Tractor Calculus and Invariants for Conformal Sub-Manifolds

Robert Stafford

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Abstract

We consider a smooth conformal n-manifold $(M, \mathbf{g})$ with an embedded co-dimension 1 submanifold $(\Sigma, \bar{\mathbf{g}})$. Using a technique called tractor calculus, we investigate the relationship between the connections of $(M, \mathbf{g})$ and $(\Sigma, \bar{\mathbf{g}})$.

We then build a family of third order differential operators Λ_3. These operators are conformally invariant when acting on functions of any conformal weight for any ambient dimension $n \geq 4$. This generalises a known operator which is conformally invariant only on sections of weight $w = \frac{4-n}{2}$, also with $n \geq 4$.

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Chapter 1

Introduction

Conform manifolds and conformally invariant differential operators have long been important to physics [9]. While much work has been invested in these fields over the past few years, but there is still much to be explored. Conformal sub-manifold geometry in particular has not yet been thoroughly investigated. Some very modern techniques have recently been used to analyse the structure of conformal sub-manifolds [6]. Other investigations have produced differential operator acting on a specified conformal weight [5]. The number of terms comprising a conformally invariant differential operator increases quickly with the order of the operator, making explicit calculations difficult. As such, there is somewhat of a gulf between what is known about conformal sub-manifold geometry and what can be calculated explicitly within it.

The recently developed method of tractor calculus can be used to construct conformally differential operators [4], even in the conformally curved case. As well as being compact and reliable, tractor calculus is in some way the canonical conceptual standpoint [2] for conformal geometry. One minor disadvantage of tractor calculus is that operators so constructed are not always of the preferred normal order. Further, it is not known in general how to adjust the normal order of the differential operators in a conformally invariant way [10].

This thesis concerns conformal n-manifolds where $n \geq 4$ with an embedded co-dimension 1 sub-manifold. Both manifolds in question are assumed to be smooth. We begin with an investigation of the basic machinery of conformal differential geometry and tractor calculus. This leads to a description of the relationship between the tractor connection intrinsic to the sub-manifold and the projected part of the ambient connection. We then construct a conformally invariant third order differential operator on a sub-manifold for all ambient dimensions $n \geq 4$. This operator acts on sections of arbitrary conformal weight and specialises to known operators, such as the Yamabe operator, when acting on sections of particular weights.

1.1 Conventions

Unless explicitly stated otherwise, this thesis will use the Penrose abstract index notation [13]. Bundles and sections thereof will be denoted by $\mathcal{E}$ adorned with appropriate abstract indices. The tangent bundle and co-tangent bundle are labeled $\mathcal{E}^a$ and $\mathcal{E}_a$ respectively. Here a is not related to a specific frame but is instead a Penrose abstract index. A contraction is implied in cases where upper and lower indices are matched. The term $u^a v_{ac}$ is the 1-form equal to the contraction of the vector field $u^a \in \mathcal{E}^a$ into the first slot of the 2-form $v_{bc} \in \mathcal{E}_{bc}$. Tensor products of spaces are represented by concatenated indices.

Further, if u and v are tensors each with a single index then the term $u{\cdot}v$ is understood to mean $u^a v_a$ or $u^a v^b g_{ab}$ as appropriate. This convention is extended to objects which are not tensors but still have a single index, for instance $(\Upsilon \cdot \nabla)\varphi = \Upsilon^a \nabla_a \varphi$. This "inner product" notation will not be used for tractor indices, nor will it be used if either tensor involved in a contraction has more than one index.

Chapter 2

Riemannian Manifolds

Throughout we shall assume that a manifold M is orientable and smooth with $n \geq 4$ and equipped with a Riemannian metric g.

2.1 Essential Riemannian Constructions

Definition 2.1. [12] *A* **connection** *on a vector bundle $\mathcal{V}$ over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$ is an operator $\nabla : \Gamma(\mathcal{V}) \to \Gamma(T^*M \otimes \mathcal{V})$ satisfying the following properties:*

$$\begin{aligned} &i.) && \nabla_u(\alpha V + W) = \alpha \nabla_u V + \nabla_u W \\ &ii.) && \nabla_{fu+v} W = f \nabla_u W + \nabla_v W \\ &iii.) && \nabla_u f W = df(u) \otimes W + f \nabla_u W \qquad \text{(Leibniz rule)} \end{aligned}$$

where $V, W \in \Gamma(\mathcal{V})$, $u, v \in \Gamma(TM)$, $\alpha \in \mathbb{F}$, $f \in C^\infty(M)$, and df is the Exterior Derivative of f.

We will deal with vector bundles and connections over $\mathbb{R}$. When introducing a connection ∇ on a manifold (M, g) without specifying a vector bundle we imply that ∇ acts on the tangent bundle of M, i.e., $\mathcal{V} = TM$. The notation will be such that connections (and indeed all other operators) will act on everything to the right.

Definition 2.2. *A connection ∇ is said to be* **metric-compatible** *with a Riemannian metric $g(\cdot, \cdot)$ if for all $u, v, w \in \Gamma(TM)$ it satisfies the equation*

$$ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$$

where we view the vector field u as a derivative on the left hand side.

In abstract indices, metric compatibility is the condition that $\nabla_c g_{ab} = 0$.

Definition 2.3. *The* **torsion** *$T(u, v)$ of a connection ∇ on M acting on the tangent bundle TM is defined as*

$$T(u, v) \quad = \quad \nabla_u v - \nabla_v u - [u, v]$$

where $u, v \in \Gamma(TM)$.

The bracket $[\cdot, \cdot]$ appearing in the definition is the Lie bracket of u and v, i.e., $[u, v]$ is the unique vector satisfying $[u, v]f = u(v(f)) - v(u(f))$ for all functions $f \in C^\infty(M)$. A connection ∇ is said to be **torsion-free** if $T(u, v) \equiv 0$.

Proposition 2.4. [7, 12] *Given a Riemannian manifold (M, g), there exists a connection which is both torsion-free and compatible with the metric g. Further, this connection is unique and hence completely determined by the metric g.*

The above proposition is well known, and is not difficult to prove using the Christoffel symbols. The unique metric-compatible torsion-free connection guaranteed by proposition 2.4 is called the **Levi-Civita connection**, and will see extensive use throughout this paper.

2.2 Curvature on Manifolds

Definition 2.5. *Given a connection ∇ on a manifold (M, g) we define the* **curvature** *tensor $R : \Lambda^2 TM \mapsto End(TM)$ as*

$$R(u, v)w = \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right) w$$

Proposition 2.6. *If a connection ∇ acting on the tangent bundle of a manifold (M, g) is torsion-free then the curvature tensor may be written using Penrose abstract indices as*

$$R_{ab}{}^{c}{}_{d} v^d = [\nabla_a, \nabla_b]\, v^c$$

Proof. If ∇ is torsion-free then $[u,v] = \nabla_u v - \nabla_v u$ whereby

$$\begin{aligned} \left(u^a \nabla_a v^b\right) \nabla_b - \left(v^b \nabla_b u^a\right) \nabla_a &= \left(\nabla_u v^b\right) \nabla_b - \left(\nabla_v u^a\right) \nabla_a \\ &= \nabla_{\nabla_u v - \nabla_v u} = \nabla_{[u,v]} \end{aligned}$$

Straightfoward calculation then yields

$$\begin{aligned} [\nabla_u, \nabla_v] - \nabla_{[u,v]} &= u^a \nabla_a v^b \nabla_b - v^b \nabla_b u^a \nabla_a - \nabla_{[u,v]} \\ &= u^a v^b \left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) + \left(u^a \nabla_a v^b\right) \nabla_b \\ &\quad - \left(v^b \nabla_b u^a\right) \nabla_a - \nabla_{[u,v]} \\ &= u^a v^b \left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) \end{aligned}$$

hence $R_{ab}{}^c{}_d u^a v^b w^d = u^a v^b \left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) w^c$ as required. $\square$

The Bianchi identities follow from simple combinations of the Lie bracket identities with various formulations of the curvature tensor.

Proposition 2.7. *The* **first Bianchi identity**

$$R_{[ab}{}^c{}_{d]} = 0$$

holds for any torsion-free connection ∇ on TM.

Proof. If ∇ is torsion-free then $\nabla_x y - \nabla_y x = [x,y]$ for all $x, y, z \in \Gamma(TM)$. Hence

$$\nabla_u \nabla_v w - \nabla_u \nabla_w v = \nabla_u [v,w] \qquad \text{and} \qquad \nabla_x [y,z] - \nabla_{[y,z]} x = [x,[y,z]]$$

Careful application of the above formulae quickly leads to

$$\begin{aligned} R_{[ab}{}^c{}_{d]} u^a v^b w^d &= R(u,v)w + R(w,u)v + R(v,w)u \\ &= \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w + \nabla_w \nabla_u v - \nabla_u \nabla_w v \\ &\quad - \nabla_{[w,u]} v + \nabla_v \nabla_w u - \nabla_w \nabla_v u - \nabla_{[v,w]} u \\ &= \nabla_u [v,w] + \nabla_w [u,v] + \nabla_v [w,u] \\ &\quad - \nabla_{[u,v]} w - \nabla_{[w,u]} v - \nabla_{[v,w]} u \\ &= [u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0 \end{aligned}$$

where the final equality follows from the Jacobi identity. $\square$

Proposition 2.8. *The* **second Bianchi identity** [12]

$$\nabla_{[e} R_{ab]}{}^{c}{}_{d} = 0$$

holds for any torsion-free connection ∇*, where square brackets around abstract indices indicate the completely anti-symmetric part over the enclosed indices.*

Definition 2.9. *The* **Riemannian Curvature** *is the curvature* $R_{ab}{}^{c}{}_{d}$ *obtained from definition 2.5 when* ∇ *is the Levi-Civita connection.*

It is well known (see e.g. [8]) that the Riemann curvature decomposes into

$$R_{abcd} = C_{abcd} + P_{ac}g_{bd} - P_{bc}g_{ad} + P_{bd}g_{ac} - P_{ad}g_{bc} \tag{2.1}$$

where C_{abcd} is the completely trace-free Weyl curvature and P_{ab} is the symmetric Weyl-Schouten tensor, also referred to as the Rho-tensor. We will occasionally abuse notation by writing $g^{bc}P_{ab} = P_a^c$ without properly offsetting the indices a and c. This will not be ambigious as the Weyl-Schouten tensor is symmetric.

A number of other tensors also feature in our calculations. The **Ricci tensor** R_{bd} is obtained by contracting the Riemannian curvature over the first and third indices. The **scalar curvature** R is obtained by further contracting the Ricci Tensor using the metric, $R = g^{bd}R_{ab}{}^{a}{}_{d}$. Applying these contractions to the above decomposition provides the relations

$$R_{bd} = Jg_{bd} + (n-2)P_{bd}$$

and

$$R = 2(n-1)J$$

These relations will find extensive use in later chapters. Also of use will be the following:

Proposition 2.10. *The second Bianchi identity implies that for* $n \geq 3$ *the following holds:*

$$\nabla^c P_{ac} = \nabla_a J$$

Proof. The second Bianchi identity is that

$$\nabla_e R_{ab}{}^c{}_d + \nabla_a R_{be}{}^c{}_d + \nabla_b R_{ea}{}^c{}_d = 0$$

Using the property $R_{be}{}^c{}_d = -R_{eb}{}^c{}_d$, a contraction over the indices e and c gives

$$\nabla_c R_{ab}{}^c{}_d - \nabla_a R_{bd} + \nabla_b R_{ad} = 0$$

Next we introduce the decomposition of R_{abcd} and relations between R_{bd}, P_{bd} and J. Contraction over the indices b and d using g^{bd} produces

$$\begin{aligned} 0 &= g^{bd}\nabla^c (P_{ac}g_{bd} - P_{bc}g_{ad} + P_{bd}g_{ac} - P_{ad}g_{bc}) \\ &\quad -2(n-1)\nabla_a J + \nabla^d (J g_{ad} + (n-2)P_{ad}) \\ &= (2n-4)\nabla^c P_{ac} + (4-2n)\nabla_a J \end{aligned}$$

from which the result follows. □

2.3 Riemannian Sub-manifolds

Throughout we assume that Σ is an orientable sub-manifold of (M, g) with co-dimension 1. As such Σ may be equipped with a smooth unit vector field $N^a \in \Gamma(TM)$ satisfying both $N \perp \Sigma$ and $N^a \nabla_b N_a = 0$ everywhere on Σ. The vector bundle $T\Sigma$ may then be identified with the sub-bundle of TM for which $g(v, N)$ vanishes identically, i.e., $\Gamma(T\Sigma)$ consists of the sections of TM orthogonal to N.

Definition 2.11. *The* **projection** *operator* Π^a_b *is defined as*

$$\Pi^a_b = \delta^a_b - N^a N_b$$

where the sub-manifold Σ *has unit normal* N^a *and* δ^a_b *is the Kronecker delta.*

An arbitrary vector field $v^a \in \Gamma(TM)$ along Σ decomposes as

$$v^a = v^a - N^a N_b v^b + N^a N_b v^b = \Pi^a_b v^b + N^a N_b v^b$$

It is well known that a sub-manifold $\Sigma \subset M$ has an intrinsic metric $\bar{g}$ obtained by restricting the ambient g to the relevant sub-bundles, as in

$$\bar{g} = g\,|_{T\Sigma}, \qquad \text{or} \qquad \bar{g}_{ab} = \Pi^{a'}_a \Pi^{b'}_b g_{a'b'}$$

along Σ. The last expression makes sense even when acting on ambient sections, which allows $\bar{g}$ to be extended outside of $T\Sigma$ to a tensor acting on S^2TM along Σ as described above.

Proposition 2.12. *The Levi-Civita connection $\overline{\nabla}$ of Σ takes the form*

$$\overline{\nabla}_a v_b = \Pi_a^{a'} \Pi_b^{b'} \nabla_{a'} v_{b'}$$

Proof. Since the projections are linear tensors, $\overline{\nabla}$ inherits linearity and the Leibniz property from ∇ and so $\overline{\nabla}$ is indeed a connection on Σ. Metric compatibility follows as

$$\begin{aligned}
\overline{\nabla}_c \bar{g}_{ab} &= \Pi_a^{a'} \Pi_b^{b'} \Pi_c^{c'} \nabla_{c'} \left(g_{a'b'} - N_{a'} N_{b'} \right) \\
&= -\Pi_a^{a'} \Pi_b^{b'} \Pi_c^{c'} \nabla_{c'} N_{a'} N_{b'} \\
&= -\Pi_a^{a'} N_{a'} \Pi_b^{b'} \Pi_c^{c'} \nabla_{c'} N_{b'} - \Pi_a^{a'} \Pi_b^{b'} N_{b'} \Pi_c^{c'} \nabla_{c'} N_{a'} = 0
\end{aligned}$$

since ∇ is the Levi-Civita connection of (M, g) and $\Pi_a^{a'} N_{a'} = 0$. It is similarly easy to demonstrate that $\overline{\nabla}$ is torsion-free. It follows from proposition 2.4 that $\overline{\nabla}$ is unique and determined by $\bar{g}$ and is therefore the Levi-Civita connection for $(\Sigma, \bar{g})$. □

Proposition 2.13. *The action of the Levi-Civita connection $\overline{\nabla}$ on sections $v \in \Gamma(T\Sigma)$ may be decomposed into*

$$\overline{\nabla}_a v^b = \Pi_a^{a'} \nabla_{a'} v^b + N^b v^{b'} L_{ab'}$$

where L_{ab} is the **second fundamental form***, which with our conventions is given by $L_{ab} = \Pi_a^c \nabla_c N_b$.*

Proof.

$$\begin{aligned}
\overline{\nabla}_a v^b &= \Pi_a^{a'} \Pi_{b'}^b \nabla_{a'} v^{b'} \\
&= \Pi_a^{a'} \nabla_{a'} v^b - \Pi_a^{a'} N^b N_{b'} \nabla_{a'} v^{b'} \\
&= \Pi_a^{a'} \nabla_{a'} v^b + N^b v^{b'} \Pi_a^{a'} \nabla_{a'} N_{b'} \\
&= \Pi_a^{a'} \nabla_{a'} v^b + N^b v^{b'} L_{ab'}
\end{aligned}$$

with $L_{ab'}$ as provided above. □

Contracting over the second fundamental form with either the ambient metric or intrinsic metric produces a multiple of the **mean curvature**, denoted H, as in

$$g^{ac}L_{ac} = \bar{g}^{ac}L_{ac} = (n-1)H \tag{2.2}$$

The trace-free part of the second fundamental form will be denoted $L_{(ab)^0}$, and is given by

$$\begin{aligned} L_{(ab)^0} &= L_{ab} - \frac{1}{n-1}\bar{g}_{ab}\bar{g}^{cd}L_{cd} \\ &= L_{ab} - \bar{g}_{ab}H \end{aligned}$$

The small circle following the bracketed indices indicates that the tensor is trace-free.

Chapter 3

Conformal Geometry

3.1 Conformal Manifolds

Definition 3.1. [8] *A* **conformal manifold** *is a pair* $(M, [g])$ *consisting of a manifold* M *and an equivalence class* $[g]$ *of Riemannian metrics on* M *under the equivalence relation*

$$g \sim g' \Leftrightarrow g' = e^{2w} g, \qquad w \in \mathcal{E}$$

The ray bundle of metrics $\mathcal{G}$ over M is then a principal bundle with fibres isomorphic to $\mathbb{R}_+$. For each representation $\rho : \mathbb{R}^+ \mapsto \mathrm{End}(\mathbb{R})$ we obtain an associated vector bundle. We denote as ρ_w the representation of $\mathcal{G}$ on $\mathbb{R}$ which maps $\lambda \in \mathbb{R}_+$ to the endomorphism $r \mapsto \lambda^{-w/2} r$, where $r \in \mathbb{R}$. The associated vector bundle, to be denoted $\mathcal{E}[w]$, is then described as

$$\mathcal{E}[w] = \mathcal{G} \times_{\rho_w} \mathbb{R} = \mathcal{G} \times_{\rho} \mathbb{R}/_{\sim_w} \tag{3.1}$$

A section $\sigma \in \mathcal{E}[w]$ is called a **density**. The relation $\sim_w$ is given by

$$(g\lambda, r) \sim_w (g, (\rho(\lambda))^{-1} r) = (g, \lambda^{w/2} r)$$

for $g \in \mathcal{G}$, $\lambda \in \mathbb{R}_+$ and $r \in \mathbb{R}$. A density $\sigma \in \mathcal{E}[w]$ is thus equivalent to a homogeneous function $\overline{\sigma} : \mathcal{G} \mapsto \mathbb{R}$ which satisfies

$$\overline{\sigma}(g\lambda, x) \;\; = \;\; \lambda^{w/2} \overline{\sigma}(g, x)$$

for every $x \in M$.

There is a tautological mapping $\underline{\mathbf{g}} : \mathcal{G} \mapsto S^2T^{\star}M$ which takes $(g_x, x) \mapsto g_x$ where $x \in M$. This mapping is homogeneous, since $\underline{\mathbf{g}}(\lambda^2 g_x, x) = \lambda^2 g_x$. Given a function $\widetilde{\sigma} : \mathcal{G} \mapsto \mathbb{R}$ homogenious of degree 1, we have $\widetilde{\sigma}(g_x\lambda^2, x) = \lambda\widetilde{\sigma}(g_x, x)$ with $x \in M$. Clearly $\widetilde{\sigma}^{-2}\underline{\mathbf{g}}$ is then homogeneous of weight zero, and is therefore equivalent to a Riemannian metric g on M. Thus $\underline{\mathbf{g}}$ is equivalent to a section $\mathbf{g}_{ab} \in \mathcal{E}_{ab}\oplus\mathcal{E}[2]$, which will be called the **conformal metric**. Any Riemannian metric $g \in [g]$ can be recovered by $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$ for some non-vanishing $\sigma \in \mathcal{E}[1]$. A section $\sigma \in \mathcal{E}[1]$ is called a conformal choice of scale, for it is equivalent to selecting a Riemannian metric $g \in [g]$.

Tensor indices can be lowered or raised in a conformally invariant way using the conformal metric $\mathbf{g}_{ab} \in \mathcal{E}_{ab}[2]$ or $\mathbf{g}^{ab} \in \mathcal{E}^{ab}[-2]$ respectively, albeit at the expense of adjusting the weight of the tensor involved. For example if $u_a \in \mathcal{E}_a[w]$ then $u^b = \mathbf{g}^{ab}u_a \in \mathcal{E}^b[w-2]$. Unless otherwise stated, all further raising and lowering of tensor indices will be performed using the conformal metric $\mathbf{g}_{ab}$. Additionally, the notation $(M, \mathbf{g})$ and $(M, [g])$ will be used interchangeably, since both structures give conformal manifolds.

Definition 3.2. *For a particular choice of metric, the action of the corresponding Levi-Civita connection on a weighted section $\rho \in \mathcal{E}[w]$ is defined to be*

$$\nabla\rho = \sigma^w d(\sigma^{-w}\rho)$$

where d is the exterior derivative on M.

This is well defined since the exterior derivative d acts on the function $\sigma^{-w}\rho$ which is unweighted.

3.2 Basic Conformal Transformations

It is essential to know how a conformal rescaling affects the action of the Levi-Civita connection. The Christoffel symbols can be used to verify that the Levi-Civita connection $\widehat{\nabla}$ for a rescaled metric $\hat{g} = e^{2w}g$ acts on one-forms and vector fields [8] as

$$\widehat{\nabla}_a u^b = \nabla_a u^b + \Upsilon_a u^b - \Upsilon^b u_a + \delta_a^b \Upsilon\cdot u \tag{3.2}$$

$$\widehat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + \mathbf{g}_{ab}\Upsilon\cdot u \tag{3.3}$$

where $\Upsilon = dw$. The conformal transformations for the action of the Levi-Civita connection on tensors with more than one index can be obtained from these using the Leibniz rule. There is also a non-trivial conformal rescaling involved when a connection acts on a weighted section.

Proposition 3.3. *A conformal rescaling $g \to \hat{g} = e^{2f}g$ of the metric results in the transformation of ∇ acting on $\rho \in \mathcal{E}[w]$ as*

$$\widehat{\nabla}_a \rho = \nabla_a \rho + w\rho\Upsilon_a$$

where $\Upsilon_a = df_a$

Proof. Let $\sigma \in \mathcal{E}[1]$ be the choice of conformal scale such that $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$. Then $\hat{g}_{ab} = e^{2f}g_{ab}$ implies that $\hat{g}_{ab} = \widehat{\sigma}^{-2}\mathbf{g}_{ab}$ is satisfied when $\widehat{\sigma} = e^{-f}\sigma$. Since both e^{wf} and $\sigma^{-w}\rho$ are both functions $M \mapsto \mathbb{R}$, we can derive

$$\begin{aligned} d(\widehat{\sigma}^{-w}\rho) &= d(e^{wf}\sigma^{-w}\rho) \\ &= e^{wf}d(\sigma^{-w}\rho) + \sigma^{-w}\rho w e^{wf} df \end{aligned}$$

using the Leibniz rule and the chain rule, so that

$$\begin{aligned} \widehat{\nabla}\rho &= \widehat{\sigma}^{w} d(\widehat{\sigma}^{-w}\rho) \\ &= e^{-wf}\sigma^{w}\left(e^{wf}d(\sigma^{-w}\rho) + \sigma^{-w}\rho w e^{wf} df\right) \\ &= \sigma^{w} d(\sigma^{-w}\rho) + w\rho df \\ &= \nabla\rho + w\Upsilon\rho \end{aligned}$$

completes the proof. □

This result can be also be combined with equations 3.2 and 3.3 using the Leibniz rule to obtain expressions for the conformal transformation of a connection acting on any weighted tensor. For instance, for $u_b \in \mathcal{E}_b[w]$,

$$\widehat{\nabla}_a u_b = \nabla_a u_b + (w-1)\Upsilon_a u_b - \Upsilon_b u_a + \mathbf{g}_{ab}\Upsilon \cdot u \tag{3.4}$$

Understanding the conformal behavior of higher order operators will also be required.

Lemma 3.4. *Let $\sigma \in \mathcal{E}[w]$. The Riemannian Laplacian transforms under a conformal rescaling of the metric according to*

$$\widehat{\Delta}\sigma = \Big[\Delta + (n+2w-2)\Upsilon\cdot\nabla + w\nabla\cdot\Upsilon + w(w+n-2)\Upsilon\cdot\Upsilon\Big]\sigma$$

Proof. Using the above results for $\widehat{\nabla}_a\varphi$ with $\phi \in \mathcal{E}[w]$ and $\widehat{\nabla}_b\varphi_a$ for $\varphi_a \in \mathcal{E}_a[w]$, expanding $\widehat{\Delta}$ yields

$$\begin{aligned}
\mathbf{g}^{ab}\widehat{\nabla}_b\widehat{\nabla}_a &= \mathbf{g}^{ab}\widehat{\nabla}_b\left(\nabla_a + w\upsilon_a\right)\sigma \\
&= \mathbf{g}^{ab}\Big[\left(\nabla_b\nabla_a + (w-1)\Upsilon_b\nabla_a - \Upsilon_a\nabla_b + \mathbf{g}_{ba}\Upsilon\cdot\nabla\right) \\
&\qquad + \left(w\nabla_b\Upsilon_a + (w-1)w\Upsilon_b\Upsilon_a - w\Upsilon_a\Upsilon_b + w\mathbf{g}_{ab}\Upsilon\cdot\Upsilon\right)\Big]\sigma \\
&= \mathbf{g}^{ab}\Big[\nabla_b\nabla_a + (w-1)\Upsilon_b\nabla_a + (w-1)\Upsilon_a\nabla_b + \mathbf{g}_{ba}\Upsilon\cdot\nabla \\
&\qquad + w\left(\nabla_b\Upsilon_a\right) + w(w-2)\Upsilon_a\Upsilon_b + w\mathbf{g}_{ab}\Upsilon\cdot\Upsilon\Big]\sigma \\
&= \Big[\Delta + (n+2w-2)\Upsilon\cdot\nabla + w\nabla\cdot\Upsilon + w(w+n-2)\Upsilon\cdot\Upsilon\Big]\sigma
\end{aligned}$$

completing the proof. □

3.3 Transformation of Curvature Tensors

Higher order operators involve terms other than the Levi-Civita connection for a particular metric. Establishing the conformal scaling of such terms is then beneficial to verify the conformal invariance of the higher order operators. Calculating the conformal scaling of the curvature terms starts with the expansion of

$$\widehat{R}_{ab}{}^{c}{}_{d}v^d = \left(\widehat{\nabla}_a\widehat{\nabla}_b - \widehat{\nabla}_b\widehat{\nabla}_a\right)v^c$$

This requires the rules of 3.2 and 3.3 to be combined using the Leibniz rule. After the terms have been expanded and collected one obtains [8]

$$\begin{aligned}
\widehat{R}_{abcd} &= R_{abcd} - g_{bd}\left(\nabla_a\Upsilon_c - \Upsilon_a\Upsilon_c + \frac{1}{2}\Upsilon\cdot\Upsilon g_{ac}\right) + g_{ad}\left(\nabla_b\Upsilon_c - \Upsilon_b\Upsilon_c + \frac{1}{2}\Upsilon\cdot\Upsilon g_{bc}\right) \\
&\quad - g_{ac}\left(\nabla_b\Upsilon_d - \Upsilon_b\Upsilon_d + \frac{1}{2}\Upsilon\cdot\Upsilon g_{bd}\right) + g_{bc}\left(\nabla_a\Upsilon_d - \Upsilon_a\Upsilon_d + \frac{1}{2}\Upsilon\cdot\Upsilon g_{ad}\right)
\end{aligned}$$

This form provides the easiest comparison with the decomposition of Riemannian curvature into Weyl curvature and Weyl-Schouten tensor parts; it is clear that the Weyl-Schouten tensor transforms conformally as

$$\widehat{P}_{ab} = P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\mathbf{g}_{ab}\Upsilon\cdot\Upsilon \qquad (3.5)$$

and also that the Weyl curvature C_{abcd} is conformally invariant. The application of g^{ab} to equation 3.5 shows that the J scalar field transforms by

$$\widehat{J} = J - \nabla\cdot\Upsilon + \frac{2-n}{2}\Upsilon\cdot\Upsilon$$

In addition to curvature terms, the following differential operator will appear in calculations of higher order differential operators on conformal manifolds.

Definition 3.5. [4] *The* **Yamabe operator** $\Box$ *is defined to act on densities* $\phi \in \mathcal{E}[w]$ *as*

$$\Box\phi \quad = \quad (\Delta + wJ)\phi$$

The Yamabe operator is defined separately for each metric $g \in [g]$. It follows from previous results that

$$\hat{\Box}\phi \quad = \quad \left(\Box + (n + 2w - 2)\left(\Upsilon\cdot\nabla + \frac{w}{2}\Upsilon\cdot\Upsilon\right)\right)\phi$$

It follows that the Yamabe operator is conformally invariant when acting on weighted sections $\phi \in \mathcal{E}[\frac{2-n}{2}]$.

3.4 Conformal properties of Sub-manifolds

Let $(M, \mathbf{g})$ denote a conformal n-manifold and Σ a sub-manifold of co-dimension 1 with a unit normal field N^a. For each choice of conformal scale g we obtain a Riemannian manifold (M, g) and an associated Riemannian sub-manifold $(\Sigma, \bar{g})$ by restriction, i.e., $\bar{g} = g|_{T\Sigma}$ along Σ as in section 2.3.

A rescaling of the ambient metric $\hat{g} = e^{2w}g$ will induce a rescaling of the sub-manifold metric given by $\hat{\bar{g}} = e^{2\bar{w}}\bar{g}$ where $\bar{w} = w|_\Sigma$. The class of metrics $[\bar{g}]$ obtained from $[g]$ in this way give Σ a conformal structure. Terms arising from a conformal scaling intrinsic to Σ will involve the one-form $\overline{\Upsilon}^a = \overline{dw}^a = \Pi^a_b\Upsilon^b$ along Σ, where $\overline{dw}^a$ is the exterior derivative of Σ applied to $\bar{w} = w|_\Sigma$. In particular, all the statements of sections 3.2 and 3.3 will have versions that apply to $(\Sigma, \bar{\mathbf{g}})$, with Υ and n replaced by $\overline{\Upsilon}$ and $n-1$ respectively, for instance

$$\hat{\overline{\Box}} \quad = \quad \overline{\Box} + (n + 2w - 3)\left(\overline{\Upsilon}\cdot\overline{\nabla} + \frac{w}{2}\overline{\Upsilon}\cdot\overline{\Upsilon}\right)$$

We can also obtain a conformally invariant counterpart to the Riemannian normal vector N^a. For a given Riemannian metric $g = \sigma^{-2}\mathbf{g}$, we define the **conformal normal vector** $\underline{N}^a \in \mathcal{E}[-1]a$ to be the solution of $\sigma\underline{N}^a = N^a$. Note that $\underline{N}^a$ is independent of the choice of $g \in [g]$. Note that σ is a parallel section for the Levi-Civita connection ∇ corresponding to g and so may be effectively suppressed in calculations; all computations based in Riemannian geometry of section 2.3 using N^a may be imported into the conformal setting by replacing N^a and g with $\underline{N}^a$ and $\mathbf{g}$ respectively. As such, from this point onwards N^a will refer to the conformal normal vector field. It follows that we may obtain conformal versions of $N_a \in \mathcal{E}_a[1]$, a conformal projection $\Pi_b^a \in \mathcal{E}_b^a[0]$ (using the conformal normal), $H \in \mathcal{E}[-1]$, and $L_{ab} \in \mathcal{E}_{ab}[1]$. Such results may still transform under conformal rescalings of the metric, for instance:

Proposition 3.6. *A sub-manifold Σ with conformal unit normal vector field $N_a \in \mathcal{E}_a[1]$ has a second fundamental form L_{ab} which transforms conformally as*

$$\widehat{L}_{ab} = L_{ab} + \bar{\mathbf{g}}_{ab}\Upsilon\cdot N$$

Proof. Using equation 3.4 with $w = 1$ gives

$$\begin{aligned}\widehat{L}_{ab} &= \Pi_a^c \widehat{\nabla}_c N_b \\ &= \Pi_a^c \left(\nabla_c N_b - N_c \Upsilon_b + \mathbf{g}_{cb}\Upsilon\cdot N\right) \\ &= \Pi_a^c \nabla_c N_b + \bar{\mathbf{g}}_{ab}\Upsilon\cdot N\end{aligned}$$

completing the proof. □

Several equations that result from simple contractions of the second fundamental form will be used throughout later chapters.

Corollary 3.7. *The mean curvature $H \in \mathcal{E}[-1]$ transforms as $\widehat{H} = H + \Upsilon\cdot N$.*

Corollary 3.8. *The trace-free second fundamental form $L_{(ab)^0}$ is conformally invariant.*

The conformal invariance of one tensor in particular will be relevant later.

Proposition 3.9. *The tensor*

$$\overline{P}_{ac} - \Pi_a^{a'}\Pi_c^{c'} P_{a'c'} - HL_{(ac)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}H^2$$

is conformally invariant.

Proof. We begin with the conformal transformation of the projected ambient Weyl-Schouten tensor.

$$\Pi_a^{a'}\Pi_c^{c'}\widehat{P}_{a'c'} = \Pi_a^{a'}\Pi_c^{c'}P_{a'c'} - \Pi_a^{a'}\Pi_c^{c'}\left(\nabla_{a'}\Upsilon_{c'}\right) - \Pi_a^{a'}\Pi_c^{c'}\Upsilon_{a'}\Upsilon_{c'} - \frac{1}{2}\mathbf{g}_{ac}\Upsilon\cdot\Upsilon$$

Since $\overline{\Upsilon}_a = \Pi_a^{a'}\Upsilon_{a'}$ we obtain

$$\begin{aligned}
\widehat{\overline{P}}_{ac} - \Pi_a^{a'}\Pi_c^{c'}\widehat{P}_{a'c'} &= \overline{P}_{ac} - \Pi_a^{a'}\Pi_c^{c'}P_{a'c'} + \Pi_a^{a'}\Pi_c^{c'}\nabla_{a'}\left(\Upsilon_{c'} - \overline{\Upsilon}_{c'}\right) + \frac{1}{2}\bar{\mathbf{g}}_{ac}\left(\overline{\Upsilon}\cdot\overline{\Upsilon} - \Upsilon\cdot\Upsilon\right) \\
&= \overline{P}_{ac} - \Pi_a^{a'}\Pi_c^{c'}P_{a'c'} + \Upsilon\cdot N L_{ac} + \frac{1}{2}\bar{\mathbf{g}}_{ac}\left(\Upsilon\cdot N\right)^2
\end{aligned}$$

The other terms in the relevant tensor transform according to

$$\begin{aligned}
-\widehat{H}L_{(ac)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}\widehat{H}^2 &= -HL_{(ac)^0} - \Upsilon\cdot N L_{(ac)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}\left(H + \Upsilon\cdot N\right)^2 \\
&= -HL_{(ac)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}H^2 - \Upsilon\cdot N L_{ac} - \frac{1}{2}\bar{\mathbf{g}}_{ac}\left(\Upsilon\cdot N\right)^2
\end{aligned}$$

whereby the tensor of the proposition is seen to be conformally invariant. $\square$

Chapter 4

Tractor Calculus

Using the expansions of chapter 3 to verify that an operator or tensor is conformally invariant requires checking equations with large numbers of terms. A different simpler method involves building larger structures out of smaller conformally invariant objects called tractors. Composition of conformally invariant objects yields yet more conformally invariant objects, and in this way checking conformal invariance is greatly simplified. In addition, the calculus of tractors allows the construction of new operators which are automatically conformally invariant.

4.1 The Tractor Bundle

Definition 4.1. [4, 8, 11] *Given a particular metric g we define a* **tractor** V^A *to be a section of* $\mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$, *i.e.,*

$$[V^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}$$

which transforms under a conformal rescaling fo the mertic $g \to \hat{g}$ as

$$[V^A]_{\hat{g}} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_b \mu^b - \frac{1}{2} \Upsilon \cdot \Upsilon \sigma \end{pmatrix}$$

The space of all tractors over a conformal manifold $(M, \mathbf{g})$ will be denoted by $\mathbb{T}$.

The Tractor bundle is in fact isomorphic to the bundle of 2-jets $\mathcal{J}^2\mathcal{E}[1]$ [2]. Capital letter subscripts and superscripts will denote tractor indices, while lower case letters will be reserved for tensor indices. The only exception is the term ∇_N which refers to the normal derivative $\nabla_N = N^a\nabla_a$.

Definition 4.2. *A* **rescaling operator** $\Xi_{g\to\hat{g}}$ *is an automorphism of* $\mathcal{E}[1]\oplus\mathcal{E}^a[-1]\oplus\mathcal{E}[-1]$ *given by*

$$\Xi_{g\to\hat{g}} : \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a\sigma \\ \rho - \Upsilon_b\mu^b - \frac{1}{2}\Upsilon\cdot\Upsilon\sigma \end{pmatrix}$$

with g and $\hat{g}$ being metrics from the conformal class satisfying $\hat{g} = e^{2w}g$, and $\Upsilon = dw$.

It is clear that for any tractor

$$\Xi_{g\to\hat{g}}\left[V^A\right]_g = \left[V^A\right]_{\hat{g}}$$

The rescaling operators provide a concise notation which which to state and prove the most fundamental properties of the tractor bundle. For instance, it can easily be shown that the set of all rescaling operators form an Abelian groupoid.

Proposition 4.3. *The tractor bundle may be equipped with the conformally invariant* **tractor metric** $h(\cdot,\cdot)$ *described by*

$$h(V^A, V'^A) \quad = \quad \sigma\rho' + \sigma'\rho + \mu_a\mu'^a \tag{4.1}$$

Proof. The invariance of the metric follows from the Polarisation of the conformally invariant quadratic $2\sigma\rho + \mu_a\mu^a$. It remains to show the quadratic is indeed invariant:

$$\begin{aligned} 2\hat{\sigma}\hat{\rho} + \hat{\mu}_a\hat{\mu}^a &= 2\sigma(\rho - \Upsilon_b\mu^b - \frac{1}{2}\Upsilon^b\Upsilon_b\sigma) + (\mu^a + \Upsilon^a\sigma)(\mu_a + \Upsilon_a\sigma) \\ &= 2\sigma\rho - 2\Upsilon_b\mu^b\sigma - \Upsilon^b\Upsilon_b\sigma^2 + \mu^a\mu_a + \Upsilon^a\mu_a\sigma + \mu^a\Upsilon_a\sigma + \Upsilon^a\Upsilon_a\sigma^2 \\ &= 2\sigma\rho - 2\Upsilon\cdot\mu\sigma - \Upsilon\cdot\Upsilon\sigma^2 + \mu\cdot\mu + \Upsilon\cdot\mu\sigma + \mu\cdot\Upsilon\sigma + \Upsilon\cdot\Upsilon\sigma^2 \\ &= 2\sigma\rho + \mu\cdot\mu \end{aligned}$$

which demonstrates the conformal invariance of both the quadratic and the metric. ☐

The tractor metric facilitates the raising and lowering of a tractor index just as a conformal (or Riemannian) metric allows the raising and lowering of a tensor index; henceforth indices of all types will be raised and lowered as required using the appropriate metric.

4.2 Injecting Operators

For a particular Riemannian metric the tractor bundle decomposes as

$$\mathcal{E}^A \quad = \quad \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$$

There is, associated with each Riemannian metric, a set of operators which facilitate computation using this decomposition of the tractor bundle $\mathcal{E}A$.

Definition 4.4. *Let $\sigma \in \mathcal{E}[1]$, $\mu^a \in \mathcal{E}^a[-1]$, and $\rho \in \mathcal{E}[-1]$. Choose a metric g and define the* **injecting operators** *Y^A, Z_a^A, and X^A for this metric as those as those taking the actions*

$$\left[Y^A\sigma\right]_g = \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix}, \quad \left[Z_a^A\mu^a\right]_g = \begin{pmatrix} 0 \\ \mu^a \\ 0 \end{pmatrix}, \quad \text{and} \quad \left[X^A\rho\right]_g = \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}$$

The notation X^A does not explicitly mention the Riemannian connection g providing the injecting operators. In cases where injecting operators corresponding to different Riemannian metrics are present in the same equations, the operators Y^A, Z_a^A, and X^A will correspond to the metric g while the operators $\widehat{Y}^A$, $\widehat{Z}_a^A$, and $\widehat{X}^A$ are those relevant to the rescaled metric $\hat{g}$.

The injecting operators are weighted tractors or tensor-tractors themselves, with $Y^A \in \mathcal{E}^A[-1]$ since Y^A maps densities of conformal weight 1 to elements of $\mathcal{E}^A$. Likewise, $Z_a^A \in \mathcal{E}_a^A[1]$ and $X^A \in \mathcal{E}^A[1]$. Definition 4.1 indicates how the injecting operators of different metrics are related; X^A is independent of the metric, whereas rescaling the metric leads to new injecting operators Z_a^A and Y^A given by

$$\begin{aligned} \hat{Z}_a^A &= Z_a^A + \Upsilon_a X^A \\ \hat{Y}^A &= Y^A - \Upsilon^b Z_b^A - \frac{1}{2}\Upsilon^b\Upsilon_b X^A \end{aligned} \tag{4.2}$$

It is clear that given a metric g an arbitrary unweighted tractor $V^A \in \mathcal{E}^A$ may be expressed as $V^A = Y^A\sigma + Z_a^A\mu^a + X^A\rho$ for some suitable choice of $\sigma \in \mathcal{E}[1]$, $\mu^a \in \mathcal{E}^a[-1]$, and $\rho \in \mathcal{E}[-1]$. Further, it is clear from the tractor metric that $Y^A X_A = X^A Y_A = 1$ and $Z_a^A Z_{Ab} = \mathbf{g}_{ab}$ are the only non-trivial inner products between the injecting operators. The injecting operators for a metric g can then also be used to recover the components of a tractor V^A with respect to g;

$$V^A X_A = \sigma$$

recovering the first component of the tractor V^A. Similarly the ρ component can be extracted by a contraction of V^A into Y^A, and μ^a can be obtained via inner products with a collection of $Z_a^A\phi^a$ terms, with suitably many linearly independent terms $\phi^a \in \mathcal{E}^a[-1]$. In light of this we may write the tractor metric entirely in terms of the injecting operators.

Proposition 4.5. *The tractor metric may be expressed as*

$$h(V^A, U^B) = (X_A Y_B + Y_A X_B + Z_A^a Z_{Ba})\, V^A U^B$$

that is, $h_{AB} = X_A Y_B + Y_A X_B + Z_A^a Z_{Ba}$.

We also say that a metric g (or the injecting operators of a metric) "splits" a tractor V^A into components σ, μ^a and ρ, indicating that we perform the decomposition

$$V^A = \sigma Y^A + \mu^a Z_a^A + \rho X^A$$

according to the Riemannian metric g.

4.3 Connections on Tractors

Definition 4.6. [2] *The* **tractor connection** *is defined to be the operator* ∇ *which acts on a tractor* V^A *as*

$$\left[\nabla_b V^A\right]_g = \begin{pmatrix} \nabla_b\sigma - \mu_b \\ \nabla_b\mu^a + \delta_b^a\rho + P_b^a\sigma \\ \nabla_b\rho - P_{cb}\mu^c \end{pmatrix}$$

It should be noted that the symbol ∇ is used to denote a Levi-Civita connection, the conformally invariant tractor connection, and the coupled connection in the case where a section carries both tractor and tensor indices. This overloaded notation will simplify arguments, particularly when the nature of a section's indices is unknown.

The tractor/coupled Laplacian, $\Delta = \mathbf{g}_{ab}\nabla_a\nabla_b$, will also be used.

Lemma 4.7. *The tractor connection acts on the injecting operators as:*

$$\begin{aligned} \nabla_b Y^A &= Z_a^A P_b^a \\ \nabla_b Z_a^A &= -X^A P_{ab} - \mathbf{g}_{ab} Y^A \\ \nabla_b X^A &= Z_b^A \end{aligned}$$

Proof. Choosing a metric g yields

$$\nabla_b V^A = \nabla_b \left(Y^A \sigma + Z_a^A \mu^a + X^A \rho \right)$$

Writing definition 4.6 in terms of the injecting operators yields

$$\nabla_b V^A = Y^A \left(\nabla_b \sigma - \mu_b \right) + Z_a^A \left(\nabla_b \mu^a + \delta_b^a \rho + P_b^a \sigma \right) + X^A \left(\nabla_b \rho - P_{cb} \mu^c \right)$$

whereas expansion using the Leibniz rule produces

$$\nabla_b V^A = \sigma \nabla_b Y^A + Y^A \nabla_b \sigma + \mu^a \nabla_b Z_a^A + Z_a^A \nabla_b \mu^a + \rho \nabla_b X^A + X^A \nabla_b \rho$$

These expressions for $\nabla_b V^A$ will agree when

$$-Y^A \mu_b + Z_a^A \delta_b^a \rho + Z_a^A P_b^a \sigma - X^A P_{ab} \mu^a = \sigma \nabla_b Y^A + \mu^a \nabla_b Z_a^A + \rho \nabla_b X^A$$

and so from the independence of the section σ, μ^a and ρ we obtain the decomposition into the required results. $\square$

Corollary 4.8. *The tractor contractions $X^A \nabla_b X_A$, $X^A \nabla_b Y_A$, $Y^A \nabla_b X_A$, $Y^A \nabla_b Y_A$ and $Z_a^A \nabla_b Z_{Ac}$ vanish identically.*

Lemma 4.9. *The tractor Laplacian acts on the injecting operators as:*

$$\begin{aligned} \Delta Y^A &= Z_a^A \nabla^a J - P^{ab} P_{ab} X^A - J Y^A \\ \Delta Z_a^A &= -X^A \nabla_a J - 2 P_a^b Z_b^A \\ \Delta X^A &= -J X^A - n Y^A \end{aligned}$$

Proof. The results follow from repeated application of lemma 4.7:

$$\begin{aligned}
\Delta Y^A &= \nabla^b \nabla_b Y^A \\
&= \nabla^b P_b^a Z_a^A \\
&= Z_a^A \nabla^b P_b^a + P_b^a \nabla^b Z_a^A \\
&= Z_a^A \nabla^a J + P_b^a (-P_a^b X^A - \delta_a^b Y^A) \\
&= Z_a^A \nabla^a J - P^{ab} P_{ab} X^A - J Y^A
\end{aligned}$$

which completes the calculations for ΔY^A. For ΔZ_a^A we require proposition 2.10 that $\nabla^b P_b^a = \nabla^a J$. We obtain

$$\begin{aligned}
\Delta Z_a^A &= g^{bc} \nabla_c \nabla_b Z_a^A \\
&= g^{bc} \nabla_c \left(-P_{ab} X^A - g_{ab} Y^A\right) \\
&= -g^{bc} \left(X^A \nabla_c P_{ab} + P_{ab} \nabla_c X^A + g_{ab} \nabla_c Y^A\right) \\
&= -g^{bc} \left(X^A \nabla_c P_{ab} + P_{ab} Z_c^A + g_{ab} P_c^d Z_d^A\right) \\
&= -X^A \nabla_c P_a^c - P_a^c Z_c^A - P_a^d Z_d^A \\
&= -X^A \nabla_a J - 2 P_a^c Z_c^A
\end{aligned}$$

The term ΔX^A is calculated as

$$\begin{aligned}
\Delta X^A &= g^{bc} \nabla_c \nabla_b X^A \\
&= g^{bc} \nabla_c Z_b^A \\
&= g^{bc} \left(-P_{bc} X^A - g_{bc} Y^A\right) \\
&= -J X^A - n Y^A
\end{aligned}$$

This concludes the required calculations. $\square$

4.4 Tractor Operators

Following [3, 8] we define the tractor operator $D^A : \mathcal{E}^\star[w] \mapsto \mathcal{E}^{A\star}[w-1]$ as taking the action

$$D^A \varphi = \begin{pmatrix} (n+2w-2)w\varphi \\ (n+2w-2)\nabla^a \varphi \\ -(\Delta + wJ)\varphi \end{pmatrix} \tag{4.3}$$

on $\varphi \in \mathcal{E}^\star[w]$, where $\star$ represents any combination of tractor indices. The D^A operator will be essential to the construction of invariant operators. The availability of some well known lemmas regarding the action of D^A on the injecting operators will simplify calculations later.

Lemma 4.10. *Let $\varphi \in \mathcal{E}^\star[w]$. The tractor D^A operator has the following action on the injecting operators X^A, Y^A, and Z^A_a:*

$$\begin{aligned} D^A Y_A \varphi &= (n+w-2)J\varphi - \Delta\varphi \\ D^A Z^a_A \varphi &= (n+2w-2)\nabla^a\varphi \\ D^A X_A \varphi &= (n+2w+2)(n+w)\varphi \end{aligned} \tag{4.4}$$

Proof. The D^A operator when acting on a section $\varphi \in \mathcal{E}[w]$ can be expressed using the injecting operators as

$$D^A\varphi \;=\; (n+2w-2)wY^A\varphi + (n+2w-2)Z^A_a\nabla^a\varphi - X^A(\Delta + wJ)\varphi$$

Lemmas 4.7 and 4.9 will be applied and terms vanishing in the tractor metric contraction will be omitted. For $\varphi \in \mathcal{E}[w]$, the term $Y_A\varphi$ will have conformal weight $w-1$ and so

$$\begin{aligned} D^A Y_A\varphi &= \Big[(n+2(w-1)-2)Z^A_a\nabla^a - X^A(\Delta + (w-1)J)\Big]Y_A\varphi \\ &= (n+2w-4)Z^A_a\left(Z^b_A P^a_b\right)\varphi - (w-1)J\varphi - X^A\Delta Y_A\varphi \\ &= (n+w-3)J\varphi - \Big(X^A Y_A\Delta\varphi + X^A(\Delta Y_A)\varphi\Big) \\ &= (n+w-3)J\varphi - \Delta\varphi - X^A(-JY_A)\varphi \\ &= (n+w-4)J\varphi - \Delta\varphi \end{aligned}$$

Similarly, $Z^a_A\varphi$ has conformal weight $w-1$. Lemma 4.9 shows that $\Delta Z^a_A = -X_A\nabla^a J - 2P^a_c Z^c_A$ so that $X^A\Delta Z^a_A$ vanishes. Therefore

$$\begin{aligned} X^A\Delta Z^a_A\varphi &= 2X^A\left(\nabla_c Z^a_A\right)(\nabla^c\varphi) \\ &= 2X^A\left(-P^a_c - \delta^a_c Y^A\right)(\nabla^c\varphi) \\ &= -2\nabla^a\varphi \end{aligned}$$

Substituting this into $D^A Z_A^a \varphi$ gives

$$\begin{aligned} D^A Z_A^a \varphi &= \Big[(n+2w-4)\,(w-1) Y^A + (n+2w-4)\, Z_b^A \nabla^b \\ &\qquad -X^A \left(\Delta + (w-1)J\right) \Big] Z_A^a \varphi \\ &= (n+2w-4)\nabla^a \varphi - X^A \Delta Z_A^a \varphi \\ &= (n+2w-2)\nabla^a \varphi \end{aligned}$$

Lastly $X_A \varphi$ is of weight $w+1$ so application of D^A yields

$$\begin{aligned} D^A X_A \varphi &= \Big[(n+2w)(w+1) Y^A + (n+2w) Z_a^A \nabla^a \\ &\qquad -X^A \left(\Delta + (w+1)J\right) \Big] X_A \varphi \\ &= (n+2w)(w+1)\varphi + (n+2w)n\varphi - X^A \left(-nY^A\right) \varphi \\ &= \left((n+2w)(w+1) + (n+2w)n + n\right) \varphi \\ &= (n+2w+2)(w+n)\varphi \end{aligned}$$

completing the required calculations. $\square$

Chapter 5

Tractors on Sub-manifolds

A sub-manifold $\Sigma \subset M$ obtains a conformal structure $(\Sigma, \bar{\mathbf{g}})$ from $(M, \mathbf{g})$ as in section 3.4, and the tractor bundle of $(\Sigma, \bar{\mathbf{g}})$ will be referred to as the intrinsic tractor bundle. This chapter will link the tractors intrinsic to $(\Sigma, \bar{\mathbf{g}})$ to ambient tractors on $(M, \mathbf{g})$ for ambient dimensions $n \geq 4$.

5.1 Decomposing Ambient Tractors

Given a particular metric $\bar{g}$, the tractor bundle for the conformal sub-manifold $(\Sigma, \bar{\mathbf{g}})$ decomposes as sections of $\mathcal{E}[1] \oplus \mathcal{E}^a_\Sigma[-1] \oplus \mathcal{E}[-1]$. Branson & Gover [3] show that the ambient tractor bundle along Σ decomposes as $\mathcal{E}^A\,|_\Sigma = \mathcal{E}^A_\Sigma \oplus \mathcal{N}^A$. Here we provide an explicit form for the embedding $\mathcal{E}^A_\Sigma \hookrightarrow \mathcal{E}^A\,|_\Sigma$ and a parameterisation of the "normal" component $\mathcal{N}^A$.

Definition 5.1. [2] *The* **normal tractor** N^A *is defined as*

$$\left[N^A\right]_g = \begin{pmatrix} 0 \\ N^a \\ -H \end{pmatrix}$$

where Σ is a co-dimension 1 sub-manifold with unit normal vector field $N^a \in \mathcal{E}^a[1]$ and mean curvature $H \in \mathcal{E}[-1]$.

It is easy to show that the normal tractor is conformally invariant and has unit length. Using the tractor metric h_{AB} we obtain a decomposition $\mathcal{E}^A\,|_\Sigma = \mathcal{E}^A_\parallel \oplus \mathrm{span}(N^A)$ along Σ

with the condition that $V^A \in \mathcal{E}_{\|}^A$ iff $V^A N_A = 0$. The following proposition verifies that $\mathcal{E}_{\Sigma}^A \cong \mathcal{E}_{\|}^A$ and that $\mathcal{N}^A = \mathrm{span}(N^A)$.

Proposition 5.2. *Let g and $\bar{g}$ be choices of conformal scale on $(M, \mathbf{g})$ and $(\Sigma, \bar{\mathbf{g}})$ such that $\bar{g} = g|_\Sigma$ along Σ. The mapping $\Cap$ defined by*

$$\left[U^A\right]_{\bar{g}} = \begin{pmatrix} \sigma \\ \nu^a \\ \tau \end{pmatrix} \quad \mapsto \quad \Cap \left[U^A\right]_{\bar{g}} = \begin{pmatrix} \sigma \\ \Pi^a_b \nu^b + H N^a \sigma \\ \tau - \frac{1}{2} H^2 \sigma \end{pmatrix}$$

for $U^A \in \mathcal{E}_{\Sigma}^A$ is a well defined tractor isomorphism from $\mathcal{E}_{\Sigma}^A$ to $\mathcal{E}_{\|}^A$.

Proof. First it will be shown that $\Cap$ is an isomorphism between the tensor product spaces $\mathcal{E}_{\Sigma}^A = \mathcal{E}[1] \oplus \mathcal{E}_{\Sigma}^a[-1] \oplus \mathcal{E}[-1]$ and $\mathcal{E}_{\|}^A$ viewed as a subspace of $\mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$, i.e., that $\Cap$ is an isomorphism of the spaces $\mathcal{E}_{\Sigma}^A$ and $\mathcal{E}_{\|}^A$ with respect to the Riemannian metrics g and $\bar{g} = g|_\Sigma$ along Σ. Simple calculation yields

$$N_A \Cap U^A \;\; = \;\; N_a \left(\Pi^a_b \nu^b + H N^a \sigma\right) - H\sigma = 0$$

so indeed $\Cap : \mathcal{E}_{\Sigma}^A \mapsto \mathcal{E}_{\|}^A$. The mapping $\Cap$ is clearly injective; since $\mathcal{E}_{\Sigma}^A$ and $\mathcal{E}_{\|}^A$ are both of dimension $n+1$ and $\Cap$ is linear, $\Cap$ is also surjective and hence an isomorphism between $\mathcal{E}[1] \oplus \mathcal{E}_{\Sigma}^a[-1] \oplus \mathcal{E}[-1]$ and the image of $\Cap$ in $\mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$. Finally we show that $\Cap$ transforms correctly so that it is well defined as an operator on tractors. Conformally rescaling $\Cap[U^A]_g$ gives

$$\Xi_{g \to \hat{g}} \Cap \left[U^A\right]_{\bar{g}} = \begin{pmatrix} \sigma \\ \Pi^a_b \nu^b + H N^a \sigma + \Upsilon^a \sigma \\ \tau - \frac{1}{2} H^2 \sigma - \Upsilon_b \Pi^b_c \nu^c - H \Upsilon \cdot N \sigma - \frac{1}{2} \Upsilon \cdot \Upsilon \sigma \end{pmatrix}$$

Alternatively, rescaling U^A on $(\Sigma, \bar{\mathbf{g}})$ gives

$$\left[U^A\right]_{\hat{\bar{g}}} = \begin{pmatrix} \sigma \\ \nu^a + \overline{\Upsilon}^a \sigma \\ \tau - \overline{\Upsilon} \cdot \nu - \frac{1}{2} \overline{\Upsilon} \cdot \overline{\Upsilon} \sigma \end{pmatrix}$$

which $\Cap$ maps to

$$\Cap \left[U^A\right]_{\hat{g}} = \begin{pmatrix} \sigma \\ \Pi^a_b \left(\nu^a + \overline{\Upsilon}^a \sigma\right) + \widehat{H} N^a \sigma \\ \tau - \overline{\Upsilon} \cdot \nu - \frac{1}{2} \overline{\Upsilon} \cdot \overline{\Upsilon} \sigma - \frac{1}{2} \widehat{H}^2 \sigma \end{pmatrix}$$

It remains to show that the second and third components agree in the expressions for $\Cap \left[U^A\right]_{\hat{g}}$ and $\Xi_{g \to \hat{g}} \Cap \left[U^A\right]_{\bar{g}}$ above. Since $\widehat{H} = H + \Upsilon \cdot N$ and $\Upsilon^a = \overline{\Upsilon}^a + \Upsilon \cdot N N^a$,

$$\begin{aligned} \Pi^a_b \left(\nu^b + \overline{\Upsilon}^b \sigma\right) + \widehat{H} N^a \sigma &= \nu^a + \overline{\Upsilon}^a \sigma + H N^a \sigma + \Upsilon \cdot N N^a \sigma \\ &= \nu^a + H N^a \sigma + \Upsilon^a \sigma \end{aligned}$$

demonstrating the equality of the second components. For the third component we note $\overline{\Upsilon} \cdot \nu = \Upsilon_b \Pi^b_c \nu^c$ and then

$$\begin{aligned} \frac{1}{2} \overline{\Upsilon} \cdot \overline{\Upsilon} \sigma + \frac{1}{2} \widehat{H}^2 \sigma &= \frac{1}{2} \overline{\Upsilon} \cdot \overline{\Upsilon} \sigma + \frac{1}{2} H^2 \sigma + H \Upsilon \cdot N \sigma + \frac{1}{2} (\Upsilon \cdot N)^2 \sigma \\ &= \frac{1}{2} \Upsilon \cdot \Upsilon \sigma + \frac{1}{2} H^2 \sigma + H \Upsilon \cdot N \sigma \end{aligned}$$

completes the proof of equality. □

The tractor bundle of a conformal manifold $(M, \mathbf{g})$ may also be dealt with using the injecting operators Y^A, Z^A_a and X^A. The sub-manifold $(\Sigma, \bar{\mathbf{g}})$ will have separate injecting operators $\overline{Y}^A$, $\overline{Z}^A_a$ and $\overline{X}^A$ which can also be used to describe $\Cap : \mathcal{E}^A_\Sigma \hookrightarrow \mathcal{E}^A|_\Sigma$ as detailed in the following proposition;

Corollary 5.3. *The injecting operators Y^A, Z^A_a and X^A of the manifold $(M, \mathbf{g})$ are related to the injecting operators $\overline{Y}^A$, $\overline{Z}^A_a$ and $\overline{X}^A$ of the sub-manifold $(\Sigma, \bar{\mathbf{g}})$ by:*

$$\begin{aligned} \overline{Y}^A &= Y^A + Z^A_a N^a H - \frac{1}{2} H^2 X^A \\ \overline{Z}^A_a &= \Pi^b_a Z^A_b \\ \overline{X}^A &= X^A \end{aligned}$$

Proof. The result follows simply from rewriting proposition 5.2 in terms of the injecting operators. □

We will also need to move in the other direction:

Corollary 5.4. *Suppose $V^A = \sigma Y^A + \mu^a Z_a^A + \rho X^A \in \mathcal{E}^A$ such that $V^A N_A = 0$, i.e., $V^A \in \mathcal{E}_{\|}^A$. The isomorphism $\Cap^{-1} : \mathcal{E}^A|_\Sigma \mapsto \mathcal{E}_\Sigma^A$ identifies $V^A \in \mathcal{E}_{\|}^A$ with the tractor $U^A \in \mathcal{E}_\Sigma^A$ described using the intrinsic metric $\bar{g}$ as*

$$\left[U^A\right]_{\bar{g}} = \begin{pmatrix} \sigma \\ \Pi_b^a \mu^b \\ \rho + \frac{1}{2}H^2\sigma \end{pmatrix}$$

Proof. Using proposition 5.2 the tractor U^A is split by the ambient metric g as

$$\left[U^A\right]_g = \begin{pmatrix} \sigma \\ \Pi_b^a \mu^b + HN^a\sigma \\ \rho + \frac{1}{2}H^2\sigma - \frac{1}{2}H^2\sigma \end{pmatrix}$$

Immediately σ and ρ are recovered for the first and third slots respectively. Further calculation yields

$$\Pi_b^a \mu^b + HN^a\sigma = \mu^a - N^a\left(N\cdot\mu - H\sigma\right)$$

By hypothesis $N\cdot\mu - H\sigma$ vanishes, so μ^a is recovered for the second slot. $\square$

Corollary 5.3 may be rearranged to express $\Cap^{-1}$ in terms of the injecting operators of $(M, \mathbf{g})$ and $(\Sigma, \bar{\mathbf{g}})$.

In future the isomorphism $\Cap$ will not be explicitly mentioned, with $\mathcal{E}_{\|}^A$ and $\mathcal{E}_\Sigma^A$ being identified as the same tractor bundle. The metric subscript will indicate whether a tractor is expressed with ambient injecting operators of $(M, \mathbf{g})$ as in $[V^A]_g$, or else as with injecting operators intrinsic to $(\Sigma, \bar{\mathbf{g}})$ as in $[V^A]_{\bar{g}}$.

5.2 Sub-Manifold Tractor Connections

A conformal sub-manifold $(\Sigma, \bar{\mathbf{g}})$ has two natural but distinct tractor connections. There is the intrinsic tractor connection $\overline{\nabla}$ which is provided by definition 4.6, treating $(\Sigma, \bar{\mathbf{g}})$ as a conformal manifold independent of $(M, \mathbf{g})$. On $(\Sigma, \bar{\mathbf{g}})$ the connection $\overline{\nabla}$ takes the form

$$\left[\overline{\nabla}_c U^A\right]_{\bar{g}} = \begin{pmatrix} \overline{\nabla}_c\sigma - \nu_c \\ \overline{\nabla}_c\nu^a + \Pi_c^a\tau + \overline{P}_c^a\sigma \\ \overline{\nabla}_c\tau - \overline{P}_{cb}\nu^b \end{pmatrix}$$

for $U^A \in \mathcal{E}^A_\Sigma$. The presence of $\overline{P}^a_c$ indicates that the assumption $n \geq 4$ is required, although an extension to $n = 3$ is possible.

Proposition 5.5. *Let $V^A = Y^A\sigma + Z^A_a\mu^a + X^A\rho \in \mathcal{E}^A$ with $V^A N_A = 0$. The action of the intrinsic connection $\overline{\nabla}_c V^A$ is described using ambient injecting operators as*

$$\left[\overline{\nabla}_c V^A\right]_g = \begin{pmatrix} \overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b \\ \overline{\nabla}_c \Pi^a_b\mu^b + \Pi^a_c(\rho + \frac{1}{2}H^2\sigma) + \overline{P}^a_c\sigma + HN^a(\overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b) \\ \overline{\nabla}_c\rho + \sigma H\overline{\nabla}_c H - \overline{P}_{bc}\mu^b + \frac{1}{2}H^2\bar{\mathbf{g}}_{bc}\mu^b \end{pmatrix}$$

Proof. Corollary 5.4 indicates that

$$\left[V^A\right]_{\bar{g}} = \begin{pmatrix} \sigma \\ \Pi^a_b\mu^b \\ \rho + \frac{1}{2}H^2\sigma \end{pmatrix}$$

Application of the intrinsic connection $\overline{\nabla}$ leads to

$$\left[\overline{\nabla}_c V^A\right]_{\bar{g}} = \begin{pmatrix} \overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b \\ \overline{\nabla}_c \Pi^a_b\mu^b + \Pi^a_c(\rho + \frac{1}{2}H^2\sigma) + \overline{P}^a_c\sigma \\ \overline{\nabla}_c(\rho + \frac{1}{2}\sigma H^2) - \overline{P}_{bc}\mu^b \end{pmatrix}$$

The ambient expression for this is obtained using proposition 5.2

$$\left[\overline{\nabla}_c V^A\right]_g = \begin{pmatrix} \overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b \\ \overline{\nabla}_c \Pi^a_b\mu^b + \Pi^a_c(\rho + \frac{1}{2}H^2\sigma) + \overline{P}^a_c\sigma + HN^a(\overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b) \\ \overline{\nabla}_c(\rho + \frac{1}{2}\sigma H^2) - \overline{P}_{bc}\mu^b - \frac{1}{2}H^2(\overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{bc}\mu^b) \end{pmatrix}$$

which is equivalent to the result. $\square$

In addition to $\overline{\nabla}$, there is also a connection on $(\Sigma, \bar{\mathbf{g}})$ which is obtained from the ambient connection ∇ on $(M, \mathbf{g})$ via the embedding $\Sigma \subset M$;

Definition 5.6. *The* **projected ambient tractor connection** *is the connection $\widetilde{\nabla}$ on $(\Sigma, \bar{\mathbf{g}})$ given by*

$$\widetilde{\nabla}_c V^A = \Pi^A_B \Pi^{c'}_c \nabla_{c'} V^B$$

where $V^A \in \mathcal{E}^A_\Sigma$.

The connection $\widetilde{\nabla}$ leads to a tractor analogue of the Riemannian second fundamental form.

Definition 5.7. *The* **projective tractor second fundamental form** *is denoted M and is defined as taking the action*

$$M_{cA}V^A \quad = \quad N_A \Pi_c^{c'} \nabla_{c'} V^A$$

where $V^A \in \mathcal{E}_\Sigma^A$ along Σ.

With this definition the projected ambient tractor connection takes the form

$$\widetilde{\nabla}_c V^A = \Pi_c^{c'} \nabla_{c'} V^A - N^A M_{cB} V^B \tag{5.1}$$

The projective tractor second fundamental form M_{cA} inherits from its constituents conformal invariance when acting on objects V^A with arbitrary tractor indices but without tensor indices. The projective tractor second fundamental form is more complicated than the Riemannian counterpart due to the non-trivial relation between the intrinsic tractor connection and the projected ambient tractor connection.

Proposition 5.8. *The projective tractor second fundamental form is explicitly*

$$M_{cA} \quad = \quad X_A(\Pi_c^{c'} \nabla_{c'} H + N_a \Pi_c^{c'} P_{c'}^a) - Z_A^{c'} L_{(cc')^0}$$

Proof. Let $V^A = \sigma Y^A + \mu^a Z_a^A + \rho X^A \in \mathcal{E}_\Sigma^A$, whereby $\mu \cdot N - \sigma H = 0$. Using the ambient connection ∇ gives

$$\left[\Pi_c^{c'} \nabla_{c'} V^A\right]_g = \begin{pmatrix} \Pi_c^{c'} (\nabla_{c'} \sigma - \mu_{c'}) \\ \Pi_c^{c'} (\nabla_{c'} \mu^a + \delta_{c'}^a \rho + P_{c'}^a \sigma) \\ \Pi_c^{c'} \left(\nabla_{c'} \rho - P_{bc'} \mu^b\right) \end{pmatrix}$$

Calculating the inner product $N_A \Pi_c^{c'} \nabla_{c'} V^A$ gives

$$\begin{aligned} N_B \Pi_c^{c'} \nabla_{c'} V^B \quad &= \quad -H \Pi_c^{c'} \nabla_{c'} \sigma + H \Pi_c^{c'} \mu_{c'} + N_a \Pi_c^{c'} \nabla_{c'} \mu^a + N_a P_{c'}^a \Pi_c^{c'} \sigma \\ &= \quad -\Pi_c^{c'} \nabla_{c'} \sigma H + \sigma \Pi_c^{c'} \nabla_{c'} H + H \Pi_c^{c'} \mu_{c'} + \Pi_c^{c'} \nabla_{c'} \mu \cdot N \\ &\qquad -\mu^a \Pi_c^{c'} \nabla_{c'} N_a + N_a P_{c'}^a \Pi_c^{c'} \sigma \\ &= \quad \sigma(\Pi_c^{c'} \nabla_{c'} H + N_a P_{c'}^a \Pi_c^{c'}) - \mu^{c'} L_{(cc')^0} \end{aligned}$$

where the hypotheses $V^A \in \mathcal{E}_\Sigma^A$ causes the term $\Pi_c^{c'} \nabla_{c'}(\mu \cdot N - \sigma H)$ to vanish. Rewriting this in terms of the injecting operators yields the result. $\square$

It is not difficult to verify the conformal invariance of M_{cA} directly using the conformal transformations of the injecting operators with the results of sections 3.3 and 3.4.

Definition 5.9. *The* **tractor contorsion** $C_{cA}{}^{B}$ *is the* $\mathrm{End}(\mathbb{T})$ *valued* 1*-form given by*

$$C_{cA}{}^{B}V^{A} \;=\; \overline{\nabla}_c V^B - \widetilde{\nabla}_c V^B$$

where $V^A \in \mathcal{E}^A_\Sigma$.

In the Riemannian setting the Levi-Civita connection $\overline{\nabla}$ of a sub-manifold may be obtained from the ambient ∇ by projections, as $\overline{\nabla}_a\varphi^b = \Pi^a_c\Pi^d_b\nabla_c\varphi^d$. In the tractor setting however, the connections $\overline{\nabla}$ and $\widetilde{\nabla}$ do not agree in general. We will find an explicit expression for $C_{cA}{}^{B}$ to demonstrate this detail.

Proposition 5.10. *The tractor contorsion is of the form*

$$\begin{aligned} C_{cA}{}^{B} \;&=\; Z^B_b X_A\left(\overline{P}^b_c - \Pi^b_{b'}\Pi^{c'}_c P^{b'}_{c'} - HL_{(cb')^0}\bar{\mathbf{g}}^{bb'} - \frac{1}{2}\Pi^b_c H^2\right)\\ &\quad - X^B Z^a_A \bar{\mathbf{g}}_{ab}\left(\overline{P}^b_c - \Pi^b_{b'}\Pi^{c'}_c P^{b'}_{c'} - HL_{(cb')^0}\bar{\mathbf{g}}^{bb'} - \frac{1}{2}\Pi^b_c H^2\right)\end{aligned}$$

Proof. Let $V^A = \sigma Y^A + \mu^a Z^A_a + \rho X^A \in \mathcal{E}^A$ with $\mu{\cdot}N - \sigma H = 0$. The tractor contorsion may be expressed as

$$C_{cA}{}^{B}V^A = \overline{\nabla}_c V^B - \Pi^{c'}_c\nabla_{c'}V^B + N^B M_{cA}V^A$$

Explicit forms of $\left[\overline{\nabla}_c V^B\right]_g$ and $\left[M_{cA}V^A\right]_g$ are obtained from propositions 5.5 and 5.8 respectively. Combining these terms with $\left[\Pi^{c'}_c\nabla_{c'}V^B\right]_g$ gives

$$\begin{aligned}\left[C_{cA}{}^{B}V^A\right]_g \;&=\; \begin{pmatrix} \overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b \\ \overline{\nabla}_c\Pi^a_b\mu^b + \Pi^a_c(\rho + \frac{1}{2}H^2\sigma) + \overline{P}^a_c\sigma + HN^a(\overline{\nabla}_c\sigma - \bar{\mathbf{g}}_{cb}\mu^b) \\ \overline{\nabla}_c\rho + \sigma H\overline{\nabla}_c H - \overline{P}_{bc}\mu^b + \frac{1}{2}H^2\bar{\mathbf{g}}_{bc}\mu^b \end{pmatrix}\\ &\quad - \begin{pmatrix} \Pi^{c'}_c\nabla_{c'}\sigma - \Pi^{c'}_c\mu_{c'} \\ \Pi^{c'}_c\nabla_{c'}\mu^a + \Pi^a_c\rho + \Pi^{c'}_c P^a_{c'}\sigma \\ \Pi^{c'}_c\nabla_{c'}\rho - \Pi^{c'}_c P_{bc'}\mu^b \end{pmatrix}\\ &\quad + \left(\sigma(\Pi^{c'}_c\nabla_{c'}H + N_b\Pi^{c'}_c P^b_{c'}) - \mu^{c'}L_{(cc')^0}\right)\begin{pmatrix} 0 \\ N^a \\ -H \end{pmatrix}\end{aligned}$$

Clearly the primary component of $\left[C_{cA}{}^{B}V^{A}\right]_g$ vanishes since $\overline{\nabla}_c\varphi = \Pi_c^{c'}\nabla_{c'}\varphi$ when φ is a (conformally weighted) function. The secondary slot is

$$\begin{aligned} Z_B^a C_{cA}{}^B V^A &= \overline{\nabla}_c \Pi_b^a \mu^b + \Pi_c^a \frac{1}{2} H^2 \sigma + \overline{P}_c^a \sigma + H N^a (\overline{\nabla}_c \sigma - \bar{\mathbf{g}}_{cb}\mu^b) - \Pi_c^{c'} \nabla_{c'} \mu^a \\ &\quad - \Pi_c^{c'} P_{c'}^a \sigma + \sigma N^a (\Pi_c^{c'} \nabla_{c'} H + N_b \Pi_c^{c'} P_{c'}^b) - N^a \mu^{c'} L_{(cc')^0} \end{aligned}$$

We note that

$$-\Pi_c^{c'} P_{c'}^a + N^a N_b \Pi_c^{c'} P_{c'}^b = -\Pi_{a'}^a \Pi_c^{c'} P_{c'}^{a'}$$

and since $\mu \cdot N = \sigma H$,

$$\begin{aligned} &\overline{\nabla}_c \Pi_b^a \mu^b + H N^a \overline{\nabla}_c \sigma - \Pi_c^{c'} \nabla_{c'} \mu^a + \sigma N^a \Pi_c^{c'} \nabla_{c'} H \\ &\qquad = \Pi_c^{c'} \nabla_{c'} \Pi_b^a \mu^b + L_{cb} \mu^b N^a + N^a \Pi_c^{c'} \nabla_{c'} \mu \cdot N - \Pi_c^{c'} \nabla_c \mu^a \\ &\qquad = -\Pi_c^{c'} \nabla_{c'} N^a \mu \cdot N + L_{cb} \mu^b N^a + N^a \Pi_c^{c'} \nabla_{c'} \mu \cdot N \\ &\qquad = -\mu \cdot N L_{cb} \bar{\mathbf{g}}^{ab} + L_{cb} \mu^b N^a \\ &\qquad = -\sigma H L_{cb} \bar{\mathbf{g}}^{ab} + L_{cb} \mu^b N^a \end{aligned}$$

These results give

$$\begin{aligned} Z_B^a C_{cA}{}^B V^A &= \left(\overline{P}_c^a - \Pi_{a'}^a \Pi_c^{c'} P_{c'}^{a'}\right) \sigma - H L_{cb} \bar{\mathbf{g}}^{ab} \sigma + \frac{1}{2} \Pi_c^a H^2 \sigma \\ &= \left(\overline{P}_c^a - \Pi_{a'}^a \Pi_c^{c'} P_{c'}^{a'}\right) \sigma - H L_{(cb)^0} \bar{\mathbf{g}}^{ab} \sigma - \frac{1}{2} \Pi_c^a H^2 \sigma \end{aligned}$$

For the third component of $\left[C_{cA}{}^{B}V^{A}\right]_g$ we calculate:

$$\begin{aligned} Y_B C_{cA}{}^B V^A &= \frac{1}{2} \overline{\nabla}_c H^2 \sigma - \overline{P}_{bc} \mu^b - \frac{1}{2} H^2 \left(\overline{\nabla}_c \sigma - \bar{\mathbf{g}}_{cb} \mu^b\right) + \Pi_c^{c'} P_{bc'} \mu^b \\ &\quad + H \mu^b L_{(cb)^0} - \sigma H \Pi_c^{c'} \nabla_{c'} H - \sigma H N_b \Pi_c^{c'} P_{c'}^b \end{aligned}$$

Here we again use $\mu \cdot N - \sigma H = 0$ to obtain

$$\begin{aligned} \Pi_c^{c'} P_{bc'} \mu^b - H \sigma N_b \Pi_c^{c'} P_{c'}^b &= \Pi_c^{c'} P_{bc'} (\mu^b - \mu \cdot N N^b) \\ &= \Pi_c^{c'} \Pi_b^{b'} P_{b'c'} \mu^b \end{aligned}$$

This leads to

$$Y_B C_{cA}{}^B V^A = \left(\Pi_c^{c'} \Pi_b^{b'} P_{b'c'} - \overline{P}_{bc}\right) \mu^b + \frac{1}{2} H^2 \bar{\mathbf{g}}_{cb} \mu^b + H \mu^b L_{(cb)^0}$$

Combining these results gives an expression for the tractor contorsion.

$$\left[C_{cA}{}^{B}V^{A}\right]_g = \begin{pmatrix} 0 \\ \left(\overline{P}^a_c - \Pi^a_{a'}\Pi^{c'}_c P^{a'}_{c'}\right)\sigma - HL_{(cb)^0}\bar{\mathbf{g}}^{ab}\sigma - \frac{1}{2}\Pi^a_c H^2\sigma \\ \left(\Pi^{c'}_c\Pi^{b'}_b P_{b'c'} - \overline{P}_{bc}\right)\mu^b + H\mu^b L_{(cb)^0} + \frac{1}{2}H^2\bar{\mathbf{g}}_{cb}\mu^b \end{pmatrix}$$

or in terms of the injecting operators,

$$\begin{aligned} C_{cA}{}^{B} &= Z^B_b X_A \bar{\mathbf{g}}^{ab}\left(\overline{P}_{ac} - \Pi^{a'}_a\Pi^{c'}_c P_{a'c'} - HL_{(ca)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}H^2\right) \\ &\quad - X^B Z^a_A\left(\overline{P}_{ac} - \Pi^{a'}_a\Pi^{c'}_c P_{a'c'} - HL_{(ca)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}H^2\right) \end{aligned}$$

which is the required result. □

Proposition 3.9 demonstrates that the tensor

$$\overline{P}_{ac} - \Pi^{a'}_a\Pi^{c'}_c P_{a'c'} - HL_{(ca)^0} - \frac{1}{2}\bar{\mathbf{g}}_{ac}H^2$$

is conformally invariant. It is then simple to see that the above expression for $C_{cA}{}^{B}$ is indeed a conformally invariant tractor.

Burstall and Calderbank [6] propose a decomposition of the ambient connection into parts normal and tangential to the sub-manifold. The results of this section detail the tangential component of such a decomposition. Specifically, the sub-manifold connection may be decomposed into

$$\overline{\nabla}_c V^A = \Pi^{c'}_c \nabla_{c'} V^A - N^A M_{cB} V^B + C_{cB}{}^{A} V^B \qquad (5.2)$$

along Σ, where each of the terms on the right hand side are discussed above.

5.3 The Ambient Tractor Second Fundamental Form

The Riemannian second fundamental form $L_{ab} = \Pi^c_a \nabla_c N_b$ is a bilinear form measuring the difference between the intrinsic connection $\overline{\nabla}$ and the projected ambient connection $\Pi_\Sigma \nabla$. The tractor setting is not as thoroughly understood and there are several candidate objects that might be called a "tractor second fundamental form". The direct tractor analogue of L_{ab} is the projective second fundamental form M_{cA}, but an altogether different object has been introduced by Grant [5] which is not yet easily related to the connonical connections $\overline{\nabla}$ and $\widetilde{\nabla}$ on $(\Sigma, \bar{g})$.

Definition 5.11. *The* **ambient tractor second fundamental form** *is defined, for a choice of metric $\bar{g}$, in terms of the intrinsic injecting operators as*

$$\begin{aligned} L_{AB} &= (n-3)\overline{Z}_A^a\overline{Z}_B^b L_{(ab)^0} - \frac{n-3}{n-2}\overline{Z}_A^a\overline{X}_B\overline{\nabla}^b L_{(ab)^0} - \frac{n-3}{n-2}\overline{Z}_B^b\overline{X}_A\overline{\nabla}^a L_{(ab)^0} \\ &\quad +\overline{X}_A\overline{X}_B(\overline{P}^{ab}L_{(ab)^0} + \frac{1}{n-2}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}) \end{aligned}$$

where $L_{(ab)^0}$ is the trace-free Riemannian second fundamental form.

Despite the overloaded notation, there will be no confusion as to which second fundamental form is intended due to the case of the indices. The ambient tractor second fundamental form is clearly symmetric; but is not obviously conformally invariant.

Proposition 5.12. *The ambient tractor second fundamental L_{AB} form is conformally invariant in ambient dimensions $n \geq 4$.*

Proof. Equation 3.3 and proposition 3.3 are combined using the Leibniz rule to derive that for a symmetric $\mu_{ab} \in \mathcal{E}_{ab}[w]$,

$$\begin{aligned} \mathbf{g}^{bc}\widehat{\nabla}_c\mu_{ab} &= \mathbf{g}^{bc}\Big[\nabla_c\mu_{ab} + (w-2)\Upsilon_c\mu_{ab} - \Upsilon_b\mu_{ac} - \Upsilon_a\mu_{cb} + \mathbf{g}_{bc}\Upsilon^d\mu_{ad} + \mathbf{g}_{ac}\Upsilon^d\mu_{db}\Big] \\ &= \nabla^b\mu_{ab} + (n+w-2)\Upsilon^b\mu_{ab} - \mathbf{g}^{cb}\Upsilon_a\mu_{cb} \end{aligned}$$

By proposition 3.8, $L_{(ab)^0}$ is conformally invariant. Since $L_{(ab)^0}$ has conformal weight $w = 1$, the term $\overline{\nabla}^b L_{(ab)^0}$ will under conformal rescalings of the metric transform according to a Σ version of the above formula, as

$$\widehat{\overline{\nabla}}^b L_{(ab)^0} = \overline{\nabla}^b L_{(ab)^0} + (n-2)\overline{\Upsilon}^b L_{(ab)^0}$$

Recalling that for weighted one-forms $u_a \in \mathcal{E}_a[w]$,

$$\begin{aligned} \mathbf{g}^{ac}\nabla_c u_a &= \mathbf{g}^{ac}\left(\nabla_c u_a - \Upsilon_a u_c + (w-1)\Upsilon_c u_a + \mathbf{g}_{ac}\Upsilon^a u_a\right) \\ &= \nabla^a u_a + (n+w-2)\Upsilon^a u_a \end{aligned}$$

Using a $(\Sigma, \bar{\mathbf{g}})$ version of this where $\overline{\nabla}^b L_{(ab)^0}$ has conformal weight -1 yields

$$\begin{aligned} \widehat{\overline{\nabla}}^a\widehat{\overline{\nabla}}^b L_{(ab)^0} &= \widehat{\overline{\nabla}}^a\left(\overline{\nabla}^b L_{(ab)^0} + (n-2)\overline{\Upsilon}^b L_{(ab)^0}\right) \\ &= \overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0} + (n-2)\overline{\nabla}^a\overline{\Upsilon}^b L_{(ab)^0} + (n-4)\overline{\Upsilon}^a\overline{\nabla}^b L_{(ab)^0} \\ &\quad +(n-4)(n-2)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} \\ &= \overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0} + (n-2)(\overline{\nabla}^a\overline{\Upsilon}^b)L_{(ab)^0} + (2n-6)\overline{\Upsilon}^a\overline{\nabla}^b L_{(ab)^0} \\ &\quad +(n-4)(n-2)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} \end{aligned}$$

Equation 3.5 indicates that

$$\widehat{P}_{ab} \;=\; P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\mathbf{g}_{ab}\Upsilon\cdot\Upsilon$$

Use of the Σ version of this rule for $n \geq 4$ shows that the term $(\overline{P}^{ab} L_{(ab)^0} + \frac{1}{n-2}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0})$ transforms according to

$$\begin{aligned}
&\widehat{\overline{P}}^{ab} L_{(ab)^0} + \frac{1}{n-2}\widehat{\overline{\nabla}}^a\widehat{\overline{\nabla}}^b L_{(ab)^0} \\
&\qquad = \left(\overline{P}_{ab} - (\overline{\nabla}_a\overline{\Upsilon}_b) + \overline{\Upsilon}_a\overline{\Upsilon}_b\right) L_{(ab)^0} + \tfrac{1}{n-2}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0} + (\overline{\nabla}^a\overline{\Upsilon}^b) L_{(ab)^0} \\
&\qquad\quad + \tfrac{2n-6}{n-2}\overline{\Upsilon}^a\overline{\nabla}^b L_{(ab)^0} + (n-4)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} \\
&\qquad = \overline{P}^{ab} L_{(ab)^0} + \tfrac{1}{n-2}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0} + \tfrac{2n-6}{n-2}\overline{\Upsilon}^a\overline{\nabla}^b L_{(ab)^0} + (n-3)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0}
\end{aligned}$$

The injecting operators transform as

$$\begin{aligned}
\widehat{\overline{X}}_A &= \overline{X}^A \\
\widehat{\overline{Z}}^a_A &= \overline{Z}^A_a + \overline{\Upsilon}_a \overline{X}^A
\end{aligned}$$

Having established the transformations of all the components, we can verify the conformal invariance of L_{AB}.

$$\begin{aligned}
\widehat{L}_{AB} \;=\; & (n-3)\left(\overline{Z}^a_A\overline{Z}^b_B L_{(ab)^0}\right) + \overline{Z}^a_A\overline{X}_B\left(-\frac{n-3}{n-2}\widehat{\overline{\nabla}}^b L_{(ab)^0} + (n-3)\overline{\Upsilon}^b L_{(ab)^0}\right) \\
& + \overline{Z}^b_B\overline{X}_A\left(-\frac{n-3}{n-2}\widehat{\overline{\nabla}}^a L_{(ab)^0} + (n-3)\overline{\Upsilon}^a L_{(ab)^0}\right) + \overline{X}_A\overline{X}_B\Big(\widehat{\overline{P}}^{ab} L_{(ab)^0} \\
& + \frac{1}{n-2}\widehat{\overline{\nabla}}^a\widehat{\overline{\nabla}}^b L_{(ab)^0} + (n-3)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} - \frac{2n-6}{n-2}\overline{\Upsilon}^a\widehat{\overline{\nabla}}^b L_{(ab)^0}\Big)
\end{aligned}$$

Clearly the off diagonal coefficients display the correct transformation, while the $\overline{Z}^a_A\overline{Z}^b_B$ term is conformally invariant. The $\overline{X}_A\overline{X}_B$ component is

$$\begin{aligned}
&\widehat{\overline{P}}^{ab} L_{(ab)^0} + \frac{1}{n-2}\widehat{\overline{\nabla}}^a\widehat{\overline{\nabla}}^b L_{(ab)^0} + (n-3)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} - \frac{2n-6}{n-2}\overline{\Upsilon}^a\widehat{\overline{\nabla}}^b L_{(ab)^0} \\
&\qquad = \overline{P}^{ab} L_{(ab)^0} + \tfrac{1}{n-2}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0} + \tfrac{2n-6}{n-2}\overline{\Upsilon}^a\overline{\nabla}^b L_{(ab)^0} + (n-3)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} \\
&\qquad\quad + (n-3)\overline{\Upsilon}^a\overline{\Upsilon}^b L_{(ab)^0} - \tfrac{2n-6}{n-2}\overline{\Upsilon}^a\left(\overline{\nabla}^b L_{(ab)^0} + (n-2)\overline{\Upsilon}^b L_{(ab)^0}\right) \\
&\qquad = \overline{P}^{ab} L_{(ab)^0} + \tfrac{1}{n-2}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}
\end{aligned}$$

Hence L_{AB} transforms correctly under conformal rescalings of the metric. □

The ambient second fundamental form plays an important role in obtaining third order conformally invariant differential operators, but is presently not well motivated. It has been conjectured [10] that the ambient second fundamental form might be recovered via the ambient construction of the tractor bundle.

Chapter 6

Sub-manifold Tractor Operators

Here we extend the results of Grant [5] on higher order conformally invariant operators to operators that are invariant on any conformal weight. The author understands that Andreas Juhl is also pursuing work in this general direction [1, 10], but differences in notation make comparison awkward.

Descriptions of conformally invariant operators require increasingly vast numbers of terms as the normal order increases. Work will be broken into a number of smaller more manageable lemmas.

Lemma 6.1. *Let* $V^A = Y^A\sigma + Z^A_a\mu^a + X^A\rho \in \mathcal{E}^A[w]$. *Then*

$$\begin{aligned}\overline{D}_B\Pi^B_A V^A &= (n+w-2)\overline{J}\sigma - \overline{\Delta}\sigma + (n+2w-1)\overline{\nabla}_a\Pi^a_b\mu^b \\ &\quad +(n+2w-1)(n+w-2)\left(\rho + H\mu\cdot N - \frac{1}{2}H^2\sigma\right)\end{aligned}$$

Proof. By lemma 5.3, the projection $\Pi^B_A V^A$ appears with respect to the intrinsic metric $\bar{g}$ as

$$\Pi^B_A V^A = \overline{Y}^B\sigma + \overline{Z}^B_a\Pi^a_b\mu^b + \overline{X}^B\left(\rho + H\mu\cdot N - \frac{1}{2}H^2\sigma\right)$$

Since V^A has conformal weight w, σ and μ^a have weight $w+1$ while ρ has conformal weight $w-1$. Careful application of lemma 4.10 yields

$$\begin{aligned}\overline{D}_B\overline{Y}^B\sigma &= \big((n-1)+(w+1)-2\big)\overline{J}\sigma - \overline{\Delta}\sigma \\ \overline{D}_B\overline{Z}^B_a\Pi^a_b\mu^b &= \big((n-1)+2(w+1)-2\big)\overline{\nabla}_a\Pi^a_b\mu^b \\ \overline{D}_B\overline{X}^B\tau &= \big((n-1)+2(w-1)+2\big)\big((n-1)+(w-1)\big)\tau\end{aligned}$$

where $\tau = \left(\rho + H\mu\cdot N - \frac{1}{2}H^2\sigma\right)$. These terms sum to the result. □

Proposition 6.2. *The conformally invariant operator* $\overline{D}_B \Pi_A^B D^A$ *takes the explicit form*

$$\begin{aligned} \overline{D}_B \Pi_A^B D^A \varphi &= (n+w-3)\Big[(n+2w-2)\overline{\Box} - (n+2w-3)\Box \\ &\qquad +(n+2w-3)(n+2w-2)(H\nabla_N - \frac{1}{2}wH^2)\Big]\varphi \end{aligned}$$

where $\varphi \in \mathcal{E}[w]$.

Proof. From the definition, $D^A\varphi$ is the tractor

$$D^A\varphi = \begin{pmatrix} (n+2w-2)w\varphi \\ (n+2w-2)\nabla^a\varphi \\ -\Box\varphi \end{pmatrix}$$

of conformal weight $w-1$. Application of lemma 6.1 yields

$$\begin{aligned} &\overline{D}_B \Pi_A^B D^A \varphi = \\ &\qquad (n+w-3)(n+2w-2)w\overline{J}\varphi - (n+2w-2)w\overline{\Delta}\varphi \\ &\qquad\quad +(n+2w-3)(n+2w-2)\overline{\nabla}_a \Pi_b^a \nabla^b\varphi \\ &\qquad\quad +(n+2w-3)(n+w-3)\left(-\Box\varphi + (n+2w-2)(HN_a\nabla^a\varphi - \tfrac{1}{2}H^2 w\varphi)\right) \end{aligned}$$

which simplifies to the result, since $\overline{\nabla}_a \Pi_b^a \nabla^b \varphi = \overline{\Delta}\varphi$ when φ has no indices. $\Box$

The second order operator $\overline{D}_B \Pi_A^B D^A$ was discussed by David Grant [5] in this form. It was also developed independently by Andreas Juhl [1] using different methods and notation. It should also be noted that this operator reduces to the Yamabe operator when acting on the appropriate Yamabe weights. Setting $w = \frac{2-n}{2}$ recovers a multiple of $\Box$, which is conformally invariant on functions of this conformal weight. Substituting instead $w = \frac{3-n}{2}$ produces a multiple of $\overline{\Box}$, the Yamabe operator of the sub-manifold, which is conformally invariant when acting on functions of weight $w = \frac{3-n}{2}$.

6.1 The δ_3 Operator

Definition 6.3. [4] *The* **conformal Robin operator** δ *is*

$$\delta\varphi = \nabla_N\varphi - wH\varphi$$

where $\varphi \in \mathcal{E}^\star[w]$ with $\star$ representing any combination of tractor indices, and ∇_N is the coupled Levi-Civita tractor connection.

It is easy to show that δ is conformally invariant using the known conformal transformations of H and ∇_N acting on tractors.

Lemma 6.4. *Let $\varphi \in \mathcal{E}[w]$. Then the conformally invariant operator δD^A has weight -2 and is explicitly*

$$\left[\delta D^A \varphi\right]_g = \begin{pmatrix} (n+2w-2)(w-1)(\nabla_N - wH)\varphi \\ (n+2w-2)\left(\nabla_N \nabla^a + wP^a_b N^b - (w-1)H\nabla^a\right)\varphi - N^a \Box \varphi \\ -(n+2w-2)N^c P_{cb} \nabla^b \varphi - \nabla_N \Box \varphi + (w-1)H\Box\varphi \end{pmatrix}$$

Proof. Applying the tractor connection ∇_N to $D^A\varphi$ yields

$$\begin{aligned} \left[\nabla_N D^A \varphi\right]_g &= \nabla_N \begin{pmatrix} (n+2w-2)w\varphi \\ (n+2w-2)\nabla_N \varphi \\ -\Box\varphi \end{pmatrix} \\ &= \begin{pmatrix} (n+2w-2)(w-1)\nabla_N \varphi \\ (n+2w-2)\left(\nabla_N \nabla^a + wN^c P^a_c\right)\varphi - N^a \Box\varphi \\ -\nabla_N \Box \varphi - (n+2w-2)N^c P_{cb} \nabla^b \varphi \end{pmatrix} \end{aligned}$$

The action of the Robin operator on $D^A\varphi$ is then

$$\left[\delta D^A \varphi\right]_g = \begin{pmatrix} (n+2w-2)(w-1)(\nabla_N - wH)\varphi \\ (n+2w-2)\left(\nabla_N \nabla^a + wP^a_b N^b - (w-1)H\nabla^a\right)\varphi - N^a \Box \varphi \\ -(n+2w-2)N^c P_{cb} \nabla^b \varphi - \nabla_N \Box \varphi + (w-1)H\Box\varphi \end{pmatrix}$$

since $D^A\varphi$ has conformal weight $w-1$. $\square$

Combining lemmas 6.1 and 6.4 we obtain an explicit expression for the 3th order conformally invariant operator $\overline{D}_B \Pi^B_A \delta D^A$;

Proposition 6.5. *Let $\varphi \in \mathcal{E}[w]$. The conformally invariant operator $\overline{D}_B \Pi_A^B \delta D^A$ has weight -3 and takes the form*

$$
\begin{aligned}
&\overline{D}_B \Pi_A^B \delta D^A \varphi = \\
&\quad (n+w-4)(n+2w-2)(w-1)\overline{J}\,(\nabla_N - wH)\,\varphi \\
&\quad -(n+2w-2)(w-1)\overline{\Delta}\,(\nabla_N - wH)\,\varphi \\
&\quad +(n+2w-2)(n+2w-5)\left(\overline{\nabla}_a \Pi_b^a \nabla_N \nabla^b + w\overline{\nabla}_a \Pi_b^a P_c^b N^c\right)\varphi \\
&\quad -(n+2w-2)(n+2w-5)(w-1)\left(\overline{\nabla}_a H \overline{\nabla}^a\right)\varphi \\
&\quad +(n+w-4)(n+2w-2)(n+2w-5)\left(-P_{cb}N^b\nabla^c + HN^bN^c\nabla_c\nabla_b + wHP^\perp\right)\varphi \\
&\quad +(n+w-4)(n+2w-5)\left(-\nabla_N\Box + (w-2)H\Box\right)\varphi \\
&\quad +(n+w-4)(n+2w-2)(n+2w-5)(w-1)\left(\tfrac{-1}{2}H^2(3\nabla_N - wH)\right)\varphi
\end{aligned}
$$

Proof. Adjusting lemma 6.1 to act on $U^A \in \mathcal{E}^A[w-2]$ yields

$$
\begin{aligned}
\overline{D}_B \Pi_A^B U^A &= (n+w-4)\overline{J}\sigma - \overline{\Delta}\sigma + (n+2w-5)\overline{\nabla}_a \Pi_b^a \mu^b \\
&\qquad +(n+2w-5)(n+w-4)\left(\rho + H\mu\cdot N - \frac{1}{2}H^2\sigma\right)
\end{aligned}
$$

Substituting into this equation the tractor $\delta D^A \varphi$ of lemma 6.4 produces

$$
\begin{aligned}
\overline{D}_B \Pi_A^B \delta D^A \varphi &= \left((n+w-4)\overline{J} - \overline{\Delta}\right)(n+2w-2)(w-1)(\nabla_N - wH)\varphi \\
&\quad +(n+2w-5)\overline{\nabla}_a \Pi_b^a (n+2w-2)\left(\nabla_N \nabla^a + wP_b^a N^b - (w-1)H\nabla^a\right)\varphi \\
&\quad +(n+2w-5)(n+w-4)\times \\
&\qquad \Big[\left((n+2w-2)N^c P_{cb}\nabla^b\varphi - (\nabla_N - (w-1)H)\,\varphi\right) \\
&\qquad +HN_a(n+2w-2)\left(\nabla_N\nabla^a + wP_b^a N^b - (w-1)H\nabla^a\right)\varphi - H\Box\varphi \\
&\qquad -\frac{1}{2}H^2(n+2w-2)(w-1)(\nabla_N - wH)\varphi\Big]
\end{aligned}
$$

which is equivalent to the result. $\square$

In the case $w = \frac{4-n}{2}$ the weighted operator above specialises to the operator δ_3 derived by Grant [5]. The δ_3 operator has the form

$$\begin{aligned}
\delta_3\varphi &= (n-4)\Bigg(\frac{2-n}{2}\overline{J}\nabla_N\varphi + \frac{(n-2)(n-4)}{4}\overline{J}H\varphi + \frac{2-n}{2}\overline{\Delta}H\varphi + \overline{\nabla}_a\Pi^a_b P^b_c N^c\varphi \\
&\quad + P_{cb}N^b\nabla^c\varphi - HN^bN^c\nabla_c\nabla_b\varphi - \frac{4-n}{2}HP^{\perp}\varphi + \frac{4-n}{4}\nabla_N J\varphi - \frac{1}{2}\nabla_N\Delta\varphi \\
&\quad - \frac{n(n-4)}{8}HJ\varphi + \frac{n}{4}H\Delta\varphi + \frac{3(n-2)}{4}H^2\nabla_N\varphi + \frac{(n-2)(n-4)}{4}H\varphi\Bigg) \\
&\quad + (n-2)\overline{\Delta}\nabla_N\varphi - (n-2)\overline{\nabla}^a H\overline{\nabla}_a\varphi + 2\overline{\nabla}_a\Pi^a_b\nabla_N\nabla^b\varphi
\end{aligned}$$

6.2 An Invariant Third Normal Order Operator

Proposition 6.6. *Given an index-free weighted section $\varphi \in \mathcal{E}[w]$, the conformally invariant operator $\overline{D}^A L_{AB} D^B$ is explicitly*

$$\begin{aligned}
\overline{D}^A L_{AB} D^B &= (n+2w-5)(n+2w-2) \times \Big[(n-3)\overline{\nabla}^a L_{(ab)^0}\overline{\nabla}^b\varphi \\
&\quad - \frac{n-3}{n-2}(n+2w-4)\left(\overline{\nabla}^a L_{(ab)^0}\right)\overline{\nabla}^b\varphi \\
&\quad + \frac{w}{n-2}(w-1)\left(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}\right)\varphi \\
&\quad + (n+w-4)w\overline{P}^{ab}L_{(ab)^0}\varphi\Big]
\end{aligned}$$

Proof. Applying the tractor projection to the D^B operator yields

$$\left[\Pi^B_C D^C\varphi\right]_{\bar{g}} = \begin{pmatrix} (n+2w-2)w\varphi \\ (n+2w-2)\Pi^b_c\nabla^c\varphi \\ -\Box\varphi + (n+2w-2)\left(H\nabla_N\varphi - \frac{1}{2}wH^2\varphi\right) \end{pmatrix}$$

The tractor L_{AB} of definition 5.11 is

$$\begin{aligned}
L_{AB} &= \overline{Z}^b_B\overline{Z}^a_A(n-3)L_{(ab)^0} - \overline{Z}^b_B\overline{X}_A\frac{n-3}{n-2}\overline{\nabla}^a L_{(ab)^0} - \overline{Z}^a_A\overline{X}_B\frac{n-3}{n-2}\overline{\nabla}^b L_{(ab)^0} \\
&\quad + \overline{X}_A\overline{X}_B\left(\overline{P}^{ab}L_{(ab)^0} + (n-2)^{-1}\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}\right)
\end{aligned}$$

so that $L_{AB}D^B\varphi$ is obtained from a contraction in the sub-manifold;

$$\begin{aligned}
L_{AB}D^B\varphi &= (n+2w-2)(n-3)\overline{Z}^a_A\left[L_{(ab)^0}\Pi^b_c\nabla^c\varphi - \frac{w}{n-2}(\overline{\nabla}^b L_{(ab)^0})\varphi\right] \\
&\qquad (n+2w-2)\overline{X}_A\Bigg[-\frac{n-3}{n-2}(\overline{\nabla}^a L_{(ab)^0})\Pi^b_c\nabla^c\varphi + w\overline{P}^{ab}L_{(ab)^0} \\
&\qquad + \frac{w}{n-2}(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0})\varphi\Bigg]
\end{aligned}$$

Here φ is assumed to be index-free and so $\Pi^b_c\nabla^c\varphi = \overline{\nabla}^b\varphi$. Since $(\Sigma, \bar{g})$ has dimension $(n-1)$, lemma 4.10 informs that the $\overline{D}^A$ operator acts on the injecting operators $\overline{Z}^a_A$ and $\overline{X}_A$ as

$$\begin{aligned}
\overline{D}^A\overline{Z}^a_A\varphi &= (n+2w-3)\overline{\nabla}^a\phi \\
\overline{D}^A\overline{X}_A\varphi &= (n+2w+1)(n+w-1)\phi
\end{aligned}$$

where in each case ϕ is assumed to be of weight w. The $\overline{Z}^a_A$ coefficient of $L_{AB}D^B\varphi$ has weight $w-1$ while the $\overline{X}_A$ coefficient has weight $w-3$ so that

$$\begin{aligned}
\overline{D}^A L_{AB}D^B\varphi &= (n+2w-5)(n+2w-2)(n-3)\overline{\nabla}^a\left[L_{(ab)^0}\overline{\nabla}^b\varphi - \frac{w}{n-2}(\overline{\nabla}^b L_{(ab)^0})\varphi\right] \\
&\quad +(n+2w-5)(n+w-4)(n+2w-2)\left[-\frac{n-3}{n-2}(\overline{\nabla}^a L_{(ab)^0})\overline{\nabla}^b\varphi\right. \\
&\quad \left. +w\overline{P}^{ab}L_{(ab)^0}\varphi + \frac{w}{n-2}(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0})\varphi\right] \\
&= (n+2w-5)(n+2w-2)\times\left[(n-3)\overline{\nabla}^a L_{(ab)^0}\overline{\nabla}^b\varphi\right. \\
&\quad -\frac{n-3}{n-2}\left(w\overline{\nabla}^a(\overline{\nabla}^b L_{(ab)^0})\varphi + (n+w-4)(\overline{\nabla}^a L_{(ab)^0})\overline{\nabla}^b\varphi\right) \\
&\quad \left. +(n+w-4)w\left(\overline{P}^{ab}L_{(ab)^0}\varphi + \frac{1}{n-2}(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0})\varphi\right)\right]
\end{aligned}$$

Using the Leibniz rule and the symmetry of $L_{(ab)^0}$ we obtain

$$\overline{\nabla}^a\left(\overline{\nabla}^b L_{(ab)^0}\right)\varphi = \left(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}\right)\varphi + \left(\overline{\nabla}^a L_{(ab)^0}\right)\overline{\nabla}^b\varphi$$

which allows further simplification to the result. $\square$

For the conformal weight $w = \frac{4-n}{2}$ the term involving $(\overline{\nabla}^a L_{(ab)^0})\overline{\nabla}^b\varphi$ vanishes and so the operator $\overline{D}^A L_{AB}D^B$ takes the form

$$\overline{D}^A L_{AB}D^B\varphi = 2(3-n)\overline{\nabla}^a L_{(ab)^0}\overline{\nabla}^b\varphi + \frac{n-4}{2}\left((n-4)\overline{P}^{ab}L_{(ab)^0} - \left(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}\right)\right)\varphi$$

which recovers the weighted operator $\overline{D}^A L_{AB}D^B\colon \mathcal{E}[\frac{4-n}{2}] \mapsto \mathcal{E}[\frac{-2-n}{2}]$ of Grant [5].

Lemma 6.7. *The term $\overline{\nabla}^a L_{(ab)^0}\overline{\nabla}^b$ may be expanded as*

$$\overline{\nabla}^a L_{(ab)^0}\overline{\nabla}^b\varphi = \overline{\Delta}\nabla_N\varphi - \overline{\nabla}^a H\overline{\nabla}_a\varphi - \nabla_a\Pi^a_b\nabla_N\nabla^b\varphi$$

for weighted functions $\varphi \in \mathcal{E}[w]$.

Proof. The Leibniz rule allows

$$\begin{aligned}\overline{\Delta}\nabla_N\varphi &= \overline{\nabla}_a\Pi^a_b\nabla^b N^c\nabla_c\varphi\\ &= \overline{\nabla}_a\Pi^a_b N^c\nabla^b\nabla_c\varphi + \overline{\nabla}_a\left(\Pi^a_b\nabla^b N^c\right)\nabla_c\varphi\end{aligned}$$

The connections ∇_a and ∇_b commute when acting on $\varphi \in \mathcal{E}[w]$. Therefore

$$\overline{\Delta}\nabla_N\varphi = \overline{\nabla}_a\Pi^a_b N^c\nabla_c\nabla^b\varphi + \overline{\nabla}^a L_{ac}\nabla^c\varphi$$

We apply $\overline{\nabla}_c\varphi = \Pi^{c'}_c\nabla_{c'}\varphi$, again because φ is without indices, so that

$$\begin{aligned}\overline{\Delta}\nabla_N\varphi - \overline{\nabla}^a H\overline{\nabla}_a\varphi &= \overline{\nabla}_a\Pi^a_b N^c\nabla_c\nabla^b\varphi + \overline{\nabla}^a L_{ac}\overline{\nabla}^c\varphi - \overline{\nabla}^a H\overline{\nabla}_a\varphi\\ &= \overline{\nabla}_a\Pi^a_b N^c\nabla_c\nabla^b\varphi + \overline{\nabla}^a L_{(ac)^0}\overline{\nabla}^c\varphi\end{aligned}$$

from which the result follows. □

Lemma 6.7 allows some simplifications when adding the operators $\overline{D}_B\Pi^B_A\delta D^A$ and $\overline{D}^A L_{AB}D^B$ of propositions 6.5 and 6.6 respectively. We obtain the conformally invariant third order operator

$$\begin{aligned}\Lambda_3(w) = {} & (n+w-4)(n+2w-2)(w-1)\overline{J}\left(\nabla_N - wH\right)\varphi\\ &+(n+2w-2)(w-1)w\overline{\Delta}H\varphi\\ &+(n+2w-2)(n+2w-5)\left(w\overline{\nabla}_a\Pi^a_b P^b_c N^c\right)\varphi\\ &+(n+2w-2)(n+2w-5)(4-n-w)\left(\overline{\nabla}_a H\overline{\nabla}^a\right)\varphi\\ &+(n+w-4)(n+2w-2)(n+2w-5)\left(-P_{cb}N^b\nabla^c + HN^bN^c\nabla_c\nabla_b + wHP^{\perp}\right)\varphi\\ &+(n+w-4)(n+2w-5)\left(-\nabla_N\Box + (w-2)H\Box\right)\varphi\\ &+(n+w-4)(n+2w-2)(n+2w-5)(w-1)\left(\frac{-1}{2}H^2(3\nabla_N - wH)\right)\varphi\\ &+(n+2w-2)\big((n+2w-4)(n-4)+w\big)\overline{\Delta}\nabla_N\varphi\\ &+(n+2w-5)(n+2w-2)(4-n)\overline{\nabla}_a\Pi^a_b\nabla_N\nabla^b\varphi\\ &+(n+2w-5)(n+2w-2)\times\left[\frac{n-3}{n-2}(n+2w-4)\left(\overline{\nabla}^a L_{(ab)^0}\right)\overline{\nabla}^b\varphi\right.\\ &\qquad\left.+\frac{w}{n-2}(w-1)\left(\overline{\nabla}^a\overline{\nabla}^b L_{(ab)^0}\right)\varphi + (n+w-4)w\overline{P}^{ab}L_{(ab)^0}\varphi\right]\end{aligned}$$

This construction is polynomial in $\mathbf{g}$, $\mathbf{g}^{-1}$, ∇, P_{ab}, N^a, and L_{ab}. It follows that we are free to cancel any common factors of rational functions of n and w without affecting the conformal invariance or the normal order of the operator.

Similar to the second order operator $\overline{D}_B \Pi^B_A D^A$, the operator $\Lambda_3(w)$ simplifies to known operators for particular conformal weights. When $w = \frac{2-n}{2}$ we recover

$$\begin{aligned}\Lambda_3\left(\frac{2-n}{2}\right) &= \frac{3(4-n)}{2}\left(-\nabla_N \Box + \frac{n-2}{2} H \Box\right)\varphi \\ &= \frac{3(n-4)}{2}\,\delta\,\Box\,\varphi\end{aligned}$$

since $\Box\varphi$ will be of weight $w-2$. When $w = \frac{5-n}{2}$ we obtain

$$\Lambda_3\left(\frac{5-n}{2}\right) = \frac{3}{2}(n-3)\,\overline{\Box}\,\delta\,\varphi$$

which is again a multiple of known conformally invariant operators. To conclude, the methods of tractor calculus provide a concise and insightful way to produce conformal invariant quantities. The setting of conformal sub-manifolds presents still more research opportunities. Further work could involve establishing tractor calculus results analogous to useful Riemannian theorems such as the Gauss-Codazzi-Mainardi theorem for the curvature of a sub-manifold.

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