# Maths 260 Lecture 7

- **Topics for today:**
  Existence and uniqueness of solutions
- **Reading for this lecture:** BDH Section 1.5
- **Suggested exercises:** BDH Section 1.5, #1,3,5,7,15.
- **Reading for next lecture:** BDH Section 1.6, pp 76-85
- **Today's handouts:** Tutorial 3 question sheet
  Assignment 1

# Existence and Uniqueness of solutions

In the theory and examples we have studied so far we have been making two major assumptions: that the DEs we study have solutions and that solutions to IVPs are unique.

On the whole we are safe in making these assumptions.
Today we shall see why.

# Existence Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

If $f(t, y)$ is a continuous function at $(t, y) = (t_0, y_0)$, then there is a constant $\epsilon > 0$, and a function $y(t)$ defined for $t$ in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ solves the IVP.

**Note:** The theorem guarantees a solution exists for a small interval in $t$, but says nothing about existence for $t$ outside this interval.

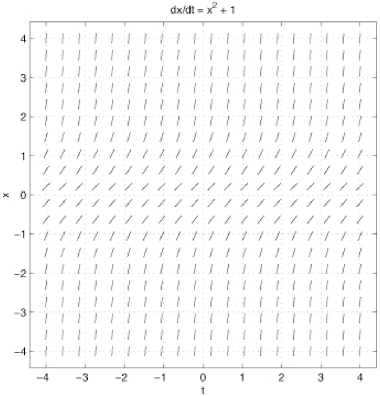
# Example 1:

Consider the IVP

$$\frac{dy}{dt} = 1 + y^2, \qquad y(0) = 0.$$

Does the IVP have a solution? If so, for what values of $t$ does the solution exist?

The slope field for the DE is:



# Example 2:

Does the following IVP have a solution?

$$\frac{dy}{dt} = \frac{1}{t}, \qquad y(0) = 1$$

# Uniqueness Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

If $f(t, y)$ and $\partial f / \partial y$ are continuous functions at $(t, y) = (t_0, y_0)$, then there is a constant $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ is the unique solution to the IVP on this interval.

**Note:** If the conditions of the Uniqueness Theorem are satisfied, different solutions cannot cross or meet in the $(t, y)$ plane.

# Example 3:

Does the following IVP have a unique solution?

$$\frac{dy}{dt} = 1 + y^2, \qquad y(0) = 0.$$

# Example 4:

Does the IVP

$$\frac{dy}{dt} = \sqrt{y}, \qquad y(2) = 0$$

have a unique solution?

What about the IVP

$$\frac{dy}{dt} = \sqrt{y}, \qquad y(0) = 2 \ ?$$

# Example 5:

For the IVP

$$\frac{dy}{dt} = ty^{\frac{1}{5}}, \qquad y(t_0) = y_0,$$

1. Find a value of $t_0$ and a value of $y_0$ so that the IVP has a unique solution. Give a reason for your answer. (Do not try to find an explicit solution to this DE to answer the question.)
2. Find a value of $t_0$ and a value of $y_0$ so that the IVP has more than one solution. For your choice of $t_0$ and $y_0$ write down two functions that satisfy the DE.

# When are the Existence and Uniqueness Theorems useful?

**Example 6:** Use *dfield* to investigate the qualitative behaviour of solutions to the DE

$$\frac{dy}{dt} = y(y-4)(2+\cos^2(y^2))$$

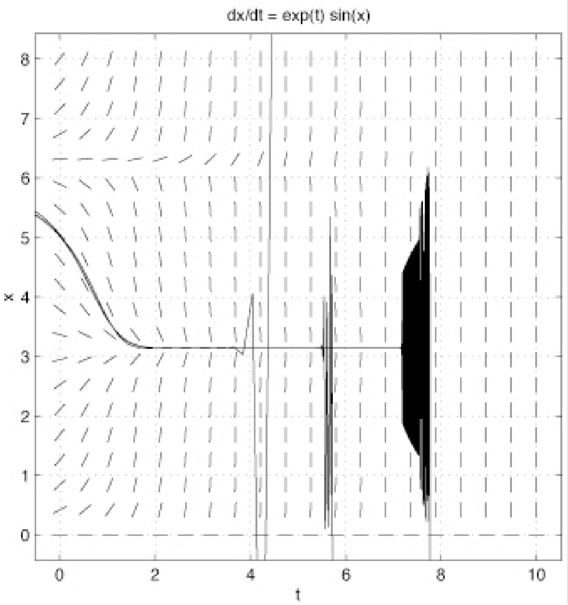
# Example 7:

For the IVP

$$\frac{dy}{dt} = e^t \sin(y), \qquad y(0) = 5$$

use the function *dfield* from Matlab and Euler's method with various step sizes to determine the behaviour of the solution to the DE.

# Example 7:



# Important ideas from today:

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

- If $f$ is 'nice', a solution to the IVP exists, at least for $t$ near $t_0$.
- Also, if $f$ and $\partial f / \partial y$ are 'nice', the solution to the IVP is unique. This implies that solution curves won't cross or touch in $(t, y)$-space.