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UNIT KILLING VECTOR FIELDS ON NEARLY KÄHLER MANIFOLDS

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We study 6-dimensional nearly Kähler manifolds admitting a Killing vector field of unit length. In the compact case, it is shown that up to a finite cover there is only one geometry possible, that of the 3-symmetric space $S^3 \times S^3$.

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1. Introduction

Nearly Kähler geometry (shortly NK in what follows) naturally arises as one of the sixteen classes of almost Hermitian manifolds appearing in the celebrated Gray–Hervella classification [8]. These manifolds were studied intensively in the seventies by Gray [7]. His initial motivation was inspired by the concept of weak holonomy [7], but very recently it turned out that this concept, as defined by Gray, does not produce any new geometric structure (see [1]) other than those coming from a Riemannian holonomy reduction. One of the most important properties of NK manifolds is that their canonical Hermitian connection has totally skew-symmetric, parallel torsion [15]. From this point of view, they naturally fit into the setup proposed in [6] towards a weakening of the notion of Riemannian holonomy. The same property suggests that NK manifolds might be objects of interest in string theory [10].

The structure theory of compact NK manifolds, as developed in [16] reduces their study to positive quaternion-Kähler manifolds and nearly Kähler manifolds of dimension 6. The last class of manifolds falls in the area of special metrics with very rigid — though not yet fully understood — properties.

Indeed, it is known since a long time that in 6 dimensions, a NK metric which is not Kähler has to be Einstein of positive scalar curvature. Moreover, such a structure is characterized by the existence of some (at least locally defined) real Killing spinor [13]. Combining these properties with the fact that the first Chern class vanishes [7], one observes that non-Kähler, nearly Kähler 6-dimensional manifolds solve most of the type II string equations [10]. Despite of all these interesting features, very little is known about these manifolds. In particular, the only known compact examples are the 3-symmetric spaces

$$S^6, S^3 \times S^3, \mathbb{C}P^3, F(1,2)$$

and moreover, Butruille recently proved that there are no other compact homogeneous examples [5].

In a recent article [14], Hitchin shows that nearly parallel G_2-structures ([11] for an account) and NK manifolds of 6 dimensions have the same variational origins. On the other hand, many examples of nearly parallel G_2-structures are available since any 7-dimensional, 3-Sasakian manifold carries such a structure [11, 12] and a profusion of compact examples of the latter were produced in [4]. Since one property of 3-Sasakian manifolds is to admit unit Killing vector fields, one might ask whether this can happen in the NK setting.

In the present paper we study 6-dimensional non-Kähler, nearly Kähler manifolds which globally admit a Killing vector field ξ of constant length. After recalling some elementary features of nearly Kähler geometry in Sec. 2, we show in Sec. 3 that any Killing vector field of unit length induces a transversal almost hyper-Hermitian structure on the manifold. This almost hyper-Hermitian structure is preserved by the Killing vector field ξ but not by $J\xi$ (here J denotes the almost complex structure of the nearly Kähler structure). We measure this in the fourth section by computing the Lie derivatives of the various geometrically significant tensors in the direction of $J\xi$. This technical part is used in Sec. 5 to perform a double reduction of the 6-dimensional nearly Kähler manifold. The resulting 4-dimensional manifold is in fact a Kähler–Einstein surface of positive scalar curvature admitting an orthogonal almost-Kähler structure inducing the opposite orientation. The geometry of the situation is completely understood in terms of this data. Moreover, if the nearly Kähler manifold is compact, a Sekigawa-type argument from [2] shows that the almost-Kähler structure is actually integrable, allowing us to prove the main result of this paper.

Theorem 1.1. *Let (M^6, g, J) be a complete nearly Kähler manifold. If g admits a unit Killing vector field, then up to a finite cover (M^6, g, J) is isometric to $S^3 \times S^3$ endowed with its canonical NK structure.*

2. Nearly Kähler Manifolds

An almost Hermitian manifold (M, g, J) is called *nearly Kähler* if $(\nabla_X J)X = 0$ is satisfied for all vector fields X. In other words, the covariant derivative of J (viewed as a $(3,0)$-tensor via the metric g) is skew-symmetric in all three arguments, not only in the last two, as it is the case for general almost Hermitian structures. This is equivalent to $d\Omega = 3\nabla\Omega$, where Ω is the fundamental 2-form, i.e. $\Omega(X,Y) := g(JX, Y)$. The following lemma summarizes some of the known identities for nearly Kähler manifolds.

Lemma 2.1 (cf. [7]). *Let (M, g, J) be a nearly Kähler manifold. Then*

(1) $(\nabla_X J)Y + (\nabla_Y J)X = 0$,
(2) $(\nabla_{JX} J)Y = (\nabla_X J)JY$,
(3) $J((\nabla_X J)Y) = -(\nabla_X J)JY = -(\nabla_{JX} J)Y$,
(4) $g(\nabla_X Y, X) = g(\nabla_X JY, JX)$,
(5) $2g((\nabla^2_{W,X} J)Y, Z) = -\sigma_{X,Y,Z}\, g((\nabla_W J)X, (\nabla_Y J)JZ)$,

where $\sigma_{X,Y,Z}$ denotes the cyclic sum over the vector fields X, Y, Z.

A nearly Kähler manifold is called to be of *constant type α* if

$$\|(\nabla_X J)(Y)\|^2 = \alpha\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(JX,Y)^2\}$$

holds for any vector fields X, Y. Gray proved that a nearly Kähler manifold of positive constant type is necessarily 6-dimensional (cf. [7]). Moreover he showed:

Proposition 2.2. *Let (M, g, J) be a 6-dimensional nearly Kähler, non-Kähler manifold, then*

(1) *M is of constant type $\alpha > 0$.*
(2) *$c_1(M) = 0$ and in particular M is a spin manifold.*
(3) *(M, g) is Einstein and $\mathrm{Ric} = 5\alpha\mathrm{Id} = 5\mathrm{Ric}^*$.*

Here the $*$-Ricci curvature Ric^* is defined as $\mathrm{Ric}^*(X,Y) = \mathrm{tr}(Z \mapsto R(X, JZ)JY)$. From this it easily follows that $\mathrm{Ric}^*(X,Y) = \mathcal{R}(\Omega)(X, JY)$, where $\mathcal{R}$ denotes the curvature operator on 2-forms. The following results are straightforward algebraic calculations using Lemma 2.1.

Lemma 2.3. *Let (M^6, gJ) be a nearly Kähler manifold of constant type α, then*

$$\begin{aligned} g((\nabla_U J)X, (\nabla_Y J)Z) = \alpha\{&g(U,Y)g(X,Z) - g(U,Z)g(X,Y) \\ &- g(U,JY)g(X,JZ) + g(U,JZ)g(X,JY)\}. \end{aligned}$$

Corollary 2.4. *Let* (M^6, g, J) *be a nearly Kähler manifold of constant type* $\alpha = 1$, *then*

$$(\nabla_X J) \circ (\nabla_X J)Y = -|X|^2 Y, \quad \textit{for } Y \perp X, JX,$$
$$\nabla^* \nabla \Omega = 4\Omega.$$

Lemma 2.5. *Let* X *and* Y *be any vector fields on* M, *then the vector field* $(\nabla_X J)Y$ *is orthogonal to* X, JX, Y, *and* JY.

This lemma allows us to use adapted frames $\{e_i\}$ which are especially convenient for local calculations. Let e_1 and e_3 be any two orthogonal vectors and define:

$$e_2 := Je_1, \quad e_4 := Je_3, \quad e_5 := (\nabla_{e_1} J)e_3, \quad e_6 := Je_5.$$

Lemma 2.6. *With respect to an adapted frame* $\{e_i\}$ *one has*

$$\nabla J = e_1 \wedge e_3 \wedge e_5 - e_1 \wedge e_4 \wedge e_6 - e_2 \wedge e_3 \wedge e_6 - e_2 \wedge e_4 \wedge e_5,$$
$$*(\nabla J) = -e_2 \wedge e_4 \wedge e_6 + e_2 \wedge e_3 \wedge e_5 + e_1 \wedge e_4 \wedge e_5 + e_1 \wedge e_3 \wedge e_6,$$
$$\Omega = e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6.$$

Corollary 2.7. *Let* Ω *be the fundamental* 2*-form and* X *an arbitrary vector field, then*

(1) $X \lrcorner \Omega = JX^\flat, \quad X \lrcorner * \Omega = JX^\flat \wedge \Omega, \quad X \lrcorner d\Omega = JX \lrcorner * d\Omega,$
(2) $|\Omega|^2 = 3, \quad *\Omega = \frac{1}{2}\Omega \wedge \Omega, \quad vol = e_1 \wedge \cdots \wedge e_6 = \frac{1}{6}\Omega^3, \quad \Omega \wedge d\Omega = 0,$
(3) $*X^\flat = \frac{1}{2} JX^\flat \wedge \Omega \wedge \Omega,$
where $X^\flat$ *denotes the* 1*-form which is metric dual to* X.

Proposition 2.8. *Let* (M^6, g, J) *be a nearly Kähler, non-Kähler manifold with fundamental* 2*-form* Ω *which is of constant type* $\alpha = 1$, *then*

$$\Delta\Omega = 12\Omega.$$

Proof. To show that Ω is an eigenform of the Laplace operator we will use the Weitzenböck formula on 2-forms, i.e. $\Delta = \nabla^*\nabla + \frac{s}{3}\mathrm{Id} - 2\mathcal{R}$. From Corollary 2.4, we know that $\nabla^*\nabla\Omega = 4\Omega$. Since we assume M to be of constant type 1 the scalar curvature is $s = 30$ and the $*$-Ricci curvature Ric^* coincides with the Riemannian metric g. Hence, $\mathcal{R}(\Omega)(X, Y) = -Ric^*(X, JY) = \Omega(X, Y)$. Substituting this into the Weitzenböck formula yields $\Delta\Omega = 12\Omega$. □

Corollary 2.9. *If* ξ *is any vector field satisfying* $L_\xi(*d\Omega) = 0$, *then*

$$d(J\xi \lrcorner d\Omega) = -12 J\xi^\flat \wedge \Omega. \tag{1}$$

Proof. We use Corollary 2.7 and the relation $L_\xi = d \circ \xi \lrcorner + \xi \lrcorner \circ d$ to obtain

$$d(J\xi \lrcorner d\Omega) = -d(\xi \lrcorner * d\Omega) = \xi \lrcorner (d * d\Omega) = -\xi \lrcorner (*d^* d\Omega) = -\xi \lrcorner (*\Delta\Omega).$$

Note that $d^*\Omega = 0$. Again applying Corollary 2.7 together with (1), we get

$$d(J\xi \lrcorner d\Omega) = -12\xi \lrcorner (*\Omega) = -6\xi \lrcorner (\Omega \wedge \Omega) = -12(\xi \lrcorner \Omega) \wedge \Omega = -12J\xi^\flat \wedge \Omega. \qquad \square$$

The structure group of a nearly Kähler manifold (M^6, g, J) reduces to SU(3) which implies the decomposition $\Lambda^2(TM) = \Lambda^{\text{inv}} \oplus \Lambda^{\text{anti}}$ with

$$\alpha \in \Lambda^{\text{inv}} \Leftrightarrow \alpha(X,Y) = \alpha(JX, JY),$$
$$\alpha \in \Lambda^{\text{anti}} \Leftrightarrow \alpha(X,Y) = -\alpha(JX, JY).$$

We will denote the projection of a 2-form α onto Λ^{inv} by $\alpha^{(1,1)}$ and the projection onto Λ^{anti} by $\alpha^{(2,0)}$. This is motivated by the isomorphisms

$$\Lambda^{\text{inv}} \otimes \mathbb{C} \cong \Lambda^{(1,1)}(TM) \cong \mathrm{u}(3), \quad \Lambda^{\text{anti}} \otimes \mathbb{C} \cong \Lambda^{(2,0)}(TM) \oplus \Lambda^{(0,2)}(TM).$$

Lemma 2.10. *The decomposition* $\Lambda^2(TM) = \Lambda^{\text{inv}} \oplus \Lambda^{\text{anti}}$ *is orthogonal and the projections of a 2-form* α *onto the two components are given by*

$$\alpha^{(1,1)}(X,Y) = \frac{1}{2}(\alpha(X,Y) + \alpha(JX,JY)) = \frac{1}{2}\mathrm{Re}(\alpha([X+iJX],[Y-iJY])),$$
$$\alpha^{(2,0)}(X,Y) = \frac{1}{2}(\alpha(X,Y) - \alpha(JX,JY)) = \frac{1}{2}\mathrm{Re}(\alpha([X+iJX],[Y+iJY])).$$

Under the isomorphism $\Lambda^2(TM) \xrightarrow{\sim} \mathrm{End}_0(TM)$ every 2-form α corresponds to a skew-symmetric endomorphism A which is defined by the equation $\alpha(X,Y) = g(AX,Y)$. Note that $|A|^2 = 2|\alpha|^2$, where $|\cdot|$ is the norm induced from the Riemannian metric on the endomorphisms and on the 2-forms.

Lemma 2.11. *Let* $A^{(2,0)}$ (*respectively*, $A^{(1,1)}$) *be the endomorphisms corresponding to the components* $\alpha^{(2,0)}$ (*respectively*, $\alpha^{(1,1)}$), *then*

(1) $A^{(2,0)} = \frac{1}{2}(A + JAJ), \quad A^{(1,1)} = \frac{1}{2}(A - JAJ)$,
(2) $J \circ A^{(1,1)} = A^{(1,1)} \circ J, \quad J \circ A^{(2,0)} = -A^{(2,0)} \circ J$.

3. The Transversal Complex Structures

In this section we consider 6-dimensional compact non-Kähler nearly Kähler manifolds (M, g, J) of constant type $\alpha \equiv 1$, i.e. with scalar curvature normalized by $s \equiv 30$. We will assume from now on that (M, g) is not isometric to the sphere with its standard metric.

Proposition 3.1. *Let ξ be a Killing vector field then the Lie derivative of the almost complex structure J with respect to ξ vanishes*

$$L_\xi J = 0.$$

Proof. Nearly Kähler structures J on (M, g) are in one-to-one correspondence with Killing spinors of unit norm on M (cf. [13]). However, if (M, g) is not isometric to the standard sphere, the (real) space of Killing spinors is 1-dimensional, so there exist exactly 2 nearly Kähler structures compatible with g: J and $-J$. This shows that the identity component of the isometry group of M preserves J, so in particular $L_\xi J = 0$ for every Killing vector field ξ. □

Since a Killing vector field ξ satisfies by definition $L_\xi g = 0$, we obtain that the Lie derivative L_ξ of all natural tensors constructed out of g and J vanishes. In particular, we have

Corollary 3.2. *If ξ is a Killing vector field on the nearly Kähler manifold (M, g, J) with fundamental 2-form Ω, then*

$$L_\xi(\Omega) = 0, \quad L_\xi(d\Omega) = 0, \quad L_\xi(*d\Omega) = 0.$$

In order to simplify the notations we denote by $\zeta := \xi^\flat$ the dual 1-form to ξ.

Corollary 3.3.

$$d(J\zeta) = -\xi \lrcorner\, d\Omega.$$

Proof. From $L_\xi\Omega = 0$, it follows: $-\xi \lrcorner\, d\Omega = d(\xi \lrcorner\, \Omega) = d(J\zeta)$. □

Lemma 3.4. *Let ξ be a Killing vector field with metric dual ζ and let $d\zeta = d\zeta^{(1,1)} + d\zeta^{(2,0)}$ be the type decomposition of $d\zeta$. Then the endomorphisms corresponding to $d\zeta^{(2,0)}$ and $d\zeta^{(1,1)}$ are $-\nabla_{J\xi}J$ and $2\nabla \cdot \xi + \nabla_{J\xi}J$ respectively.*

Proof. The equation $L_\xi J = 0$ applied to a vector field X yields $[\xi, JY] = J[\xi, Y]$, which can be written as

$$\nabla_\xi JY - \nabla_{JY}\xi = J\nabla_\xi Y - J\nabla_Y \xi.$$

From this equation we obtain

$$(\nabla_\xi J)Y = \nabla_{JY}\xi - J\nabla_Y\xi. \tag{2}$$

Let A $(= 2\nabla \cdot \xi)$ denote the skew-symmetric endomorphism corresponding to the 2-form $d\zeta$. Then (2) can be written

$$\nabla_\xi J = \frac{1}{2}(A \circ J - J \circ A).$$

From Lemma 2.1, we get $\nabla_{J\xi}J = (\nabla_\xi J)\circ J$, whence

$$\nabla_{J\xi}J = \frac{1}{2}(A\circ J - J\circ A)\circ J = -\frac{1}{2}(A + J\circ A\circ J) = -A^{(2,0)}.$$

For the corresponding 2-form we obtain: $d\zeta^{(2,0)} = -\nabla_{J\xi}\Omega$. Finally we compute $A^{(1,1)}$ as

$$A^{(1,1)} = A - A^{(2,0)} = 2\nabla\cdot\xi + \nabla_{J\xi}J\ . \qquad \square$$

From now on we will mainly be interested in compact nearly Kähler manifolds admitting a Killing vector field ξ of constant length (normalized to 1). We start by collecting several elementary properties

Lemma 3.5. *The following relations hold*:

(1) $\nabla_\xi\xi = \nabla_{J\xi}\xi = \nabla_\xi J\xi = \nabla_{J\xi}J\xi = 0,\ [\xi, J\xi] = 0,$
(2) $\xi\lrcorner\, d\zeta = J\xi\lrcorner\, d\zeta = 0.$

In particular, the distribution $V := \mathrm{span}\{\xi, J\xi\}$ *is integrable.*

We now define two endomorphisms which turn out to be complex structures on the orthogonal complement $H := \{\xi, J\xi\}^\perp$:

$$I := \nabla_\xi J, \quad \text{and} \quad K := \nabla_{J\xi}J\ . \tag{3}$$

Note that $-K$ is the endomorphism corresponding to $d\zeta^{(2,0)}$. Later on, we will see that the endomorphism $\hat{J} := \nabla\cdot\xi + \frac{1}{2}\nabla_{J\xi}J$, corresponding to $\frac{1}{2}d\zeta^{(1,1)}$, defines a complex structure on H, too.

Lemma 3.6. *The endomorphisms I and K vanish on* $\mathrm{span}\{\xi, J\xi\}$ *and define complex structures on $H = \{\xi, J\xi\}^\perp$ compatible with the metric g. Moreover they satisfy $K = I\circ J$, $0 = I\circ J + J\circ I$ and for any $X, Y \in H$, we have the equation*

$$(\nabla_X J)Y = \langle Y, IX\rangle\xi + \langle Y, KX\rangle J\xi.$$

We will call endomorphisms of TM, which are complex structures on $H = \{\xi, J\xi\}^\perp$ *transversal complex structures.* The last equation shows that ∇J vanishes in the direction of H. It turns out that the same is true for transversal complex structures I and J.

Lemma 3.7. *The transversal complex structures I and K are parallel in the direction of the distribution H, i.e.*

$$\langle(\nabla_X I)Y, Z\rangle = \langle(\nabla_X K)Y, Z\rangle = 0$$

holds for all vector fields X, Y, Z in H.

Proof. First of all we compute for any vector fields X, Y, Z the covariant derivative of I.

$$\langle(\nabla_X I)Y, Z\rangle = \langle(\nabla_X(\nabla_\xi I))Y, Z\rangle = \langle(\nabla^2_{X,\xi} I)Y, Z\rangle + \langle(\nabla_{\nabla_X \xi} I)Y, Z\rangle.$$

If X is a vector field in H, then Lemma 2.5 implies that $\nabla_X J$ maps H to V and vice versa.

Let now X, Y, Z be any vector fields in H. Then both summands in the above formula for $\nabla_X I$ vanish, which is clear after rewriting the first summand using formula (5) of Lemma 2.1 and the second by using formula (1) of Lemma 2.1.

The proof for the transversal complex structure K is similar. □

Our next goal is to show that $\frac{1}{2} d\zeta^{(1,1)}$ defines an complex structure on the orthogonal complement of $\mathrm{span}\{\xi, J\xi\}$.

Lemma 3.8. *Let ξ be a Killing vector field of length* 1, *then*

$$\|d\zeta^{(1,1)}\|^2 = 8, \quad \|d\zeta^{(2,0)}\|^2 = 2.$$

Proof. We already know that the skew-symmetric endomorphism K corresponding to $-d\zeta^{(2,0)}$ is an complex structure on $\{\xi, J\xi\}^\perp$ with $K(\xi) = K(J\xi) = 0$. Hence it has norm 4 and $\|d\zeta^{(2,0)}\|^2 = \frac{1}{2}\|K\|^2 = 2$. Next, we have to compute the (pointwise) norm of $d\zeta^{(1,1)}$. Since ξ is a Killing vector field and M is Einstein with Einstein constant 5, we have $\Delta\zeta = 10\zeta$. Moreover, using Lemma 3.5 and the fact that $d^*\zeta = 0$, we get

$$\|d\zeta\|^2 = \langle d^* d\zeta, \zeta\rangle + d^*(\xi \lrcorner d\zeta) = \langle\Delta\zeta, \zeta\rangle,$$

whence

$$\|d\zeta^{(1,1)}\|^2 = \|d\zeta\|^2 - \|d\zeta^{(2,0)}\|^2 = \langle\Delta\zeta, \zeta\rangle - 2 = 8.$$ □

Notice that the square norm of a p-form is, as usual, equal to $\frac{1}{p!}$ times the square of the tensorial norm. For example, the 2-form $e_1 \wedge e_2$ has unit norm as a form, and it is identified with the tensor $e_1 \otimes e_2 - e_2 \otimes e_1$, whose square norm is 2.

Corollary 3.9. *The square* (*tensorial*) *norm of the endomorphism $\hat{J}$ corresponding to $\frac{1}{2} d\zeta^{(1,1)}$ is equal to* 4.

Proposition 3.10. *Let (M^6, g, J) be a compact nearly Kähler, non-Kähler manifold of constant type* 1 *and let ξ be a Killing vector field of constant length* 1 *with dual* 1*-form ζ. Then*

$$d^*(J\zeta) = 0 \quad \text{and} \quad \Delta(J\zeta) = 18 J\zeta.$$

In particular, the vector field $J\xi$ is never a Killing vector field.

Proof. We start to compute the L^2-norm of the function $d^*(J\zeta)$:

$$\|d^*(J\zeta)\|^2 = (d^*J\zeta, d^*J\zeta) = (\Delta(J\zeta), J\zeta) - (d^*d(J\zeta), J\zeta).$$

Since $\Delta = \nabla^*\nabla + \mathrm{Ric}$ on 1-forms and $\mathrm{Ric} = 5\mathrm{Id}$, we obtain

$$\|d^*(J\zeta)\|^2 = \|\nabla(J\zeta)\|^2 + 5\|J\zeta\|^2 - \|dJ\zeta\|^2.$$

To compute the norm of $\nabla(J\zeta)$ we use the formula $2(\nabla X^\flat) = dX^\flat + L_X g$ which holds for any vector field X.

Note that the decomposition $T^*M \otimes T^*M \cong \Lambda^2(TM) \oplus \mathrm{Sym}^2(TM)$ is orthogonal. Together with Lemma 4.1, this yields

$$\|\nabla(J\zeta)\|^2 = \frac{1}{4}\big(\|dJ\zeta\|^2 + \|L_{J\xi}g\|^2\big) = \frac{1}{4}\big(\|dJ\zeta\|^2 + 4\|\hat{J}\|^2\big).$$

From Lemma 3.8, it follows: $\|\hat{J}\|^2 = \frac{1}{2}\|d\zeta^{(1,1)}\|^2 = 4$. Using Corollary 3.3, the formula $d\Omega = 3\nabla\Omega$ and the fact that $I = \nabla_\xi J$ is again a transversal complex structure, we find

$$\|dJ\zeta\|^2 = 9\|\xi \lrcorner \nabla\Omega\|^2 = 36.$$

Combining all these computations yields $\|d^*(J\zeta)\|^2 = 0$.

Next we want to compute $\Delta J\zeta$. We start by using the Weitzenböck formula on 1-forms and the equation $\nabla^*\nabla(J\tau) = (\nabla^*\nabla J)\tau + J(\nabla^*\nabla\tau) - 2\sum(\nabla_{e_i}J)(\nabla_{e_i}\tau)$ for any 1-from τ. This gives

$$\begin{aligned}\Delta J\zeta &= \nabla^*\nabla J\zeta + \mathrm{Ric}(J\zeta) = (\nabla^*\nabla J)\zeta + J(\nabla^*\nabla\zeta) + 5(J\zeta) - 2\sum(\nabla_{e_i}J)(\nabla_{e_i}\zeta)\\ &= 14J\zeta - 2\sum(\nabla_{e_i}J)(\nabla_{e_i}\zeta).\end{aligned}$$

For the last equation we used Corollary 2.4 and the assumption that ξ is a Killing vector field, hence $\nabla^*\nabla\zeta = \Delta\zeta - \mathrm{Ric}(\zeta) = 5\zeta$. Since $d^*(J\zeta) = 0$, we can compute $\Delta J\zeta$ by

$$\begin{aligned}\Delta J\zeta &= d^*d(J\zeta) = \sum e_i \lrcorner \nabla_{e_i}(\xi \lrcorner d\Omega) = \sum e_i \lrcorner (\nabla_{e_i}\xi) \lrcorner d\Omega + e_i \lrcorner \xi \lrcorner (\nabla_{e_i} d\Omega)\\ &= -3\sum(\nabla_{e_i}\Omega)(\nabla_{e_i}\xi) + 12J\zeta.\end{aligned}$$

Comparing these two equations for $\Delta J\zeta$ we get $\sum(\nabla_{e_i}\Omega)(\nabla_{e_i}\xi) = -2J\zeta$, so finally $\Delta J\zeta = 18J\zeta$.

Since $\Delta X = 2\mathrm{Ric}(X) = 10X$ for every Killing vector field X, $J\zeta$ cannot be Killing. □

Notice that if the manifold M is not assumed to be compact a local calculation still shows that $d^*(J\zeta)$ is a constant.

Corollary 3.11. *If ξ is a Killing vector field of unit length, then*

$$\langle d\zeta, \Omega\rangle = \langle d\zeta^{(1,1)}, \Omega\rangle = 0.$$

Proof. Using Corollary 2.7, we compute

$$\langle d\zeta, \Omega\rangle \mathrm{vol} = d\zeta \wedge *\Omega = \frac{1}{2} d\zeta \wedge \Omega \wedge \Omega = \frac{1}{2} d(\zeta \wedge \Omega \wedge \Omega) = -d(*J\zeta) = *(d^* J\zeta) = 0.$$

This proves the corollary since the decomposition $\Lambda^2(TM) = \Lambda^{\mathrm{inv}} \oplus \Lambda^{\mathrm{anti}}$ is orthogonal and $\Omega \in \Lambda^{\mathrm{inv}}$. □

We are now ready to prove that the $(1,1)$-part of $d\zeta$ defines a fourth complex structure on $H = \{\xi, J\xi\}^{\perp}$. Indeed we have

Proposition 3.12. *Let ξ be a Killing vector field of constant length* 1*. Then*

$$\hat{J} := \nabla \cdot \xi + \frac{1}{2} K$$

defines a transversal complex structure on H which is compatible with the metric g. Moreover, $\hat{J}$ is the skew-symmetric endomorphism corresponding to $\frac{1}{2} d\zeta^{(1,1)}$ and it commutes with I, J and K:

$$[\hat{J}, J] = [\hat{J}, K] = [\hat{J}, I] = 0.$$

Proof. Lemma 3.4 shows that $\hat{J}$ corresponds to $\frac{1}{2} d\zeta^{(1,1)}$. Since $\langle d\zeta^{(1,1)}, \Omega\rangle = 0$ and since $\hat{J}$ vanishes on $\mathrm{span}\{\xi, J\xi\}$, it follows that $d\zeta^{(1,1)} \in \Lambda_0^{(1,1)}(H) = \Lambda_-^2(H)$. Hence, Corollary 3.9 yields

$$\hat{J}^2 = -\frac{1}{4} \|\hat{J}\|^2 \mathrm{Id}_H = -\mathrm{Id}_H.$$

Finally, $\hat{J}$ commutes with I, J and K since endomorphisms corresponding to self-dual and anti-self-dual 2-forms in dimension 4 commute. □

4. Projectable Tensors

In this section we want to study which of the above defined tensors descend to the space of leaves of the integrable distribution $V = \mathrm{span}\{\xi, J\xi\}$. For doing this we have to compute Lie derivatives in the direction of ξ and $J\xi$. We first remark that the flow of ξ preserves both the metric g and the almost complex structure J, thus it also preserves $I, K, d\zeta, \hat{J}$ etc.

The situation is more complicated for $J\xi$. Since $J\xi$ is not Killing for g, we have to specify, when computing the Lie derivative of a tensor with respect to $J\xi$, whether the given tensor is regarded as endomorphism or as bilinear form.

Lemma 4.1. *Let $\alpha \in \Gamma(T^*M \otimes T^*M)$ be a $(2,0)$-tensor and A be the corresponding endomorphism. Then the Lie derivatives of A and α with respect to $J\xi$ are related by*

$$(L_{J\xi}\alpha)(X,Y) = g((L_{J\xi}A)X,Y) + 2g(J\hat{J}A(X),Y). \tag{4}$$

In particular, the Lie derivative of the Riemannian metric g with respect to $J\xi$ is

$$L_{J\xi}g = 2g(J\hat{J}\cdot,\cdot). \tag{5}$$

Proof. Taking the Lie derivative in $\alpha(X,Y) = g(AX,Y)$ yields

$$(L_{J\xi}\alpha)(X,Y) = (L_{J\xi}g)(AX,Y) + g((L_{J\xi}A)X,Y).$$

Thus (5) implies (4). Taking $\alpha = g$ in (4) yields (5), so the two assertions are equivalent.

Using Proposition 3.12, we can write

$$\begin{aligned} L_{J\xi}g(X,Y) &= g(\nabla_X J\xi, Y) + g(X, \nabla_Y J\xi) \\ &= \nabla J(X,\xi,Y) + \nabla J(Y,\xi,X) + g(J\nabla_X\xi,Y) + g(J\nabla_Y\xi,X) \\ &= g\left(J\left(\hat{J} - \frac{1}{2}K\right)X,Y\right) + g\left(J\left(\hat{J} - \frac{1}{2}K\right)Y,X\right) = 2g(J\hat{J}X,Y). \end{aligned}$$

□

From Lemma 3.5, we see that $L_{J\xi}\xi = L_{J\xi}J\xi = 0$. Thus, if ζ and $J\zeta$ denote as before the metric duals of ξ and $J\xi$, (5) shows that $L_{J\xi}\zeta = L_{J\xi}J\zeta = 0$, hence

$$L_{J\xi}(d\zeta) = L_{J\xi}(dJ\zeta) = 0. \tag{6}$$

Lemma 4.2. *If ξ is a Killing vector field of constant length 1, then*

$$L_{J\xi}(\Omega) = J\xi \lrcorner\, d\Omega - d\zeta = 4\omega_K - 2\omega_{\hat{J}}, \quad L_{J\xi}(J) = 4K, \tag{7}$$

$$L_{J\xi}(\omega_K) = -4\Omega + 4\zeta \wedge J\zeta = -4\omega_J, \quad L_{J\xi}(K) = -4J|_H - 2I\hat{J}, \tag{8}$$

$$L_{J\xi}(\omega_{\hat{J}}) = -2\Omega + 2\zeta \wedge J\zeta, \quad L_{J\xi}(\hat{J}) = 0, \tag{9}$$

$$L_{J\xi}(\omega_I) = 0, \quad L_{J\xi}(I) = 2\hat{J}K, \tag{10}$$

where $\omega_I, \omega_K, \omega_{\hat{J}}$ are the 2-forms corresponding to $I, K, \hat{J}$ and ω_J denotes the projection of Ω onto $\Lambda^2 H$, i.e. $\omega_J = \Omega - \xi \wedge J\xi$.

Proof. We will repeatedly use the formula $L_X\alpha = X \lrcorner\, d\alpha + dX \lrcorner\, \alpha$, which holds for any vector field X and any differential form α.

Proposition 3.12 shows that

$$2\omega_{\hat{J}} = d\zeta + \omega_K \tag{11}$$

hence

$$L_{J\xi}\Omega = J\xi \lrcorner\, d\Omega + d(J\xi \lrcorner\, \Omega) = J\xi \lrcorner\, d\Omega - d\zeta = 3\omega_K - d\zeta = 4\omega_K - 2\omega_{\hat{J}}.$$

The second equation in (7) follows by taking $A = J$ in Lemma 4.1.

Using Corollary 2.9, we get

$$L_{J\xi}(\omega_K) = \frac{1}{3}L_{J\xi}(J\xi\lrcorner\, d\Omega) = \frac{1}{3}J\xi\lrcorner\, d(J\xi\lrcorner\, d\Omega) = -4J\xi\lrcorner\,(J\zeta\wedge\Omega) = -4\Omega + 4\zeta\wedge J\zeta.$$

The second equation in (8) follows directly from the first one, by taking $A = K$ in Lemma 4.1.

Using (6), (8) and (11), we obtain

$$2L_{J\xi}\omega_{\hat{J}} = L_{J\xi}(d\zeta + \omega_K) = L_{J\xi}\omega_K = -4\Omega + 4\zeta\wedge J\zeta.$$

The second part of (9) follows from Lemma 4.1.

Finally, in order to prove (10), we use (6) twice:

$$L_{J\xi}\omega_I = \frac{1}{3}L_{J\xi}(\xi\lrcorner\, d\Omega) = \frac{1}{3}\xi\lrcorner\, L_{J\xi}(d\Omega) = -4\xi\lrcorner\,(J\zeta\wedge\Omega) = 0,$$

and the second part follows from Lemma 4.1 again. □

5. The Transversal Involution

We define a transversal orthogonal involution $\sigma \in \mathrm{End}(TM)$ by

$$\sigma = K\circ\hat{J},$$

i.e. we have $\sigma^2 = \mathrm{Id}$ on H and $\sigma = 0$ on $V = \mathrm{span}\{\xi, J\xi\}$. Hence, the distribution H splits into the (±1)-eigenspaces of σ and we can define a new metric g_0 on M as

$$g_0 = g + \frac{1}{2}g(\sigma\cdot,\cdot), \tag{12}$$

i.e. we have $g_0 = g$ on V, $g_0 = \frac{1}{2}g$ on the (-1)-eigenspace of σ and $g_0 = \frac{3}{2}g$ on the $(+1)$-eigenspace of σ. The reason for introducing g_0 is the fact that, in contrast to g, this new metric is preserved by the flow of $J\xi$ (cf. Corollary 5.2 below).

Lemma 5.1. *If $A^\flat$ denotes the $(2,0)$-tensor corresponding to an endomorphism A, and $\alpha^\sharp$ denotes the endomorphism corresponding to a $(2,0)$-tensor α with respect to the metric g, then*

$$L_{J\xi}\sigma^\flat = -4(J\hat{J})^\flat.$$

Proof. The right parts of (8) and (9) read

$$L_{J\xi}(K) = -4J + 4(\zeta\wedge\hat{J}\zeta)^\sharp - 2I\hat{J}, \quad L_{J\xi}(\hat{J}) = 2(\zeta\wedge\hat{J}\zeta)^\sharp.$$

We clearly have $(\zeta\wedge\hat{J}\zeta)^\sharp\circ K = K\circ(\zeta\wedge\hat{J}\zeta)^\sharp = (\zeta\wedge\hat{J}\zeta)^\sharp\circ\hat{J} = \hat{J}\circ(\zeta\wedge\hat{J}\zeta)^\sharp = 0$, therefore

$$L_{J\xi}\sigma = (L_{J\xi}K)\hat{J} + K(L_{J\xi}\hat{J}) = (-4J + 4(\zeta\wedge\hat{J}\zeta)^\sharp - 2J\hat{J}K)\hat{J} = -4J\hat{J} - 2I.$$

Thus Lemma 4.1 gives

$$L_{J\xi}\sigma^\flat = (-4J\hat{J} - 2I)^\flat + 2(J\hat{J}(K\hat{J}))^\flat = -4(J\hat{J})^\flat. \qquad \square$$

Corollary 5.2. *The metric g_0 is preserved by the flow of $J\xi$:*

$$L_{J\xi}g_0 = 0.$$

Proof. Direct consequence of (5):

$$L_{J\xi}g_0 = L_{J\xi}\left(g + \frac{1}{2}\sigma^\flat\right) = 2(J\hat{J})^\flat + \frac{1}{2}(-4(J\hat{J})^\flat) = 0. \qquad \square$$

Proposition 5.3. *For every horizontal vector fields $X,\ Y,\ Z \in H$, the Levi–Civita connection ∇^{g_0} of g_0 is related to the Levi–Civita connection ∇ of g by the formula*

$$g_0\big(\nabla^{g_0}_X Y, Z\big) = g_0\big(\nabla_X Y, Z\big) + \frac{1}{3}g_0\left(\left(1 - \frac{1}{2}\sigma\right)\big[(\nabla_X\sigma)Y + \big(\nabla_{KX}\hat{J}\big)Y\big], Z\right). \quad (13)$$

Proof. Since the expression $g_0(\nabla^{g_0}_X Y, Z) - g_0(\nabla_X Y, Z)$ is tensorial, we may suppose that X, Y, Z are ∇-parallel at some point where the computation is performed. The Koszul formula for $g_0(\nabla^{g_0}_X Y, Z)$ yields directly

$$\begin{aligned} 2g_0(\nabla^{g_0}_X Y, Z) &= 2g_0(\nabla_X Y, Z) \\ &\quad + \frac{1}{2}[\langle(\nabla_X\sigma)Y, Z\rangle + \langle(\nabla_Y\sigma)X, Z\rangle - \langle(\nabla_Z\sigma)X, Y\rangle], \end{aligned} \quad (14)$$

where $\langle\cdot,\cdot\rangle$ denotes the metric g. Since $\sigma = K \circ \hat{J} = \hat{J} \circ K$ and since — according to Lemma 3.7 — K is ∇-parallel in direction of H, we obtain $\langle(\nabla_X\sigma)Y, Z\rangle = \langle(\nabla_X\hat{J})KY, Z\rangle$. Now, (11) shows that $2\omega_{\hat{J}} - \omega_K = d\xi$ is a closed 2-form. Hence,

$$\big\langle\big(\nabla_{X_1}\hat{J}\big)X_2, X_3\big\rangle + \big\langle\big(\nabla_{X_2}\hat{J}\big)X_3, X_1\big\rangle + \big\langle\big(\nabla_{X_3}\hat{J}\big)X_1, X_2\big\rangle = 0$$

for all vectors X_i. Using this equation for $X_1 = Y, X_2 = KX$ and $X_3 = Z$, we obtain

$$\begin{aligned} &\langle(\nabla_X\sigma)Y, Z\rangle + \langle(\nabla_Y\sigma)X, Z\rangle - \langle(\nabla_Z\sigma)X, Y\rangle \\ &\quad = \langle(\nabla_X\hat{J})KY, Z\rangle + \langle(\nabla_Y\hat{J})KX, Z\rangle - \langle(\nabla_Z\hat{J})KX, Y\rangle \\ &\quad = \langle(\nabla_X\hat{J})KY + (\nabla_{KX}\hat{J})Y, Z\rangle = \langle(\nabla_X\sigma)Y + (\nabla_{KX}\hat{J})Y, Z\rangle. \end{aligned}$$

The desired formula then follows from (12) and (14) using

$$\left(\mathrm{id}_H + \frac{1}{2}\sigma\right)\left(\mathrm{id}_H - \frac{1}{2}\sigma\right) = \frac{3}{4}\mathrm{id}_H. \quad (15)$$

$\square$

We consider the space of leaves, denoted by N, of the integrable distribution $V = \mathrm{span}\{\xi, J\xi\}$. The 4-dimensional manifold N is *a priori* only locally defined. It can be thought of as the base space of a locally defined principal torus bundle $\mathbb{T}^2 \hookrightarrow M \to N$. The local action of the torus is obtained by integrating the vector fields ξ and $\xi' = \frac{1}{2\sqrt{3}}J\xi$. Moreover, if one considers the 1-forms ζ and ζ' on M associated via the metric g to the vector fields ξ and $2\sqrt{3}J\xi$ it follows that $\zeta(\xi) = \zeta'(\xi') = 1$ and the Lie derivatives of ζ and ζ' in the directions of ξ and ξ' vanish by (6). Therefore ζ and ζ' are principal connection 1-forms in the torus bundle $\mathbb{T}^2 \hookrightarrow M \to N$.

A tensor field on M projects to N if and only if it is horizontal and its Lie derivatives with respect to ξ and $J\xi$ both vanish. All horizontal tensors defined above have vanishing Lie derivative with respect to ξ. Using (10) together with Corollary 5.2, we see that ω_I and g_0 project down to N. Moreover, ω_I is compatible with g_0 in the sense that

$$\omega_I(X,Y) = \frac{2}{\sqrt{3}}g_0(I_0X,Y), \quad \forall\, X,Y \in H,$$

where I_0 is the g_0-compatible complex structure on H given by

$$I_0 = \frac{2}{\sqrt{3}}\left(I - \frac{1}{2}\sigma I\right).$$

This follows directly from (12) and (15). Keeping the same notations for the projections on N of projectable tensors (like g_0 or I_0) we now prove

Theorem 5.4. *(N^4, g_0, I_0) is a Kähler manifold.*

Proof. In order to simplify notations we will denote by $\tilde{\nabla}$ and $\tilde{\nabla}^{g_0}$ the partial connections on the distribution H given by the H-projections of the Levi–Civita connections ∇ and ∇^{g_0}.

Then Proposition 5.3 reads

$$\tilde{\nabla}^{g_0}_X = \tilde{\nabla}_X + \frac{1}{3}\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)\left(\tilde{\nabla}_X\sigma + \tilde{\nabla}_{KX}\hat{J}\right). \tag{16}$$

We have to check that $\tilde{\nabla}^{g_0}_X I_0 = 0$ for all X in H. We first notice the tautological relation $\tilde{\nabla}_X \mathrm{id}_H = 0$. From Lemmas 3.6 and 3.7, we have $\tilde{\nabla}_X I = \tilde{\nabla}_X J = \tilde{\nabla}_X K = 0$ for all X in H. Moreover, the fact that I, J and K commute with $\hat{J}$ and the relation $\hat{J}^2 = \mathrm{id}_H$ easily show that $\tilde{\nabla}_X\hat{J}$ commutes with I, J, K and anti-commutes with $\hat{J}$ and σ. Consequently, $\tilde{\nabla}_X\sigma$ $(= K\tilde{\nabla}_X\hat{J})$ commutes with K and anti-commutes with I, J, $\hat{J}$ and σ for all $X \in H$.

We thus get

$$\tilde{\nabla}_X I_0 = \tilde{\nabla}_X \frac{2}{\sqrt{3}}\left(I - \frac{1}{2}\sigma I\right) = -\frac{1}{\sqrt{3}}(\tilde{\nabla}_X\sigma)I. \tag{17}$$

On the other hand, the commutation relations above show immediately that the endomorphism I_0 commutes with $(\mathrm{id}_H - \frac{1}{2}\sigma)\tilde{\nabla}_{KX}\hat{J}$ and anti-commutes with

$(\mathrm{id}_H - \frac{1}{2}\sigma)\tilde{\nabla}_X\sigma$. Thus the endomorphism $\frac{1}{3}(\mathrm{id}_H - \frac{1}{2}\sigma)(\tilde{\nabla}_X\sigma + \tilde{\nabla}_{KX}\hat{J})$ acts on I_0 by

$$\begin{aligned}\frac{1}{3}\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)(\tilde{\nabla}_X\sigma + \tilde{\nabla}_{KX}\hat{J})(I_0) &= 2\frac{1}{3}\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)(\tilde{\nabla}_X\sigma)I_0 \\ &= \frac{4}{3\sqrt{3}}\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)\tilde{\nabla}_X\sigma\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)I \\ &= \frac{4}{3\sqrt{3}}\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)\left(\mathrm{id}_H + \frac{1}{2}\sigma\right)(\tilde{\nabla}_X\sigma)I \\ &= \frac{1}{\sqrt{3}}(\tilde{\nabla}_X\sigma)I.\end{aligned}$$

This, together with (16) and (17), shows that $\tilde{\nabla}^{g_0}_X I_0 = 0$. □

We will now look closer at the structure of the metric g. Since

$$g = g_0 + \frac{1}{2}\omega_K(\hat{J}\cdot, \cdot),$$

the geometry of N, together with the form ω_K and the almost complex structure $\hat{J}$ determine completely the nearly Kähler metric g. But the discussion below will show that ω_K depends also in an explicit way on the geometry of the Kähler surface (N^4, g_0, I_0).

If α is a 2-form on H we shall denote by α' (respectively α'') the invariant (respectively anti-invariant) parts of α with respect to the almost complex structure I_0. An easy algebraic computation shows that ω_J is I_0-anti-invariant whilst

$$\omega_K' = -\frac{1}{3}(\omega_K - 2\omega_{\hat{J}}), \quad \text{and} \quad \omega_K'' = \frac{2}{3}(2\omega_K - \omega_{\hat{J}}). \tag{18}$$

Consider now the complex valued 2-form of H given by

$$\Psi = \sqrt{3}\omega_K'' + 2i\omega_J. \tag{19}$$

It appears then from Lemma 4.2 and (18) that

$$L_{\xi'}\Psi = i\Psi. \tag{20}$$

Thus Ψ is not projectable on N, but it can be interpreted as a $\mathcal{L}$-valued 2-form on N, where $\mathcal{L}$ is the complex line bundle over N associated to the (locally defined) principal S^1-bundle

$$M/\{\xi\} \to N := M/\{\xi, \xi'\}$$

with connection form ζ'.

Corollary 3.3, together with (3) implies that the curvature form of $\mathcal{L}$ equals

$$d\zeta' = -6\sqrt{3}\omega_I = -12 g_0(I_0\cdot, \cdot). \tag{21}$$

Notice that, since the curvature form of $\mathcal{L}$ is of type $(1,1)$, the Koszul–Malgrange theorem implies that $\mathcal{L}$ is holomorphic.

The following proposition computes the Ricci curvature of the Kähler surface (N^4, g, I_0) by identifying the line bundle $\mathcal{L}$ with the anti-canonical bundle of (N, I_0).

Proposition 5.5. *(N^4, g, I_0) is a Kähler–Einstein surface with Einstein constant equal to 12. Moreover, $\mathcal{L}$ is isomorphic to the anti-canonical line bundle $\mathcal{K}$ of (N^4, g, I_0).*

Proof. We first compute $\omega_J(I_0\cdot,\cdot) = -\frac{\sqrt{3}}{2}\omega''_K$ and $(\omega''_K)(I_0\cdot,\cdot) = \frac{2}{\sqrt{3}}\omega_J$. These lead to

$$\Psi(I_0\cdot,\cdot) = -i\Psi$$

in other words Ψ belongs to $\Lambda^{0,2}_{I_0}(H,\mathbb{C})$. We already noticed that by (20), Ψ defines a section of the holomorphic line bundle

$$\Lambda^{0,2}_{I_0}(N) \otimes \mathcal{L} = \mathcal{K}^{-1} \otimes \mathcal{L}. \tag{22}$$

Since Ψ is non-vanishing, this section induces an isomorphism $\Psi\colon \mathcal{K} \to \mathcal{L}$. We now show that Ψ is in fact $\tilde{\nabla}^{g_0}$*-parallel.*

Notice first that K commutes with $\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)(\tilde{\nabla}_X\sigma + \tilde{\nabla}_{KX}\hat{J})$ (it actually commutes with each term of this endomorphism), and $\tilde{\nabla}K = 0$ by Lemma 3.7. Thus (16) shows that

$$\tilde{\nabla}^{g_0}K = 0. \tag{23}$$

Furthermore, using the relation

$$g(\cdot,\cdot) = \frac{4}{3}g_0\left(\left(1 - \frac{\sigma}{2}\right)\cdot,\cdot\right),$$

Ψ can be expressed as

$$\Psi = \frac{4}{\sqrt{3}}g_0((K - iI_0K)\cdot,\cdot). \tag{24}$$

Since $\tilde{\nabla}^{g_0}g_0 = 0$ and $\tilde{\nabla}^{g_0}I_0 = 0$ (by Theorem 5.4), (23) and (24) show that Ψ is $\tilde{\nabla}^{g_0}$-parallel.

Hence the Ricci form of (N^4, g_0, I_0) is opposite to the curvature form of $\mathcal{L}$. From (21), we obtain $\mathrm{Ric}_{g_0} = 12g_0$, thus finishing the proof. □

Proposition 5.6. *The almost complex structure $\hat{J}$ on H is projectable and defines an almost Kähler structure on (N, g_0) commuting with I_0.*

Proof. Lemma 4.2 shows that $\hat{J}$ is projectable onto N. Let us denote the associated 2-form with respect to g_0 by $\omega^0_{\hat{J}}$. Identifying forms and endomorphisms via the metric g we can write

$$\omega^0_{\hat{J}} := g_0(\hat{J}\cdot,\cdot) = \left(1 + \frac{1}{2}\sigma\right)\hat{J} = \left(1 + \frac{1}{2}K\hat{J}\right)\hat{J} = \hat{J} - \frac{1}{2}K = \frac{1}{2}d\zeta.$$

This shows that $\omega^0_{\hat{J}}$ is closed, so the projection of $(g_0, \hat{J})$ onto N defines an almost Kähler structure. □

Together with Proposition 5.5, we see that the locally defined manifold N carries a Kähler structure (g_0, I_0) and an almost Kähler structure $(g_0, \hat{J})$, both obtained by projection from M. Moreover g_0 is Einstein with positive scalar curvature. If N were compact, we could directly apply Sekigawa's proof of the Goldberg conjecture in the positive curvature case in order to conclude that $(g_0, \hat{J})$ is Kähler. As we have no information on the global geometry of N, we use the following idea. On any almost Kähler Einstein manifold, a Weitzenböck-type formula was obtained in [2], which in the compact case shows by integration that the manifold is actually Kähler provided the Einstein constant is non-negative. In the present situation, we simply interpret on M the corresponding formula on N, and after integration over M we prove a pointwise statement which down back on N just gives the integrability of the almost Kähler structure.

The following result is a particular case (for Einstein metrics) of [2, Proposition 2.1].

Proposition 5.7. *For any almost Kähler Einstein manifold* (N^{2n}, g_0, J, Ω) *with covariant derivative denoted by* ∇ *and curvature tensor* R, *the following pointwise relation holds*:

$$\Delta^N s^* - 8\delta^N(\langle \rho^*, \nabla \cdot \Omega \rangle) = -8|R''|^2 - |\nabla^* \nabla \Omega|^2 - |\phi|^2 - \frac{s}{2n}|\nabla \Omega|^2, \qquad (25)$$

where s *and* s^* *are respectively the scalar and* $*$*-scalar curvature,* $\rho^* := R(\Omega)$ *is the* $*$*-Ricci form,* $\phi(X,Y) = \langle \nabla_{JX}\Omega, \nabla_Y \Omega \rangle$, *and* R'' *denotes the projection of the curvature tensor on the space of endomorphisms of* $[\Lambda^{2,0}N]$ *anti-commuting with* J.

We apply this formula to the (locally defined) almost Kähler Einstein manifold $(N, g_0, \hat{J})$ with Levi–Civita covariant derivative denoted ∇^0 and almost Kähler form $\hat{\Omega}$ and obtain

$$F + \delta^N \alpha = 0, \qquad (26)$$

where

$$F := 8|R''|^2 + |(\nabla^0)^* \nabla^0 \hat{\Omega}|^2 + |\phi|^2 + \frac{s}{4}|\nabla^0 \hat{\Omega}|^2$$

is a non-negative function on N and

$$\alpha := ds^* - 8g_0(\rho^*, \nabla^0 \cdot \hat{\Omega})$$

is a 1-form, both α and F depending in an explicit way on the geometric data $(g_0, \hat{J})$. Since the Riemannian submersion π: $(M, g_0) \to N$ has minimal (actually totally geodesic) fibers, the codifferentials on M and N are related by $\delta^M(\pi^*\alpha) = \pi^*\delta^N\alpha$ for every 1-form α on N. Thus (26) becomes

$$\pi^*F + \delta^M(\pi^*\alpha) = 0. \tag{27}$$

Notice that the function π^*F and the 1-form $\pi^*\alpha$ are well-defined *global* objects on M, even though F, α and the manifold N itself are just local. This follows from the fact that F and α only depend on the geometry of N, so π^*F and $\pi^*\alpha$ can be explicitly defined in terms of g_0 and $\hat{J}$ on M.

When M is *compact*, since π^*F is non-negative, (27) yields, after integration over M, that $\pi^*F = 0$. Thus $F = 0$ on N and this shows, in particular, that $\phi = 0$, so $\hat{J}$ is parallel on N.

6. Proof of Theorem 1.1

By the discussion above, when M is compact, $\hat{J}$ is parallel on N with respect to the Levi–Civita connection of the metric g_0, so $\hat{J}$ is $\tilde{\nabla}^{g_0}$-parallel on H.

Lemma 6.1. *The involution σ is $\tilde{\nabla}$-parallel.*

Proof. Since $\sigma = \hat{J}K$, (23) shows that $\tilde{\nabla}^{g_0}\sigma = 0$. Using (16) and the fact that σ anti-commutes with $\tilde{\nabla}_X\sigma$ and $\tilde{\nabla}_X\hat{J}$ for every $X \in H$, we obtain

$$\tilde{\nabla}_X\sigma + \frac{2}{3}\left(\mathrm{id}_H - \frac{1}{2}\sigma\right)(\tilde{\nabla}_X\sigma + \tilde{\nabla}_{KX}\hat{J})\sigma = 0$$

for all X in H. Since I commutes with $\tilde{\nabla}_X\hat{J}$ and anti-commutes with σ and $\tilde{\nabla}_X\sigma$, the I-invariant part of the above equation reads

$$\frac{2}{3}(\tilde{\nabla}_X\sigma)\sigma + \frac{1}{3}\tilde{\nabla}_{KX}\hat{J} = 0. \tag{28}$$

But $\sigma = \hat{J}K$ and $\tilde{\nabla}K = 0$, so from (28), we get

$$2(\tilde{\nabla}_X\hat{J})\hat{J} = \tilde{\nabla}_{KX}\hat{J}.$$

Replacing X by KX and applying this formula twice yields

$$\tilde{\nabla}_X\hat{J} = -2(\tilde{\nabla}_{KX}\hat{J})\hat{J} = 4\tilde{\nabla}_X\hat{J},$$

thus proving the lemma. □

We now recall that the first canonical Hermitian connection of the NK structure (g, J) is given by

$$\bar{\nabla}_U = \nabla_U + \frac{1}{2}(\nabla_U J)J$$

whenever U is a vector field on M. We will show that (M^6, g) is a homogeneous space actually by showing that $\bar{\nabla}$ is a Ambrose–Singer connection, that is $\bar{\nabla}\bar{T} = 0$ and $\bar{\nabla}\bar{R} = 0$, where $\bar{T}$ and $\bar{R}$ denote the torsion and curvature tensor of the canonical connection $\bar{\nabla}$.

Let $H_\pm$ be the eigen-distributions of the involution σ on H, corresponding to the eigenvalues ± 1. We define the new distributions

$$E = \langle \xi \rangle \oplus H_+ \quad \text{and} \quad F = \langle J\xi \rangle \oplus H_-.$$

Obviously, we have a g-orthogonal splitting $TM = E \oplus F$, with $F = JE$.

Lemma 6.2. *The splitting $TM = E \oplus F$ is parallel with respect to the first canonical connection.*

Proof. For every tangent vector U on M we can write

$$\bar\nabla_U \xi = \nabla_U \xi + \frac{1}{2}(\nabla_U J)J\xi = \hat J U - \frac{1}{2}KU + \frac{1}{2}JIU = (\sigma + 1)\hat J U,$$

showing that $\bar\nabla_U \xi$ belongs to E (actually to H_+) for all U in TM.

Let now Y_+ be a local section of H_+. We have to consider three cases. First,

$$\bar\nabla_\xi Y_+ = \nabla_\xi Y_+ + \frac{1}{2}(\nabla_\xi J)JY_+ = L_\xi Y_+ + \nabla_{Y_+}\xi + \frac{1}{2}IJY_+ = L_\xi Y_+ + \hat J Y_+$$

belongs to H_+ since L_ξ and $\hat J$ both preserve H_+. Next, if X belongs to H, then

$$\bar\nabla_X Y_+ = \tilde\nabla_X Y_+ + \langle \bar\nabla_X Y_+, \xi\rangle \xi + \langle \bar\nabla_X Y_+, J\xi\rangle J\xi.$$

But $\langle \bar\nabla_X Y_+, J\xi\rangle = \langle JY_+, \bar\nabla_X \xi\rangle = 0$ by the above discussion and the fact that JY_+ is in H_-, and $\tilde\nabla_X Y_+$ is an element of H_+ by Lemma 6.1. Thus $\bar\nabla_X Y_+$ belongs to E.

The third case to consider is

$$\begin{aligned}
\bar\nabla_{J\xi} Y_+ &= \nabla_{J\xi} Y_+ + \frac{1}{2}(\nabla_{J\xi} J)JY_+ = L_{J\xi} Y_+ + \nabla_{Y_+} J\xi - \frac{1}{2}\nabla_\xi Y_+ \\
&= L_{J\xi} Y_+ + (\nabla_{Y_+} J)\xi + J\nabla_{Y_+}\xi - \frac{1}{2}IY_+ \\
&= L_{J\xi} Y_+ - IY_+ + J\left(\hat J Y_+ - \frac{1}{2}KY_+\right) - \frac{1}{2}IY_+ \\
&= L_{J\xi} Y_+ - 2IY_+ + J\hat J Y_+.
\end{aligned}$$

On the other hand

$$L_{J\xi} Y_+ = L_{J\xi} \sigma Y_+ = \sigma L_{J\xi} Y_+ + (L_{J\xi}\sigma) Y_+,$$

so the H_--projection of $L_{J\xi} Y_+$ is

$$\pi_{H_-} L_{J\xi} Y_+ = \frac{1 - \sigma}{2} L_{J\xi} Y_+ = \frac{1}{2}(L_{J\xi}\sigma) Y_+.$$

Using (8) and (9) and the previous calculation, we get

$$\begin{aligned}
\pi_{H_-}(\bar\nabla_{J\xi} Y_+) &= \pi_{H_-}(L_{J\xi} Y_+ - 2IY_+ + J\hat J Y_+) \\
&= \pi_{H_-}\left(\frac{1}{2}(L_{J\xi}\sigma) Y_+ - 2IY_+ + J\hat J Y_+\right) \\
&= \pi_{H_-}((I - 2J\hat J - 2I + J\hat J) Y_+) = -\pi_{H_-}((1 + \sigma)(Y_+)) = 0.
\end{aligned}$$

Thus E is $\bar\nabla$-parallel, and since $F = JE$ and $\bar\nabla J = 0$ by definition, we see that F is $\bar\nabla$-parallel, too. □

Therefore the canonical Hermitian connection of (M^6, g, J) has reduced holonomy, more precisely complex irreducible but real reducible. Using [16, p. 487, Corollary 3.1], we obtain that $\bar{\nabla}\bar{R} = 0$. Moreover, the condition $\bar{\nabla}\bar{T} = 0$ is always satisfied on a NK manifold (see [3, lemma 2.4], for instance). The Ambrose–Singer theorem shows that if M is simply connected, then it is a homogeneous space. To conclude that (M, g, J) is actually $S^3 \times S^3$ we use the fact that the only homogeneous NK manifolds are $S^6, S^3 \times S^3, \mathbb{C}P^3, F(1, 2)$ (see [5]) and among these spaces only $S^3 \times S^3$ has vanishing Euler characteristic. If M is not simply connected, one applies the argument above to the universal cover of M which is compact and finite by Myers' theorem. The proof of Theorem 1.1 is now complete. □

7. The Inverse Construction

The construction of the (local) torus bundle $M^6 \to N^4$ described in the previous sections gives rise to the following Ansatz for constructing local NK metrics.

Let (N^4, g_0, I_0) be a (not necessarily complete) Kähler surface with $\mathrm{Ric} = 12g_0$ and assume that g_0 carries a compatible almost-Kähler structure $\hat{J}$ which commutes with I_0. Let $\mathcal{L} \to N$ be the anti-canonical line bundle of (N^4, g_0, I_0) and let $\pi_1\colon M_1 \to N$ be the associated principal circle bundle. Fix a principal connection form θ in M_1 with curvature $-12\omega_{(g_0, I_0)}$. Let H be the horizontal distribution of this connection and let Φ in $\Lambda^{0,2}_{I_0}(H, \mathbb{C})$ be the "tautological" 2-form obtained by the lift of the identity map $1_{\mathcal{L}^{-1}}\colon \mathcal{L}^{-1} \to \mathcal{L}^{-1}$.

Give M_1 the Riemannian metric

$$g_1 = \theta \otimes \theta + \frac{2}{3}\pi_1^\star g_0 - \frac{1}{2\sqrt{3}}(\mathrm{Re}\Phi)(\hat{J}\cdot, \cdot). \tag{29}$$

Let now M denote the principal S^1-bundle $\pi\colon M \to M_1$ with first Chern class represented by the closed 2-form $\Omega = 2\pi_1^\star g_0(\hat{J}\cdot, \cdot)$. Since we work locally we do not have to worry about integrability matters. Let μ be a connection 1-form in M and give M the Riemannian metric

$$g = \mu^2 + \pi^\star g_1.$$

We consider on M the 2-form

$$\omega = \frac{1}{2\sqrt{3}}\mu \wedge \pi^\star\theta + \frac{1}{2}\pi^\star(\mathrm{Im}\Phi). \tag{30}$$

By a careful inspection of the discussion in the previous sections, we obtain

Proposition 7.1. (M^6, g, ω) *is a nearly Kähler manifold of constant type equal to* 1. *Moreover, the vector field dual to* μ *is a unit Killing vector field.*

Notice that the only *compact* Kähler–Einstein surface (N^4, g_0, I_0) with $\mathrm{Ric} = 12g_0$ possessing an almost Kähler structure commuting with I_0 is the product of two spheres of radius $\frac{1}{2\sqrt{3}}$ (see [2]), which corresponds, by the above procedure, to the nearly Kähler structure on $S^3 \times S^3$. Thus the new NK metrics provided by our Ansatz cannot be compact, which is concordant with Theorem 1.1.

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