Tight Frames Produced by Group Orbits

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The Platonic solids are highly regular shapes in $\mathbb{R}^3$. Their vertices can be generated as an orbit of a specific vector under their symmetry group (considered as a matrix group of unitary transformations). The collection of vertices of a Platonic solid is an example of a tight frame where a tight frame is a collection of n vectors $(\phi_i)_{i=1}^n$ in a d-dimensional vector space V such that;

(i). $\text{span}\{\phi_i : 1 \leq i \leq n\} = V$;

(ii). $\exists c \; \forall f \in V \quad f = c \sum_{i=1}^n < f, \phi_i > \phi_i$.

More generally the collection of vectors produced as the orbit under an irreducible matrix group forms a tight frame. One area of research into tight frames is concerned with the angles between vectors within a frame. In partiuclar there is a lot of interest in equiangular tight frames where any pair of distinct vectors are at a constant angle much like the case of the vertices of Platonic solids.

In 1999 Gerhard Zauner conjectured that there exists d^2 equiangular lines in $\mathbb{C}^d$ for all values of d. There has been a lot of interest in this question from quantum information theory, design theory and frame theory. Although there are numerical solutions in all dimensions $d \leq 67$ and analytic results exist in some small dimensions Zauner's conjecture remains open.

In looking for a general analytic result a lot of work has focused on the Heisenberg group. In particular by finding a suitable initial vector an orbit of that vector under the group has been shown to give equiangular frames with d^2 vectors in $\mathbb{C}^d$ for small d. The problem with this approach is finding a general way to specify which initial vector to start with.

As an alternative to the Heisenberg group I have been looking at the class of irreducible complex reflection groups. These were classified into three inifinte familes and thirty four exceptions by Shephard and Todd in 1954. Among the irreducible complex reflection groups are the symmetry groups for the regular complex polytopes which are generalisations of regular polyhedra.

In analogy with the case of the Platonic solids we were looking for small orbits of regularly spaced familes of vectors in a complex space. Although this does not seem to provide maximal equiangular tight frames there are tight frames with only a small number of angles.

In my talk I will discuss our current findings about relationships between irreducible complex reflection groups and orbits with a small number of angles.