TO.................... FROM Nelson DATE 6^{th} January 2008 NO 1

Overall Title: Symmetries and prolongation of partial differential equations

Lecture 1: Symmetries of the Laplacian ...

Question: Which linear differential operators preserve harmonic functions on $\mathbb{R}^n$?

NB $n \geqslant 3$ (because..): Answer on $\mathbb{R}^3$ $\quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

0^{th} order $\quad f \mapsto \text{constant} \times f \quad$ i.e. spanned by $f \mapsto f$. 1-dimension

1^{st} order $\quad f \mapsto \frac{\partial f}{\partial x}$ etc. 3-dimensions

$$f \mapsto x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} \quad \text{etc. 3-dimensions}$$

Reason:

Let $\mathcal{D}$ by any of these operators. Then $[\Delta, \mathcal{D}] \equiv \Delta\mathcal{D} - \mathcal{D}\Delta = 0$.

(compute $\frac{\partial^2}{\partial x^2} x \frac{\partial}{\partial y} = x\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y}$ etc. etc. DO IT)

$$f \mapsto x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \qquad \text{Reason} \quad \frac{\partial^2}{\partial x^2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x^2}$$

$$\Delta\mathcal{D} = (\mathcal{D}+2)\Delta \quad \Longleftarrow$$

i.e. $\underline{\Delta\mathcal{D} = \delta\Delta}$ for some other operator δ. 1-dimension

Finally, there's

$$f \mapsto (x^2-y^2-z^2)\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y} + 2xz\frac{\partial f}{\partial z} + xf \quad \text{etc. 3-dimensions}$$

2^{nd} order Boyer-Kalnins-Miller 1976 (Symmetry and Separation of Variables ...)
(see also Willard Miller's book ↑)

0^{th} & 1^{st} order: 11-dimensional vector space $\mathfrak{I}$. It's a Lie algebra! Which one?

$[\mathfrak{I}, \mathfrak{I}]$ is 10-dimensional $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{1}{2}\right)$

$\cong$

$\mathfrak{so}(4,1)$ shock horror! Conformal geometry out of thin air.

Explain! Why? Consider $\Delta\mathcal{D} = \delta\Delta$ and let's say $\mathcal{D}$ is a symmetry.

(Generally, if $L: E \to F$ is a linear differential operator between two vector bundles E and F let's say that a linear differential operator $\mathcal{D}: E \to E$ is a symmetry of L iff there is a linear differential operator $\delta: F \to F$ s.t. $L\mathcal{D} = \delta L$

$$\begin{array}{ccc} E & \xrightarrow{L} & F \\ \mathcal{D}\downarrow & & \downarrow\delta \\ E & \xrightarrow{L} & F \end{array} \quad \text{commutes.}$$

NB: $\{L\phi = 0\}$ is preserved.)

Anyway, consider $\Delta\mathcal{D} = \delta\Delta$ on $\mathbb{R}^n$, for $\mathcal{D}$ of first order.

Write $\mathcal{D} = V^a\nabla_a + \mathrm{Lots}_m = V^a\nabla_a + W$ (summation convention)

~~Let~~ Then $\nabla_b\mathcal{D} = V^a\nabla_b\nabla_a + (\nabla_bV^a)\nabla_a + W\nabla_b + \nabla_bW$

so $\Delta\mathcal{D} = \cancel{V^a\nabla_a\Delta}\; V^a\Delta\nabla_a + 2(\nabla^bV^a)\nabla_b\nabla_a + W\Delta + \mathrm{Lots}$

$= V^a\nabla_a\Delta + \underbrace{2(\nabla^bV^a)\nabla_b\nabla_a} + \cancel{\mathrm{Lots}} + W\Delta + \mathrm{Lots}$

even on a Riemannian manifold

(explain $A^{(ab)}$ etc.) forces $\underline{\nabla^{(a}V^{b)} = g^{ab}\nu}$ for some ν ✓

This is the conformal Killing equation also written as

trace-free part of $\nabla^{(a}V^{b)} = 0$

or $\nabla^{(a}V^{b)} - \frac{1}{n}g^{ab}\nabla_cV^c = 0.$

Explain how it arises or at least how the <u>Killing equation</u> arises ✓

$\mathcal{L}_Vg = 0$

$\mathcal{L}_V\phi_{ab} = V^c\nabla_c\phi_{ab} + (\nabla_aV^c)\phi_{cb} + (\nabla_bV^c)\phi_{ac}$

so $\mathcal{L}_Vg_{ab} = 0 \iff \nabla_aV_b + \nabla_bV_a = 0$

Conformal Killing is same argument for $\mathcal{L}_Vg = 2\nu g$. ✓

Anyway, <u>it can be solved</u> and can also grind out what W above <u>has to be</u>.
[on $\mathbb{R}^n$ and check that it works!]

<u>Th</u>$^{\underline{m}}$: If V^a is a conformal Killing field on $\mathbb{R}^n$ then

$$\mathcal{D}_Vf \equiv V^a\nabla_af + \frac{n-2}{2n}(\nabla_aV^a)f \quad ✓$$

up to a constant

is a symmetry of the Laplacian

(specifically $\Delta\mathcal{D}_V = \delta_V\Delta$ where $\delta_Vf = V^a\nabla_af + \frac{n+2}{2n}(\nabla_aV^a)f$).

(and, together with $f \mapsto \text{constant} \times f$ yield <u>all symmetries</u>). NEXT TIME

Higher order symmetries?

Other differential operators? Δ^2 instead – first order symmetries

$$f \mapsto V^a\nabla_af + \frac{n-4}{2n}(\nabla_aV^a)f + Cf$$

NEXT TIME

conformal Killing vector   constant

Dirac operator – 1st order Benn & Kress

– higher ME & Somberg & Souček ✓

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<u>Higher order for Δ^2</u> second order symmetries

$$f \mapsto \mathcal{D}_V f + \mathcal{D}_W f + 1^{\text{st}} \text{ order symmetry}$$

where

$$\mathcal{D}_V f = V^{ab}\nabla_a\nabla_b f + \frac{n-2}{n+2}(\nabla_a V^{ab})\nabla_b f + \frac{(n-2)(n-4)}{4(n+1)(n+2)}(\nabla_a\nabla_b V^{ab})f$$

symmetric trace-free

where trace-free part of $\nabla^{(a}V^{bc)} = 0$

conformal Killing tensor

$$\mathcal{D}_W f = W\Delta f - (\nabla^a W)\nabla_a f - \frac{n-4}{2(n+2)}(\Delta W)f$$

where trace-free part of $\nabla^a\nabla^b\nabla^c W = 0$

i.e. $\nabla^a\nabla^b\nabla^c W = g^{(ab}\phi^{c)}$ for some ϕ^c.

DO THESE FIRST

<u>Higher order for Δ?</u> second order and higher - there are <u>trivial symmetries</u> of the form $\mathcal{D} = \mathcal{P}\Delta$ for any linear differerential operator $\mathcal{P}$. ✓

BKM showed that in $\mathbb{R}^3$ there are

0^{th} order: dimension 1

1^{st} order: dimension 10 (coming from conformal Killing fields)

2^{nd} order: trivial ones + <u>dimension 35</u>

explain

$$\mathcal{D}_V f \equiv V^{ab}\nabla_a\nabla_b f + \frac{n}{n+2}(\nabla_a V^{ab})\nabla_b f + \frac{n(n-2)}{4(n+1)(n+2)}(\nabla_a\nabla_b V^{ab})f$$

where V^{ab} is symmetric trace-free (else absorb into trivial ones)

and trace-free part of $\nabla^{(a}V^{bc)} = 0$

(or $\nabla^{(a}V^{bc)} = g^{(ab}\nu^{c)}$ or $\nabla^{(a}V^{bc)} - \frac{2}{n+2}g^{(ab}\nabla_d V^{c)d} = 0$).

Evidently, we are coming up with

- special (Killing) equations
- complicated formulae

both of which are crying out for some explanation. How to solve?

How to understand?

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Obvious questions – what about arbitrarily high order symmetries?
– what about on a Riemannian manifold?
– what about composition of symmetries

(Witten)

<u>More precise question here</u> We've been calling symmetries $P\Delta$ for the Laplacian <u>trivial</u>. Make this into a definition saying symmetries $\mathcal{D}_1$ and $\mathcal{D}_2$ are <u>equivalent</u> iff $\mathcal{D}_1 - \mathcal{D}_2 = P\Delta$. Then look at the <u>algebra</u> $\mathcal{A}_n$ of symmetries of Δ on $\mathbb{R}^n$ up to equivalence with algebra structure induced by composition. ✓

<u>Question</u>: What is $\mathcal{A}_n$ as an abstract algebra ?
(e.g. generators and relations) !

To answer these questions on $\mathbb{R}^n$ we shall use the "AdS/CFT correspondence" $\mathbb{R}^n \hookrightarrow S^n \subset \mathbb{RP}_{n+1}$ from Robin's talk. ✓
(Fefferman-Graham) ambient metric construction

9:45 – 10:30 M'Tea
10:30 – 11:30 Robin
11:45 – 12:45 Me
13:00 – 14:00 Lunch

Free

18:30 – 19:30 Dinner
19:45 – 20:45 Willard

<u>So</u>: Yet to do next time. Recall $\mathcal{A}_n$ and acknowledge Witten w.r.t. questions and suggestions.

Higher order symmetries as yet untouched in any explicit manner so explain $V^{ab}\nabla_a\nabla_b f + \frac{n}{n+2}(\nabla_a V^{ab})\nabla_b f + \frac{n(n-2)}{4(n+1)(n+2)}(\nabla_a\nabla_b V^{ab}) f$

$$\text{where } \underbrace{\nabla^{(a}V^{bc)} - \frac{2}{n+2}g^{(ab}\nabla_d V^{c)d} = 0}_{\text{HOW TO SOLVE ?}}.$$

And do Δ^2 case too.