and if the function to which it converges is equal to $f$ on some interval of $x$. 
Taylor’s formula and Taylor series can be defined for functions of more than one variable in a similar way.

For example, the Taylor series for \( f(x,y) \) about \((a,b)\) is:

\[
f(x,y) = f(a, b) + f_x(x - a) + f_y(y - b) + \frac{1}{2!} \left[ f_{xx}(x - a)^2 + 2f_{xy}(x - a)(y - b) + f_{yy}(y - b)^2 \right] + \ldots.
\]

where all the derivatives are evaluated at \((a,b)\).
Example 5: Find the Taylor expansion about (1,3,-2) for the function $f(x,y,z) = x^3yz$. 
Important ideas from today:

• partial derivatives
• second partial derivatives
• Taylor’s formula for functions of one or more variables
• remainder terms
• Taylor series for functions of one or more variables