THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2009
Campus: City

MATHEMATICS

Great Ideas Shaping our World

(Time allowed: TWO hours)

NOTE: This paper contains **6** questions. Answer **ALL** questions. Show **ALL** your working. There are 100 marks available in total.

1. (15 marks)

(a) (6 marks) Claire has a large box of tennis balls. There are 24 red balls, 26 blue balls and 20 yellow balls in the box. Claire take balls out of the box and gives them to Vivien without looking at the colours. How many balls does Claire need to take out of the box to be **certain** that Vivien has:

(i) At least two balls of the same colour.

(ii) At least four balls of the same colour.

(iii) At least two balls of one colour, and at least two balls of a *different* colour.

(b) (9 marks)

(i) Prove that $\sqrt[3]{2}$ is irrational.

(ii) The first few digits of the decimal expansion of $\sqrt[3]{2}$ are

$$\sqrt[3]{2} = 1.25992104989487\ldots$$

Write down a *rational* number which is greater than $\sqrt[3]{2}$, but less than 1.26.

(iii) Now write down a rational number which is greater than $\sqrt[3]{2}$ but less than the answer you gave for (ii).

(iv) If a is an irrational number and b is a rational number greater than a, can you always find a rational number between a and b? Justify your answer carefully.

2. (10 marks)

(a) What is a prime number?

(b) Let $p_1, p_2, \ldots, p_{100}$ be the first 100 prime numbers. What can you say about the factors of the number $q = p_1 \times p_2 \times \cdots \times p_{100} + 1$? Is q necessarily prime?

(c) Carefully prove that there are infinitely many prime numbers.

3. (20 marks) Let N be the set of natural numbers, i.e.,

$$N = \{1, 2, 3, 4, \dots\}.$$

For each of the following sets, state whether the set has cardinality that is less than the cardinality of N or equal to the cardinality of N or greater than the cardinality of N.

In each case, prove that the cardinality of the set is as you claim. In your answers you may assume that the cardinality of the set of points in the interval $[0, 1]$ has greater cardinality than N.

(a) Set M is the set of all mobile phones in New Zealand.

(b) Set C is the set of points on the boundary of a square of side length 2 cm.

(c) Set O is the set of numbers between 0 and 1 for which the first n digits after the decimal point in the decimal expansion are 1's and all the following digits are 0's. For example, $0.11100000\dots$ may be in O but $0.111001000\dots$ is not.

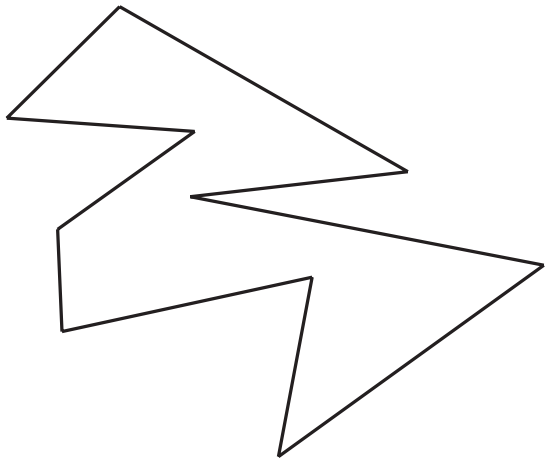
(d) Set F is the set of numbers between 0 and 1 for which the decimal expansion contains only the digits 4 and 5 after the decimal point. For example, $0.44555454\dots$ may be in F but $0.445525454\dots$ is not.

4. (20 marks)

(a) (12 marks)

(i) State the Art Gallery Theorem.

(ii) The figure below shows the floor plan for an art gallery. Using the copy of the figure provided on the answer sheet at the end of the exam paper, demonstrate your understanding of the proof of the theorem. Simply draw and label things, show where the proof tells you to locate guards, and explain why this gives a limit on the number of guards needed.



(iii) Find the minimum number of guards required so that when the guards are placed at appropriate vertices, each point of the gallery can be viewed by at least one guard. On the answer sheet provided, show one way in which this number of guards could be placed so that each point of the gallery can be seen by at least one guard.

(b) (8 marks) A cube is an example of a regular polyhedron, also known as a Platonic solid.

(i) What makes a polyhedron a *regular* polyhedron?

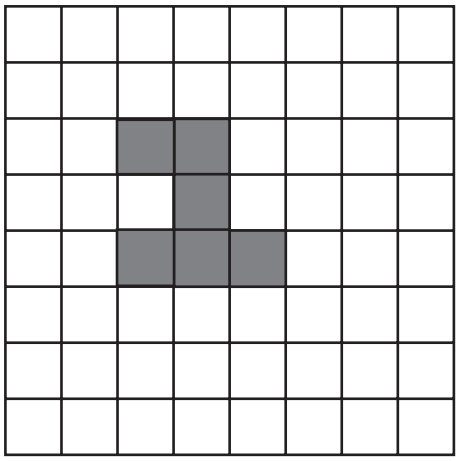
(ii) How many regular polyhedra are there?

(iii) Describe two other regular polyhedra (not including the cube). You should give the name of the polyhedron and list the shape of its faces, the number of edges, faces and vertices, and say how many faces meet at each vertex.

5. (10 marks) The Game of Life is played on a square grid, where each cell has eight neighbours. The rules are:

- A dead cell comes alive if it has exactly three live neighbours.
- A live cell remains alive if it has two or three live neighbours.
- A live cell dies if it has fewer than two or more than three live neighbours.

(a) Find the next three generations from the following starting configuration. (Grey cells are alive, white cells are dead.) Use the grids on the answer sheet at the end of the exam paper.

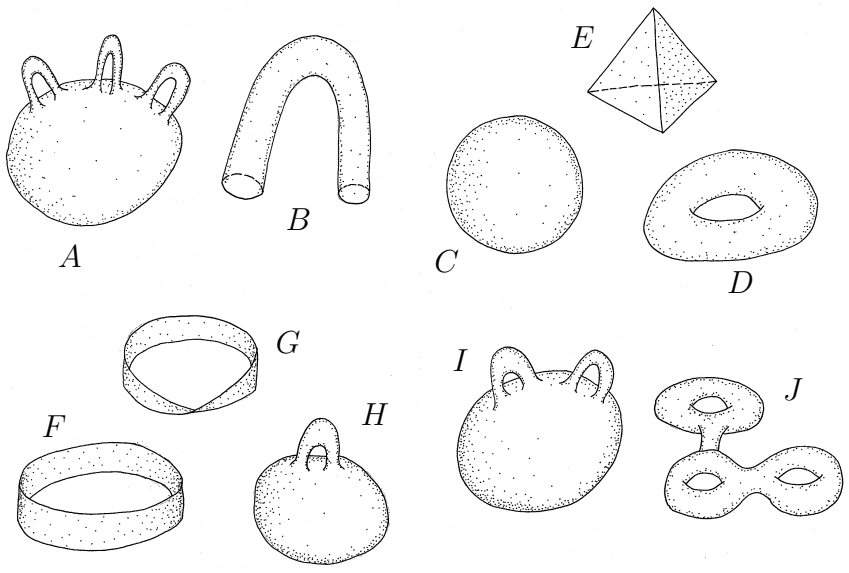


(b) Find a starting configuration with at least three cells that goes extinct after three or fewer generations. Use the grids on the answer sheet at the end of the exam paper.

(c) Find a configuration with at least three live cells that is stable, i.e., that stays the same at each generation. Use the grids on the answer sheet at the end of the exam paper.

6. (25 marks)

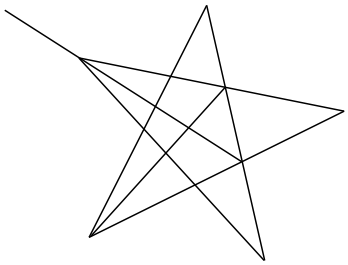
(a) (8 marks) Referring to the following **surfaces** by their letter labels, A, B, C, ..., group the surfaces together that are equivalent by distortion. *Note*: The tube labelled B is open-ended.



(b) (9 marks)

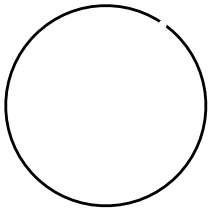
(i) State the formula defining the Euler characteristic of a graph in the plane, explaining all symbols used.

(ii) Calculate the Euler characteristic of the following graph:



(iii) What does the Euler Characteristic Theorem state about connected graphs in the plane?

(c) (8 marks) Explain, with a picture if you like, why the Hot Loop Theorem does *not* hold when the loop is replaced by a loop with a small gap, as shown below:

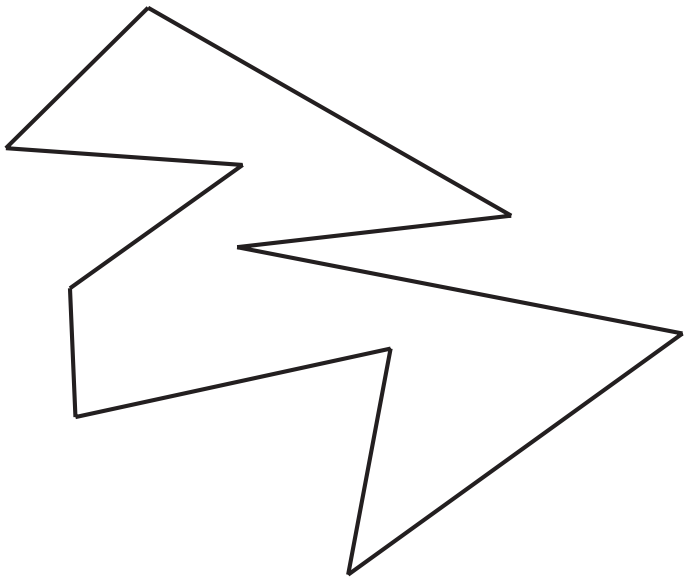


Family Name: _____ ID No: _____

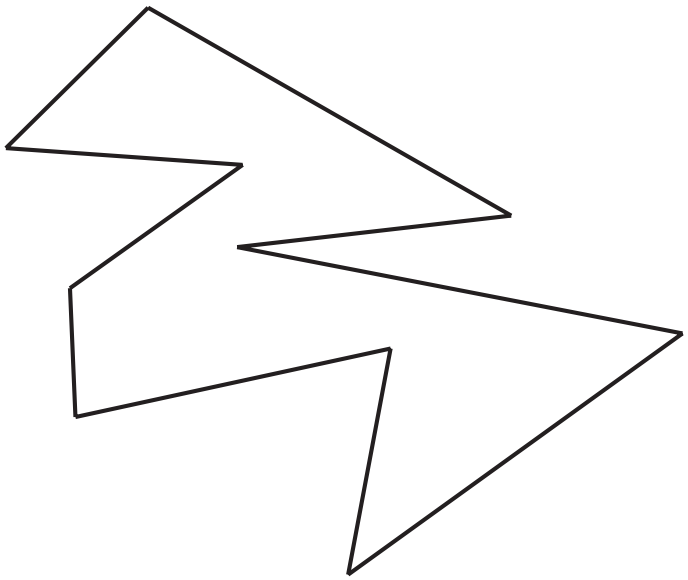
Given Name: _____

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

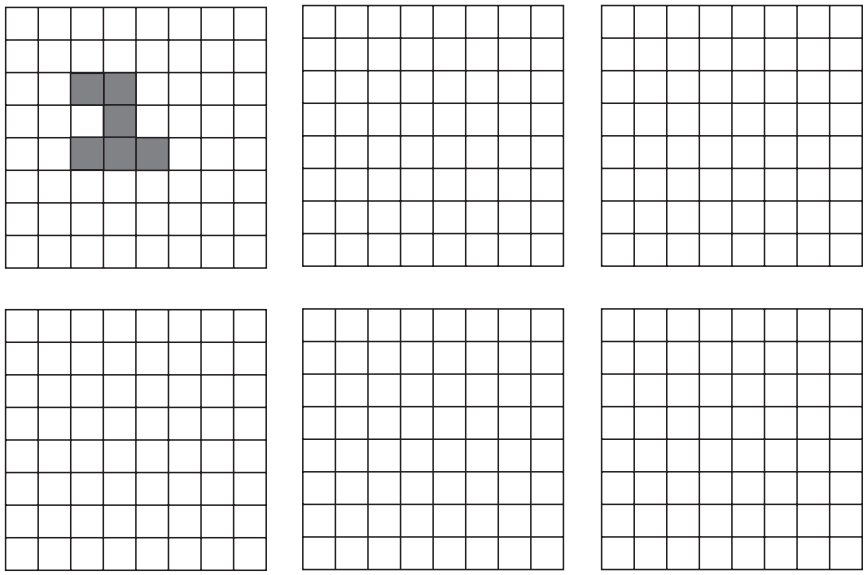
Answer sheet for Question 3(a)(ii)



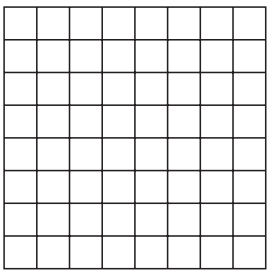
Answer sheet for Question 3(a)(iii)



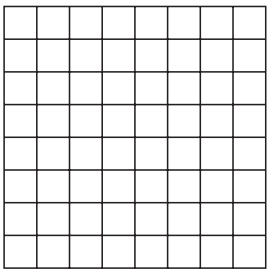
Answer sheet for Question 5(a)



Answer sheet for Question 5(b)



Answer sheet for Question 5(c)



Use this sheet for your workings for questions 5(b) and (c).

YOU DO NOT NEED TO HAND THIS SHEET IN